ON A TAMELY RAMIFIED LOCAL RELATIVE LANGLANDS CONJECTURE VIA CATEGORICAL REPRESENTATIONS

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Abstract. Let G be a complex reductive group. For a smooth affine spherical G-variety X, assume that the unramified relative local Langlands conjecture of [BSV24, Conjecture 7.5.1] for X holds, the loop space LX is an L^+G–placid ind–scheme, and there exists a dimension theory for LX, we give a spectral description of a full subcategory of Iwahori equivariant D-modules on LX in terms of the relative Langlands dual of X, confirming a slight variant of the tamely ramified local relative Langlands conjecture proposed by Devalapurkar [Dev24, Conjecture 3.4.14.].

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1. INTRODUCTION

Sakellaridis-Venkatesh [SV17] propose a framework of the relative Langlands program, in which the local harmonic analysis of spherical G-varieties is studied using Langlands parameters of the Langlands dual group $\check{G}$, together with the global analogue, and establish relations between period integrals of automorphic forms for such varieties with L-functions of Galois representations. These conjectures were later formulated by Ben–Zvi–Sakellaridis-Venkatesh [BSV24] in terms of dualities between graded Hamiltonian G-spaces M and $\check{G}$-spaces $\check{M}$, known as relative Langlands duality.

In particular, the local harmonic analysis story admits a categorification for its unramified part [BSV24, Conjecture 7.5.1] in terms of this duality (see also subsection 3.7). Beyond the unramified case, some tamely ramified relative local Langlands conjectures are formulated in [Dev24, Conjecture 3.4.14], [FGT25, Conjecture 1.1.3], and Travkin–Yang's upcoming work[1]. The main result of our paper proves a slight variant of [Dev24, Conjecture 3.4.14] for a class of spherical G-varieties.

Let us first briefly review the unramified relative local Langlands conjecture (see also Conjecture 3.29 for more details). Let G be a complex reductive group and $\check{\mathfrak{g}}$ denote the Lie algebra of the Langlands dual group $\check{G}$. Let X be a spherical G-variety (see Definition 3.27). Let L^+G, resp. L^+X, denote the positive loop group, resp. positive loop space, and let LG, resp. LX denote the loop group resp. loop space. Denote by $\mathcal{D}_c(-)$ the (small) stable ∞-category of *coherent* $\mathcal{D}$-modules, see Definition 3.23. The monoidal category $\mathcal{D}_c(L^+G\backslash \mathrm{Gr})$ acts on $\mathcal{D}_c(L^+G\backslash LX)$ by convolution from the left, and we consider the full subcategory $\mathcal{D}_c(L^+G\backslash LX)^{\mathrm{Sat}} \subseteq \mathcal{D}_c(L^+G\backslash LX)$ generated by the unramified basic object IC_0, the !-extension of constant $\mathcal{D}$-module on $L^+G\backslash L^+X$, under this action, see Definition 3.28.

Conjecture 1.1 (Unramified conjecture in the de-Rham setting)**.** There is an equivalence of categories

$$\mathcal{D}_c(L^+G\backslash LX)^{\mathrm{Sat}} \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\check{M})/\check{G}),$$

which satisfies the *pointing condition*: IC_0 is sent to $\mathcal{O}_{\mathrm{sh}^{1/2}(\check{M})}$; and is compatible with the action coming from Bezrukavnikov–Finkelberg's derived geometric Satake equivalence [BF08]

$$\mathcal{D}_c(L^+G\backslash LG/L^+G) \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G}).$$

The shearing functor $\mathrm{sh}^{1/2}$ is defined in subsection 3.1. We discuss the known cases of this conjecture in subsection 3.7.

We assume that our smooth affine spherical G variety X satisfies the following standing assumptions:

Assumption 1.2.

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[1] We were informed by Ruotao Yang about an upcoming joint work with Roman Travkin on the Iwahori Gaiotto conjecture.

(a) the colimit of L^+G-orbit closures in LX gives rise to a presentation of LX as an L^+G-placid ind-scheme,

(b) LX admits a dimension theory in the sense of [Ras],

(c) the unramified local relative Langlands Conjecture 1.1 holds.

Let $I \subset L^+G$ be an Iwahori subgroup and $\check{\mathfrak{n}}$ be the Lie algebra of the nilpotent radical of $\check{B}$. Let $\tilde{\check{\mathfrak{g}}}^*(2) := \check{G} \times^{\check{B}} \check{\mathfrak{n}}^{\perp}(2) = T^*(2)(\check{G}/\check{N})/\check{T}$ be the Grothendieck–Springer resolution, with the $\mathbb{G}_m$-action given by weight 2 on $\check{\mathfrak{n}}^{\perp}$ see Definition 3.5. We define the Iwahori level Satake category $\mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}}$ in Definition 3.28 to be the full subcategory of $\mathcal{D}_c(I\backslash LX)$ generated by $\mathrm{IC}_0 \in \mathcal{D}(L^+G\backslash LX)$ under the left action of $\mathcal{D}_c(I\backslash LG/L^+G)$.

Theorem 1.3 (Theorem 5.1)**.** Under Assumption 1.2, there is an equivalence

$$\mathbb{L}^{\mathrm{Sat}} : \mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}} \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\tilde{\check{\mathfrak{g}}}^*(2) \times_{\check{\mathfrak{g}}^*(2)} \check{M})/\check{G}), \tag{1.1}$$

which is equivariant with respect to

$$\mathrm{End}_{\mathcal{D}_c(L^+G\backslash LG/L^+G)}(\mathcal{D}_c(I\backslash LG/L^+G)) \simeq \mathrm{End}_{\mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G})}(\mathrm{Perf}(\tilde{\check{\mathfrak{g}}}^*[2]/\check{G})). \tag{1.2}$$

We make the following remarks on the relation between our result with [Dev24, Conjecture 3.4.14].The equivalence (1.1) is conjectured in *loc. cit*, which gives a spectral description of the Satake full subcategory $\mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}} \subseteq \mathcal{D}_c(I\backslash LX)$. It is unclear how to formulate the conjecture for the entire category $\mathcal{D}_c(I\backslash LX)$ for general X, and it would be an interesting question to study such a formulation.

In *loc. cit*, equivalence (1.1) is expected to be equivariant with respect to the actions that come from Bezrukavnikov's equivalence [Bez16]. Our equivalence (1.2) may be interpreted as the "self-tensor product" of the Arkhipov–Bezrukavnikov–Ginzburg's equivalence [ABG04; Gai18] over the derived geometric Satake equivalence of Bezrukavnikov-Finkelberg [BF08]. We do not prove the actions of 1.2 are compatible with those of Bezrukavnikov's equivalence.

With the machinery developed in [Zhu25], we expect that our strategy can be applied to establish our main theorem in the ℓ-adic setting[2].

1.1. **Strategy of the proof of the main theorem.** Our proof is motivated by the following idea in categorical representation theory. Let $K \subset H$ be affine group schemes. For a DG category $\mathcal{C}$ admitting an action of $\mathcal{D}(LH)$, its K-invariants

$$\mathcal{C}^K := \mathrm{Hom}_{\mathcal{D}(K)}(\mathrm{Vect}, \mathcal{C})$$

naturally carries a right action of $\mathcal{H} := \mathrm{End}_{\mathcal{D}(H)}(\mathcal{D}(H)_K)$. If the prounipotent radical of K has finite codimension, we have identification $\mathcal{H} \simeq \mathcal{D}(K\backslash H/K)$ and the functor of taking

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[2] We thank Xinwen Zhu for helpful discussions.

K-invariants admits a left adjoint which sends any $\mathcal{C}$ admitting a right $\mathcal{D}(H)$-action to $\mathcal{C} \otimes_{\mathcal{H}} \mathcal{D}(H)_K$.

If in addition H/K is ind-proper, an important observation of Campbell–Dhillon [CD21, Theorem 3.1.5] allows us to reconstruct a full subcategory of $\mathcal{C}$ from its K-invariants. More precisely, the natural action functor

$$\mathcal{D}(H/K) \otimes_{\mathcal{H}} \mathcal{C}^K \to \mathcal{C}$$

is a fully faithful functor of $\mathcal{D}(H)$-modules.

Let Gr resp. Fl denote the affine Grassmannian resp. affine flag variety of G. Adapting Campbell–Dhillon's theorem to our setting yields a fully faithful functor

$$F : \mathcal{D}(I \backslash \mathrm{Gr}) \otimes_{\mathcal{D}(L^+G \backslash \mathrm{Gr})} \mathcal{D}(L^+G \backslash LX) \to D(I \backslash LX),$$

which induces a functor

$$\iota : \mathcal{D}_c(I \backslash \mathrm{Gr}) \otimes_{\mathcal{D}_c(L^+G \backslash \mathrm{Gr})} \mathcal{D}_c(L^+G \backslash LX) \to \mathcal{D}(I \backslash \mathrm{Gr}) \otimes_{\mathcal{D}(L^+G \backslash \mathrm{Gr})} \mathcal{D}(L^+G \backslash LX).$$

Establishing the full faithfulness of ι is the central step in the proof of our main theorem which we discuss in section 4. We recall that the left hand side of ι is the relative tensor product of *small* categories which brings technical difficulties to work with. We prove the desired full faithfulness of ι by a calculation of monads in Proposition 4.8 and a Künneth type formula in Lemma 4.5. With the embedding ι, we derive an equivalence

$$F_c^{\mathrm{Sat}} : \mathcal{D}_c(I \backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(K \backslash LX)^{\mathrm{Sat}} \simeq \mathcal{D}_c(I \backslash LX)^{\mathrm{Sat}},$$

of categories by considering generators of the spherical and Iwahori Satake subcategories in Theorem 4.1.

To pass to the spectral side, we make crucial use of the integral transform of Ben–Zvi–Francis–Nadler [BFN10]. An integral transform is the categorical avatar of integration against a kernel: a correspondence $Z \subset X \times_S Y$ equipped with a kernel $\mathcal{K}$ encodes a push–pull functor on the derived categories of bounded complexes of coherent sheaves $\Phi_{\mathcal{K}} : \mathrm{Coh}^b(X) \to \mathrm{Coh}^b(Y)$, $\Phi_{\mathcal{K}}(\mathcal{F}) = p_{Y*}\big(\mathcal{K} \otimes p_X^* \mathcal{F}\big)$, where p_X and p_Y are the natural projections. This "correspondences act by kernels" principle underlies Fourier–Mukai theory and the Hecke/Whittaker kernels that implement passage between automorphic and spectral categories, see [NY19; Gai21].

For perfect derived stacks X, Y over a base S, Ben–Zvi–Francis–Nadler [BFN10] establishes canonical equivalences

$$\mathrm{QCoh}(X \times_S Y) \simeq \mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(Y) \simeq \mathrm{Fun}_{\mathrm{QCoh}(S)}\big(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)\big),$$

which realize continuous $\mathrm{QCoh}(S)$-linear functors as integral transforms with kernels in $\mathrm{QCoh}(X \times_S Y)$. Passing to compact objects, there is an identification

$$\mathrm{Perf}(X \times_S Y) \simeq \mathrm{Perf}(X) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(Y). \tag{1.3}$$

With the Arkhipov–Bezrukavnikov–Ginzburg's equivalence [ABG04; Gai18], derived geometric Satake equivalence [BF08], and the unramified local geometric Langlands conjecture for X (see Conjecture 1.1), we identify the left hand side of F_c^{Sat} with

$$\mathrm{Perf}(\mathrm{sh}^{1/2}(\tilde{\check{\mathfrak{g}}}(2) \times_{\check{\mathfrak{g}}^*(2)} \check{M})/\check{G}) \simeq \mathrm{Perf}(\tilde{\check{\mathfrak{g}}}^*[2]/\check{G}) \otimes_{\mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G})} \mathrm{Perf}(\mathrm{sh}^{1/2}(\check{M})/\check{G}).$$

The exact same argument establishes (1.2) and it follows from our construction that (1.1) is compatible with the action of (1.2) which completes the proof of our main theorem.

1.2. **Organization.** In Section 3, we recall some preliminaries for this paper: the sheafification functor, higher algebra, and the theory of $\mathcal{D}$-modules. In Section 3.6, we recall the material from the categorical actions literature that we require. In Section 3.7, we recall the unramified local relative duality and state the known cases. Section 4 is the technical core of the paper. We construct a fully faithful functor F_c^{Sat} from the automorphic side to the spectral side in Theorem 3.25. In Section 5, we complete the proof of our main theorem Theorem 5.1.

1.3. **Notations.** We introduce the following notations, which will be used throughout the paper, unless otherwise specified.

We let k denote an algebraically closed field of characteristic 0.

Fix G a connected reductive group over k, with a chosen Borel subgroup B, and a maximal torus $T \subset B$. Let I be the Iwahori subgroup corresponding to B. We denote by $\mathbb{X}^*(T)$, resp. $\mathbb{X}_*(T)$, the character, resp. cocharacter, lattice. We let $W := N_G(T)/T$ denote the finite Weyl group.

Let $\check{G}$ denote the Langlands dual group of G and $\check{\mathfrak{g}}$ the Lie algebra of $\check{G}$.

For a finite type scheme X over k, let L^+X denote the positive loop space and LX the loop space. We write $K = L^+G$ for simplicity. Let Gr resp. Fl denote the affine Grassmannian resp. affine flag variety of G.

We denote by $\mathcal{D}(-)$ the DG-category of $\mathcal{D}$-modules, $\mathcal{D}_c(-)$ the small full subcategory of coherent $\mathcal{D}$-modules, and $\mathcal{D}^{\mathrm{ren}}(-) := \mathrm{Ind}(\mathcal{D}_c(-))$ the renormalized category of $\mathcal{D}$-modules.

We denote by $\mathcal{H}_I := \mathcal{D}(I\backslash \mathrm{LG}/I)$ the affine Hecke category, $\mathcal{H}_{c,I}$ the (small) affine Hecke category of coherent $\mathcal{D}$-modules, and $\mathcal{H}_I^{\mathrm{ren}}$ the renormalized affine Hecke category. Let $\mathcal{H}_K := \mathcal{D}(L^+G\backslash \mathrm{Gr})$ be the spherical Hecke category, $\mathcal{H}_{c,K}$ the small spherical Hecke category of $\mathcal{D}$-modules, and $\mathcal{H}_K^{\mathrm{ren}}$ the renormalized spherical Hecke category.

2. Acknowledgments

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3. CONVENTIONS

Throughout this paper, we will substantially use the language of ∞-categories, as developed in [Lur09b; Lur09a]. Let Ani denote the ∞-category of ∞-groupoids, or *anima*. Given an ∞-category $\mathcal{C}$, and a pair of objects $c_1, c_2 \in \mathcal{C}$, we let

$$\mathrm{Map}_{\mathcal{C}}(c_1, c_2) \in \mathrm{Ani}$$

be the mapping anima between them. Let $\widehat{\mathrm{Cat}}_\infty$ be the category of all (not necessarily small) categories. We will frequently use the following subcategories of $\widehat{\mathrm{Cat}}_\infty$. Let us first describe the "large" setting.

Let $\left(\mathrm{Pr}^L, \otimes, \mathrm{Ani}\right)$ denote the subcategory of (not necessarily small) categories $\widehat{\mathrm{Cat}}_\infty$, spanned by presentable ∞-categories and colimit preserving functors, with symmetric monoidal structure, as in [Lur09a, Section 4.8.1], denoted $\otimes$, and unit Ani [Lur09b].

Let $\mathrm{Pr}^L_{\mathrm{st}} \hookrightarrow \mathrm{Pr}^L$ be the subcategory of presentable stable categories with colimit preserving functors, which is *lax* monoidal, cf. [Lur09a, Proposition 4.8.2.18], which we again abusively denote $\otimes$.

Let R be an $\mathbb{E}_\infty$ ring spectrum then $\mathrm{LMod}_R \in \mathrm{CAlg}(\mathrm{Pr}^L)$, allowing us consider left module objects:

$$\mathrm{LinCat}_R := \mathrm{LMod}_{\mathrm{LMod}_R}(\mathrm{Pr}^L)$$

is the ∞-category of R-linear categories. Following conventions of [GR17] (see also [Coh16]), we denote

$$\mathrm{DGCat} := \mathrm{LinCat}_k,$$

whose unit object is $\mathrm{Vect} := \mathrm{LMod}_k$. Let $\mathcal{C} \in \mathrm{DGCat}$ and $c_1, c_2 \in \mathrm{DGCat}$. We use

$$\underline{\mathrm{Map}}_{\mathcal{C}}(c_1, c_2)$$

to denote the the Vect-enriched mapping space. The ∞-groupoid $\mathrm{Map}_{\mathcal{C}}(c_1, c_2)$ is obtained from $\underline{\mathrm{Map}}_{\mathcal{C}}(c_1, c_2)$ as the underlying anima.

3.1. **Shearing.** In this section, we recall the shearing functor from [Dev24; BSV24]. We let k denote our base field.

Throughout the paper, we adopt cohomological degrees, i.e. if $M \in \mathrm{Mod}_k$, viewed as a chain complex whose i-th degree is denoted M^i, then $(M[n])^i = M^{i+n}$. For example, $k[n]$ denotes a complex with k sitting in degree $-n$.

For $(\mathcal{C}, \otimes, 1)$ a symmetric monoidal category, we denote $\mathcal{C}^{\mathrm{gr}} := \mathrm{Fun}(\mathbb{Z}_{\mathrm{ds}}, \mathcal{C})$ the category of graded objects in $\mathcal{C}$, where $\mathbb{Z}_{\mathrm{ds}}$ is $\mathbb{Z}$ regarded as a discrete symmetric monoidal category via its addition. $\mathcal{C}^{\mathrm{gr}}$ equipped with Day convolution as a monoidal structure, [Lur09a, Cor 4.8.1.12]. In particular, for two objects $(X_\bullet), (Y_\bullet) \in \mathcal{C}^{\mathrm{gr}}$, we have

$$(X_\bullet \otimes Y_\bullet)_{n\in\mathbb{Z}} = \bigsqcup_{i+j=n} X_i \otimes Y_j$$

Definition 3.1. [Dev24, Construction 2.1.1] Denote $\frac{1}{2}$-*shearing* to be the $\mathbb{E}_1$ monoidal equivalence as

$$\mathrm{sh}^{1/2} : \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$$

characterized by

$$M = \bigoplus_{w \in \mathbb{Z}} M_w \mapsto \mathrm{sh}^{1/2}(M) := \bigoplus_{w \in \mathbb{Z}} M_w[w]$$

where M_w is the weight w part of M.

Proposition 3.2. [Dev24, Remark 2.1.9] The shearing functor $\mathrm{sh}^{1/2}$ restricted to even graded modules is symmetric monoidal functor

$$\mathrm{sh}^{1/2} : \mathrm{Mod}^{\mathrm{gr}}_{\mathbb{Z},\mathrm{even}} \to \mathrm{Mod}^{\mathrm{gr}}_{\mathbb{Z},\mathrm{even}}$$

Thus, when we refer graded $\mathbb{E}_\infty$ connective dg-k-algebras A, concenrated in even cohomological degrees, $\mathrm{sh}^{1/2}(A)$ is again an $\mathbb{E}_\infty$ connective dg-k-algebra. Another way to upgrade to symmetric monoidal equivalence is to work with supervector spaces, see [BSV24, Chapter 6].

Example 3.3. If $M = \bigoplus_{i \in \mathbb{Z}} M_i \in \mathrm{Mod}_k^{\mathrm{gr},\heartsuit}$, be a discrete graded module, where $M_i \in \mathrm{Mod}_k^{\heartsuit}$ is weight i component of M, then $M_i[i]$ lives in degree $-i$ and weight i.

Definition 3.4. Let $V \in \mathrm{Mod}_k$. We denote $V(n) \in \mathrm{Mod}_k^{\mathrm{gr}}$ be the graded module of V where $\mathbb{G}_m$ acts via weight n, i.e. $t \cdot v = t^n v$ for $t \in \mathbb{G}_m, v \in V$.

Now let us briefly recall the formal construction for symmetric algebra.

Let $(\mathcal{C}, \otimes, 1)$ be a presentable stable unital symmetric monoidal category where $\otimes$ preserves colimits separately in each variable. Let $\mathrm{CAlg}(\mathcal{C})$ denote the ∞-category of commutative algebra objects in $\mathcal{C}$, and $U_{\mathcal{C}} : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. We have an adjunction

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C}) \arrow[d, bend left, "U_{\mathcal{C}}"] \\
\mathcal{C}, \arrow[u, bend left, "\mathrm{Sym}^\bullet_{\mathcal{C}}"]
\end{tikzcd}$$

which is given by the following explicit formula

$$\mathrm{Sym}^\bullet(M) \simeq \bigoplus_{n \geq 0} \mathrm{Sym}^n(M), \qquad \mathrm{Sym}^n(M) \simeq \left(M^{\otimes_{\mathcal{C}} n}\right)_{\Sigma_n}$$

where $(-)_{\Sigma_n}$ denotes homotopy orbits for the permutation action of Σ_n. We will simply denote $\mathrm{Sym}^\bullet := \mathrm{Sym}^\bullet_{\mathcal{C}}$ when the context is clear: we either work with $\mathcal{C} = \mathrm{Mod}_k^{\mathrm{gr}}$, or more generally, $\mathcal{C} = \mathrm{QCoh}(X/\mathbb{G}_m) \simeq \mathrm{QCoh}(X)^{\mathrm{gr}}$ for a scheme X.

We now globalize the construction for a $\mathcal{O}_X$ -modules for a scheme X. For a locally free sheaf $\mathcal{F}$ over $X \in \mathrm{Sch}_k$ and $n \in \mathbb{Z}$, we denote $\mathcal{F}(n)$ to be the graded locally free sheaf over X where

$\mathbb{G}_m$ acts by weight n on the fibers. We denote $\mathcal{F}[n]$ (or $\mathrm{sh}^{1/2}\mathcal{F}(n)$) to be the corresponding graded locally free sheaf where we shear fiberwise of $\mathcal{F}(n)$ by $\mathrm{sh}^{1/2}$.

Definition 3.5. For a smooth scheme X and $j \in \mathbb{Z}$, we denote $\mathcal{T}_X(j)$ to be the tangent sheaf of X, $T^*(j)X$ to be the cotangent bundle of X, where $\mathbb{G}_m$ acts on the stalks/fibers by weight j.

Definition 3.6 (Shifted vector bundles)**.** Let V be a finite-dimensional k-module. Let $(-)^*$ denote the k-linear dual. We define $V[n] := \mathrm{sh}^{1/2}V(n)$ to be the derived affine k-scheme whose coordinate rings is

$$\mathrm{sh}^{1/2}\left(\mathrm{Sym}^{\bullet} V(n)^*\right) = \bigoplus_{j\geq 0} \mathrm{sh}^{1/2}\,\mathrm{Sym}^j(V^*)(-nj) = \bigoplus_{j\geq 0} \mathrm{Sym}^j(V^*)[-nj].$$

Similarly, for a scheme X and $n \in \mathbb{Z}$, we defined $T^*[n]X$ (see also [Pan+13, Definition 1.20]) to be the relative spectrum $\underline{\mathrm{Spec}}_X$ (cf. [Lur18, p. 2.5.1.3] for the spectral analogue for definition of relative spectrum) of

$$\mathrm{sh}^{1/2}\left(\mathrm{Sym}^{\bullet}\mathcal{T}_X(-n)\right) = \bigoplus_{j\geq 0} \mathrm{sh}^{1/2}\,\mathrm{Sym}^j\,\mathcal{T}_X(-nj) = \bigoplus_{j\geq 0} \mathrm{Sym}^j\,\mathcal{T}_X[-nj].$$

3.2. **Compact generation.** As we will be passing between large and small categories, we briefly recall the machinery needed in the below.

Definition 3.7. Let $\mathcal{C} \in \widehat{\mathrm{Cat}}_\infty$. Let $\{c_\alpha\}$ be a collection of objects in $\mathcal{C}$, denote $\langle c_\alpha \rangle$ the smallest cocomplete stable subcategory of $\mathcal{C}$ containing all c_α. Equivalently, by [GR17, Proposition 5.4.5], we have

$$\mathrm{Map}_{\mathcal{C}}(c_\alpha[-i], c) = 0,\ \forall i \geq 0 \implies c = 0.$$

Definition 3.8. We call an object $c \in \mathcal{C}$ *compact* if the Yoneda functor

$$\mathrm{Maps}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Ani}$$

preserves filtered colimits. We denote $\mathcal{C}^\omega \subset \mathcal{C}$ be full subcategory spanned by compact objects.

Definition 3.9. Let $\mathcal{C} \in \widehat{\mathrm{Cat}}_\infty$. Then $\mathcal{C}$ is *compactly* generated if it satisfies the following equivalent conditions:

(1) $\mathcal{C}$ admits small filtered colimits and every object can be realized as a colimit of small filtered diagram $\{c_\alpha\}$ where $c_\alpha \in \mathcal{C}$ is a compact object.

(2) The natural functor $\mathrm{Ind}(\mathcal{C}^\omega) \to \mathcal{C}$ is an equivalence, where $\mathrm{Ind}(\mathcal{D})$ for a small category $\mathcal{D}$ is the *ind-completion* of $\mathcal{D}$ characterized by the universal property of admitting a map $\mathcal{D} \to \mathrm{Ind}(\mathcal{D})$, such that $\mathrm{Ind}(\mathcal{D})$ admits small filtered colimit, and precomposition induces an equivalence

$$\mathrm{Fun}^\omega(\mathrm{Ind}(\mathcal{D}), \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{D}, \mathcal{E})$$

where $\mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ is the full subcategory of functors commuting with small filtered colimits, see also [Lur09b, §5.3].

Definition 3.10. Let $\mathrm{Pr}^L_{\mathrm{st},\omega} \hookrightarrow \mathrm{Pr}^L$ denote the full subcategory of compactly generated stable presentable categories.

Proposition 3.11. Let $\mathcal{C}$ be a stable category admitting all small colimits. Then the following are equivalent:

(1) $\mathcal{C}$ is compactly generated.

(2) $\mathcal{C}$ admits a small set of compact object $S \hookrightarrow \mathcal{C}^{\omega}$ such that for

$$\mathrm{Map}(s[-i], c) \simeq 0 \quad \forall i \geq 0, s \in S.$$

Proof. This is [GR17, Lem 7.2.3(4)] and universal property of stabilization, [Lur09a, §1.4.4.5], where we can identify

$$\mathrm{Ind}(\mathcal{C}_0) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Sp})$$

for a small stable category $\mathcal{C}_0$, where $\mathrm{Fun}^{\mathrm{ex}}(-,-)$ denotes the full subcategory of $\mathrm{Fun}(-,-)$ spanned by exact functors. ☐

Definition 3.12.

(1) Let $\mathrm{Cat}^{\mathrm{perf}} \hookrightarrow \mathrm{Cat}$ denote the subcategory small idempotent complete stable ∞-categories with right exact functor.

(2) The category $\mathrm{Cat}^{\mathrm{perf}}$ admits a symmetric monoidal structure given by the tensor product $\otimes$ which is defined as

$$\mathcal{A} \otimes \mathcal{B} \simeq (\mathrm{Ind}\mathcal{A} \otimes \mathrm{Ind}\mathcal{B})^{\omega}\,, \quad \mathcal{A}, \mathcal{B} \in \mathrm{Cat}^{\mathrm{perf}},$$

which gives rise to symmetric monoidal adjunction equivalence

$$\begin{tikzcd} \left(\mathrm{Cat}^{\mathrm{perf}}, \otimes\right) \arrow[r, shift left, "\mathrm{Ind}"] & \mathrm{Pr}^L_{\mathrm{st},\omega}. \arrow[l, shift left, "(-)^{\omega}"] \end{tikzcd}$$

(3) We define the relative tensor product in $\mathrm{Cat}^{\mathrm{perf}}$ as

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} := (\mathrm{Ind}\mathcal{M} \otimes_{\mathrm{Ind}\mathcal{A}} \mathrm{Ind}\mathcal{N})^{\omega}\,.$$

(4) Let R be an $\mathbb{E}_{\infty}$ ring spectrum. Then $\mathrm{Perf}(R) \in \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$, and we define

$$\mathrm{Cat}^{\mathrm{perf}}_R := \mathrm{LMod}_{\mathrm{Perf}(R)}\left(\mathrm{Cat}^{\mathrm{perf}}\right)$$

of small R-linear idempotent complete stable categories[3].

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[3] Note that the functor $\mathrm{Cat}^{\mathrm{perf}}_R \to \mathrm{Cat}^{\mathrm{perf}}$ preserves finite limits and colimits, however the monoidal structures are not the same.

3.3. **Module categories.** We adopt the definitions from [Lur09a]: a monoidal category is an algebra object in DGCat. For a given monoidal category $\mathcal{A}$, we have the corresponding notions of left and right module categories (cf. [Lur09a, Sections 4.2, 4.3]). We write $\mathrm{LMod}_{\mathcal{A}}$ (resp. $\mathrm{RMod}_{\mathcal{A}}$) the category of left (resp. right) $\mathcal{A}$-module objects in DGCat. There is a canonical identification

$$\mathrm{LMod}_{\mathcal{A}} \simeq \mathrm{RMod}_{\mathcal{A}^{\mathrm{rev}}},$$

where $\mathcal{A}^{\mathrm{rev}}$ is the monoidal category $\mathcal{A}$ with reversed multiplication c.f.[GR17, Chapter 1, 3.1.4]. For monoidal categories $\mathcal{A}$ and $\mathcal{B}$, we denote by ${}_{\mathcal{A}}\mathrm{BMod}_{\mathcal{B}}$ the category of $(\mathcal{A}, \mathcal{B})$-bimodules. There is a similar identification

$${}_{\mathcal{A}}\mathrm{BMod}_{\mathcal{B}} \simeq \mathrm{LMod}_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{rev}}}.$$

Let $\mathcal{M} \in {}_{\mathcal{A}}\mathrm{BMod}_{\mathcal{B}}$, there is a functor

$$\mathcal{M} \otimes_{\mathcal{B}} (\bullet) : \mathrm{LMod}_{\mathcal{B}} \to \mathrm{LMod}_{\mathcal{A}} \tag{3.1}$$

given by Lurie's relative tensor product cf. [Lur09a, Section 4.8.1].

3.4. **Relative compactness.** Let $\mathcal{A}$ be a monoidal category.

Definition 3.13. ([GR17, Def. 8.8.2]) An object $m \in \mathcal{M}$ is called *compact relative to* $\mathcal{A}$ if

$$\underline{\mathrm{Map}}_{\mathcal{A}}(m, -) : \mathcal{M} \to \mathcal{A}$$

commutes with filtered colimits

Lemma 3.14. ([GR17, Lem. 9.3.4]) If $\mathcal{A}$ is a rigid monoidal category, m is compact if and only if m is compact relative to $\mathcal{A}$.

We will use the following result to perform renormalization of sheaf categories.

Proposition 3.15. ([GR17, Proposition 8.7.4]) Let $\mathcal{A}$ be a stable monoidal category, and $\mathcal{M}$, resp. $\mathcal{N}$, is a right, resp. left, $\mathcal{A}$-module. Assume that $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{N}$ are compactly generated, and the functors

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A},\ \mathcal{A} \otimes \mathcal{M} \to \mathcal{M},\ \mathcal{N} \otimes \mathcal{A} \to \mathcal{N}$$

preserve compact objects. Then the insertion functor

$$\mathrm{ins} : \mathcal{M} \otimes \mathcal{N} \to \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M}$$

preserves compact objects and $\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}$ is compactly generated by objects of the form $\mathrm{ins}(m \boxtimes n)$ for $m \in \mathcal{M}^{\omega}$ and $n \in \mathcal{N}^{\omega}$.

In this paper, the following special case of the above proposition will be useful to us.

Corollary 3.16. ([GR17, Corollary 9.3.3]) Let $\mathcal{A}$ be rigid stable monoidal category, and $\mathcal{M}$, resp. $\mathcal{N}$, is a right, resp. left, $\mathcal{A}$-module. Then the insertion functor

$$\mathrm{ins} : \mathcal{M} \otimes \mathcal{N} \to \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M}$$

preserves compact objects. If $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{N}$ are compactly generated, $\mathcal{N} \otimes_{\mathcal{A}} \mathcal{M}$ is compactly generated by objects of the form $\mathrm{ins}(m \boxtimes n)$ for $m \in \mathcal{M}^{\omega}$ and $n \in \mathcal{N}^{\omega}$.

3.5. **Sheaf theory.** Throughout the paper, we work with $\mathcal{D}$-modules. To make our exposition self-contained, we briefly summarize the machinery developed in [Ber17; Ras] and refer the reader to *op. cit.* for more details.

Let AffSch be the $(1,1)$-category of classical affine schemes over k and $\mathrm{AffSch}^{f.t.}$ its full subcategory of finite type affine schemes. Denote by $\mathrm{PreStk} := \mathrm{Hom}(\mathrm{AffSch}, \mathrm{Gpd})$ the category of classical prestacks. Let $\mathcal{D}^! : \mathrm{AffSch}^{op} \to \mathrm{DGCat}$ be the left Kan extension of the functor $\mathcal{D} : \mathrm{AffSch}^{f.t.} \to \mathrm{DGCat}$ which assigns to each finite type affine scheme S its DG-category of $\mathcal{D}$-modules, and attaches to each morphism $S_1 \to S_2$ the functor $f^!$. Right Kan extension gives rise to a functor

$$\mathcal{D}^! : \mathrm{PreStk}^{op} \to \mathrm{DGCat}.$$

Similarly, we have a dual version of $*$-$\mathcal{D}$-modules. Let $\mathcal{D}^* : \mathrm{AffSch} \to \mathrm{DGCat}$ be the right Kan extension of the functor $\mathcal{D} : \mathrm{AffSch}^{f.t.} \to \mathrm{DGCat}$ which assigns to a finite type affine scheme S the DG-category of $\mathcal{D}$-modules $\mathcal{D}(S)$, and attaches to a morphism $f : S_1 \to S_2$ the corresponding lower $*$ functor. Left extension produces a functor

$$\mathcal{D}^* : \mathrm{PreStk} \to \mathrm{DGCat}.$$

In this paper, we will mostly work with categories of $\mathcal{D}$-modules on the following infinite dimensional varieties:

Definition 3.17 (Placidness)**.**

- A scheme/algebraic space X *admits a placid presentation* if $X \simeq \lim_i X_i$ for a cofiltered limit of finite type schemes/algebraic spaces, such that transition maps $X_i \to X_j$ are smooth and affine. In this case, we call X a *placid* scheme/algebraic space.
- An ind-scheme X is *placid* if $X = \varinjlim X_i$ is a filtered colimit of placid schemes along closed embeddings of finite presentation. Let H be a placid group scheme acting on a placid ind-scheme X. We call X a *H-placid ind-scheme* if X is placid and each placid X_i in the placid presentation is stable under the H-action.

Example 3.18 (Loops and positive loops are placid)**.** Let $X \in \mathrm{AffSch}^{f.t.}$ be smooth affine of finite type. The n-jets $L^n X$, given by $L^n X(R) := X(R[t]/t^n)$ for $R \in \mathrm{CAlg}_k$ is represented by an affine scheme of finite type. The presentation $L^+X \simeq \lim_n L^n X$, exhibits L^+X as a placid scheme.

The loop space LX is a placid ind-scheme. In fact, let $X \subset \mathbb{A}^n$ be a closed embedding. Define $LX^i := LX \cap t^{-i} L^+ \mathbb{A}^n$, then each LX^i is a placid scheme with $LX \simeq \varinjlim_i LX^i$ an ind-placid presentation of LX. Note that if we further require the transition map in a placid scheme to be surjective (for example, in [Ras, §4.2.]), then LX^i is not necessarily placid in this sense (see [BSV24, Example 7.2.2]).

Example 3.19. Let $X = \varinjlim_i X_i$ be a placid scheme. There is an equivalence $\mathcal{D}^*(X) \simeq \varinjlim_i \mathcal{D}(X_i)$, where the connecting morphism $\mathcal{D}(X_i) \to \mathcal{D}(X_j)$ is given by the $*$-pullback along the smooth morphism $X_j \to X_i$. Similarly, there is an equivalence $\mathcal{D}^!(X) \simeq \varprojlim_i \mathcal{D}(X_i)$ with the connecting morphism given by the right adjoint of !-pullbacks. The two categories $\mathcal{D}^*(X)$ and $\mathcal{D}^!(X)$ are compactly generated and are canonically dual to each other.

Example 3.20. Let $X = \varinjlim_i X_i$ be a placid ind-scheme. We have $\mathcal{D}^!(X) = \varinjlim_i \mathcal{D}^!(X_i)$ with connecting morphism given by the left adjoint of the !-pullback $\mathcal{D}^!(X_j) \to \mathcal{D}^!(X_i)$. Similarly, we have $\mathcal{D}^*(X) = \varinjlim_i \mathcal{D}^*(X_i)$ with connecting morphism given by the right adjoint of the $*$-pushforward $\mathcal{D}(X_i) \to \mathcal{D}(X_j)$. If X admits a dimension theory, there is a canonical duality $\mathcal{D}^!(X) \simeq \mathcal{D}^*(X)$ (cf. [Ras]). In this case, we will not distinguish between the ! and $*$ versions of $\mathcal{D}$-modules.

Definition 3.21 (Iwahori and spherical orbits on loop space)**.** Let X be a smooth affine spherical G-variety. Let $\{\mathcal{O}_v\}_{v \in W_{X,\mathrm{ext}}}$ and $\{LX_\lambda\}_{\Lambda_X^+}$ be two partially ordered sets of I-orbits of LX and L^+G-orbits of LX, respectively, with the indexing sets denoted as $W_{X,\mathrm{ext}}$ and Λ_X^+, respectively [4]. Denote their orbit closures to be $\overline{\mathcal{O}}_v$ and $\overline{LX}_\lambda$ correspondingly. We also denote the embeddings as

$$\begin{aligned} i_v :& \mathcal{O}_v \xrightarrow{j_v} \overline{\mathcal{O}}_v \xrightarrow{\bar{i}_v} LX, \\ i_\lambda :& LX_\lambda \xrightarrow{j_\lambda} \overline{LX}_\lambda \xrightarrow{\bar{i}_\lambda} LX \end{aligned}$$

where for $\bullet$ be either in $W_{X,\mathrm{ext}}$ or Λ_X^+, $j_\bullet, \bar{i}_\bullet, i_\bullet = \bar{i}_\bullet \circ j_\bullet$ are open, closed, and locally closed embeddings, respectively.

The following placidness result is useful to us.

Theorem 3.22. [CY25, Theorem 36] The presentation $\varinjlim_{\lambda \in \Lambda_X^+} \overline{LX}_\lambda$, resp. $\varinjlim_{v \in W_{X,\mathrm{ext}}} \overline{\mathcal{O}}_v$ makes LX an L^+G, resp. I-placid ind-scheme.

A central object of study in our paper is the full subcategory of coherent $\mathcal{D}$-modules.

Definition 3.23. Let $\mathcal{Y}$ be a prestack, the full subcategory $\mathcal{D}_c(\mathcal{Y}) \subset \mathcal{D}(\mathcal{Y})$ of *coherent* $\mathcal{D}$-modules consists of objects $\mathcal{F}$ such that $f^!\mathcal{F}$ is a finite complex of $\mathcal{D}$-modules with coherent cohomology for any smooth morphism $f : S \to \mathcal{Y}$ where S is a dg-scheme.

In fact, $f^!$ and f^* only differ by a shift of degree, therefore coherent $\mathcal{D}$-modules may also be characterized by f^*.

For a finite type scheme, X, the canonical embedding

$$\mathcal{D}_s(X) \hookrightarrow \mathcal{D}_c(X)$$

---

[4] this notation is suggested by the fact that in group case $X = G$ acted by $H \times H$, we recover the dominant coweights Λ^+ and extended affine Weyl group W_{ext} as indexing sets for and $L^+G\backslash LG/L^+G$ and $I\backslash LG/I$, respectively

of the compact i.e. *safe* $\mathcal{D}$-modules is an equivalence. However, this embedding is usually not an equivalence for stacks. For example, the constant $\mathcal{D}$-modules on the classifying stack for a complex reductive group G is coherent but not compact.

Definition 3.24. Let $\mathcal{Y}$ be a prestack, the *renormalized* category of $\mathcal{D}$-modules $\mathcal{D}^{\mathrm{ren}}(\mathcal{Y})$ is defined to be $\mathrm{Ind}(\mathcal{D}_c(\mathcal{Y}))$.

3.6. **Categorical Actions.**

In this section, we briefly recall the machinery in categorical representation theory which is crucial to our construction.

Let H be a group placid ind-scheme. The multiplication map $\mathrm{m} : H \times H \to H$ endows $\mathcal{D}(H)$ with a monoidal structure. Set

$$\mathcal{D}(H)\text{-mod} := \mathrm{LMod}_{\mathcal{D}(H)}$$

to be the associated $(\infty, 2)$-category of left $\mathcal{D}(H)$-modules. For any $\mathcal{C} \in \mathcal{D}(H)$-mod, its H-invariants and co-invariants are given by

$$\mathcal{C}^H := \mathrm{Hom}_{\mathcal{D}(H)\text{-mod}}(\mathrm{Vect}, \mathcal{C}), \text{ and } \mathcal{C}_H := \mathrm{Vect} \otimes_{\mathcal{D}(H)} \mathcal{C}.$$

If the prounipotent radical of H is of finite codimension, the tautological functor Oblv "forgetting the H-invariants" admits a right adjoint

$$\mathrm{Oblv} : \mathcal{C}^H \rightleftarrows \mathcal{C} : \mathrm{Av}_*^H. \tag{3.2}$$

It is well-known that any affine group scheme admits a Levi decomposition, thus Av_*^H is continuous by [Ber17, Section 4.2.1].

Let $K \subset H$ be an affine group subscheme and the prounipotent radical of K of finite codimension. For any $\mathcal{C} \in \mathcal{D}(H)$-mod,

$$\mathcal{C}^K := \mathrm{Hom}_{\mathcal{D}(K)-\mathrm{mod}}(\mathrm{Vect}, \mathcal{C}) \simeq \mathrm{Hom}_{\mathcal{D}(H)-\mathrm{mod}}(\mathcal{D}(H)_K, \mathcal{C}),$$

which naturally admits an action of

$$\mathcal{H} := \mathrm{Hom}_{\mathcal{D}(H)}(\mathcal{D}(H)_K, \mathcal{D}(H)_K) \simeq \mathcal{D}(K\backslash H/K).$$

We have the adjunction

$$(-) \otimes_{\mathcal{H}} \mathcal{D}(H)_K : \text{mod-}\mathcal{H} \rightleftarrows \mathcal{D}(H)\text{-mod} : (-)^K \tag{3.3}$$

Theorem 3.25. [CD21, Proposition 3.1.3, Theorem 3.1.5] If H/K is ind-proper,

(1) the left adjoint in the adjunction (3.3) is fully faithful.

(2) the counit of the adjunction (3.3)

$$\mathcal{D}(H/K) \otimes_{\mathcal{H}} \mathcal{C}^K \xrightarrow{c} \mathcal{C} \tag{3.4}$$

is fully faithful and admits a continuous $\mathcal{D}(H)$-equivariant right adjoint c^R.

Lemma 3.26. Let $\mathcal{C}, \mathcal{D} \in \mathcal{D}(L^+G)$-mod, and $F : \mathcal{C} \to \mathcal{D}$ a fully faithful functor of $\mathcal{D}(L^+G)$-modules. For any $P \subset L^+G$ a parahoric subgroup, the associated P-invariant functor

$$F^P : \mathcal{C}^P \to \mathcal{D}^P$$

is also fully faithful. If in addition, F admits a right adjoint, then so does F^P.

Proof. The first statement is formal. We prove the second statement. Note that F^P equals the composition of the following functors

$$\mathcal{C}^P \xrightarrow{\text{Oblv}} \mathcal{C} \xrightarrow{F} D \xrightarrow{\text{Av}_*^P} \mathcal{D}^P.$$

By [Ber17, Section 4.2.1], Av_*^P is continuous and therefore admits a right adjoint by the adjoint functor theorem. The statement thus follows. ☐

3.7. **The unramified relative local Langlands equivalence.** In this section, we recall the conjectural local relative unramified Langlands equivalence Conjecture 3.29 coming from relative Langlands duality. Let G be a connected reductive group over a field k. We also summarize a class of Hamiltonian G-varieties M that is expected, as in [BSV24, §4], to be dual to Hamiltonian $\check{G}$-varieties $\check{M}$.

Definition 3.27 (Untwisted polarized hyperspherical varieties)**.** Let X be a *spherical G-variety*, i.e. a normal G-variety, with a right G-action, over k so that for any Borel subgroup B of G, X has a Zariski open B-orbit. We further assume the following two conditions:

(1) X is smooth and affine,

(2) The B-stabilizers of the points in the open B-orbit of X are connected. [5]

Under these conditions, $M = T^*(2)X$ is a *polarized hyperspherical G-variety*, with the weight-2 $\mathbb{G}_m$ action on fiber, in the sense of [BSV24, §3.5.1.] (see [BSV24, Proposition 3.7.4] for the proof). In particular, M is a *graded Hamiltonian G-variety.*

For a polarized hyperspherical variety $M = T^*(2)X$, [BSV24, §4.] constructed a graded Hamiltonian $\check{G}$-space $\check{M}$, which is expected [BSV24, Expectation 5.2.1] to be a hyperspherical $\check{G}$-variety (in particular, it is affine and smooth). In nice cases, $\check{M} = T^*(2)\check{X}$ is polarized.

Rather than recalling the construction of $\check{M}$, we focus on one of the main evidences for such duality: the categorical unramified relative local Langlands conjecture, which we restated as below, for the special case of $M = T^*X$ as defined in Definition 3.27 (we refer to [BSV24, §7] for the full statement of the conjecture). Let us recall the following definition on Satake full subcategory for our conjectures.

Definition 3.28. Let X be as in Definition 3.27. We define the following categories:

---

[5] for example, if X has no roots of type N, in the sense of [BSV24, Remark 4.2.1], then all B-stabilizers of X are connected

(1) $\mathcal{D}_c(K\backslash LX)^{\mathrm{Sat}}$: the full subcategory generated by (as in Definition 3.7) objects

$$\langle \mathcal{F} * \mathrm{IC}_0 | \mathcal{F} \in \mathcal{D}_c(K\backslash \mathrm{Gr})\rangle,$$

which are given by the convolution action of $\mathcal{D}_c(K\backslash \mathrm{Gr})$.

(2) $\mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}}$: the full subcategory of $\mathcal{D}_c(I\backslash LX)$ generated by

$$\langle \mathcal{F} * \mathrm{IC}_0 | \mathcal{F} \in \mathcal{D}_c(I\backslash \mathrm{Gr})\rangle.$$

Conjecture 3.29 (Unramified relative local Langlands)**.** Let $M = T^*(2)(X)$ be a polarized hyperspherical G-variety with the corresponding hyperspherical dual $(\check{G}, \check{M})$, then there is an equivalence of categories

$$\mathcal{D}_c(L^+G\backslash LX)^{\mathrm{Sat}} \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\check{M})/\check{G})$$

such that

(1) the equivalence is compatible with the action of derived geometric Satake equivalence [BF08]

$$\mathcal{D}_c(L^+G\backslash LG/L^+G) \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G}).$$

(2) Pointing condition is satisfied, i.e. the unramified basic object IC_0, the !-extension from the constant $\mathcal{D}$-module on L^+X, is sent to $\mathcal{O}_{\mathrm{sh}^{1/2}(\check{M})}$.

We list representative (G, X) with dual $(\check{G}, \check{M})$ and the status of the unramified equivalence.

Example 3.30.

(1) The group case: If $X = G$ acted on by $G \times G$ then then $\check{M} = T^*(2)\check{X}$ where $\check{X} = \check{G}$. This is derived geometric Satake, cf. [BF08, Theorem 5].

(2) Non-homogeneous cases:

  (a) Godement-Jacquet/Gaiotto conjecture for $m = n$: If $X = M_n$ with $G = \mathrm{GL}_n \times \mathrm{GL}_n$, then $\check{M} = T^*(2)\check{X}$ where $\check{X} = \mathrm{GL}_n \times \mathbb{A}^n$, acted on by $G = \mathrm{GL}_n \times \mathrm{GL}_n$ via by $(g, v) \cdot (g_1, g_2) = (g_1^{-1}gg_2, vg_2)$. The unramified equivalence is proven by Braverman–Finkelberg–Ginzburg–Travkin in [Bra+21, Theorem 3.6.1].

(3) Homogeneous spherical varieties:

  (a) The Shalika Model: $X = \mathrm{Sp}_{2n}\backslash\mathrm{GL}_{2n}$ then $\check{M} = T^*((\mathrm{GL}_n^{\mathrm{diag}}U, \psi)\backslash\mathrm{GL}_{2n}) = \mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n^*$ is proven in [Che+22, §1.6.].

  (b) Linear periods: If $X = \mathrm{GL}_n \times \mathrm{GL}_n\backslash\mathrm{GL}_{2n}$ then $\check{X} = \mathrm{GL}_{2n} \times_{\mathrm{Sp}_{2n}} \mathbb{A}^{2n}$. The unramified equivalence is the upcoming work of Chen-Yi.

  (c) Gan-Gross-Prasad, theta correspondence: For $G = \mathrm{SO}_{n-1} \times \mathrm{SO}_n$ and $X = \mathrm{SO}_{n-1}\backslash\mathrm{SO}_{n-1} \times \mathrm{SO}_n$. Then $\check{G} = \mathrm{SO}(V_0) \times \mathrm{Sp}(V_1)$. Here $\dim V_0 = n - 1$ if n odd and n if n even, $\dim V_1 = n - 1$ if n odd and $\dim V_1 = n - 2$ if n even. Then $\check{M} = (V_0 \otimes V_1)(1)$. This is proven by Braverman-Finkelberg-Travkin in [BFT22].

(d) X is affine homogeneous spherical of rank 1: $\check{M}$ is described in [Dev24, Table 4]. Unramified equivalences is also proved in *loc. cit.*, under certain hypothesis.

4. RELATIVE TENSOR PRODUCTS OF AUTOMORPHIC CATEGORIES

Let X be as in Definition 3.27. Specialize Theorem 3.25 to the case $H = LG$, $K = L^+G$, and $\mathcal{C} = \mathcal{D}(LX)$, we get a fully faithful functor of $\mathcal{D}(H)$-modules

$$\mathcal{D}(\mathrm{Gr}) \otimes_{\mathcal{H}_K} \mathcal{D}(K\backslash LX) \to \mathcal{D}(LX).$$

Taking I-invariants, we have by Lemma 3.26 a fully faithful functor

$$F : \mathcal{D}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K} \mathcal{D}(K\backslash LX) \to \mathcal{D}(I\backslash LX). \tag{4.1}$$

A crucial step in the proof of our main theorem is to establish the following equivalence

Theorem 4.1. The functor (4.1) induces an equivalence

$$F_c^{\mathrm{Sat}} : \mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(K\backslash LX)^{\mathrm{Sat}} \simeq \mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}}.$$

The rest of this section will devote to the proof of this theorem.

4.1. Relative tensor product of coherent $\mathcal{D}$-modules.

We first construct a fully faithful embedding

$$\iota : \mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(K\backslash LX) \to \mathcal{D}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K} \mathcal{D}(K\backslash LX). \tag{4.2}$$

By the universal property of ind-completion (cf. [GR17, Lemma 7.2.4]), the canonical embedding $\mathcal{D}_c(I\backslash \mathrm{Gr}) \hookrightarrow \mathcal{D}(I\backslash \mathrm{Gr})$ induces the *unrenormalization functor*

$$\theta_1 : \mathcal{D}^{\mathrm{ren}}(I\backslash \mathrm{Gr}) \to \mathcal{D}(I\backslash \mathrm{Gr}).$$

Similarly, we obtain unrenormalizations

$$\theta_2 : \mathcal{H}_K^{\mathrm{ren}} \to \mathcal{H}_K, \quad \theta_3 : \mathcal{D}^{\mathrm{ren}}(K\backslash LX) \to \mathcal{D}(K\backslash LX).$$

Recall that relative tensor product of small categories cf. Definition 3.12

$$\mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(L^+G\backslash LX) := (\mathcal{D}^{\mathrm{ren}}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K^{\mathrm{ren}}} \mathcal{D}^{\mathrm{ren}}(K\backslash LX))^{\omega}.$$

Lemma 4.2. Unrenormalization functors θ_i's induce a functor

$$\Theta : \mathcal{D}^{\mathrm{ren}}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K^{\mathrm{ren}}} \mathcal{D}^{\mathrm{ren}}(K\backslash LX) \to \mathcal{D}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K} \mathcal{D}(L^+G\backslash LX).$$

Proof. Recall that the monoidal structure on $\mathcal{H}_K$ and $\mathcal{H}_K^{\mathrm{ren}}$ are induced by $*$-pullback and $*$-pushforward along the convolution diagram

$$K\backslash \mathrm{Gr} \times K\backslash \mathrm{Gr} \xleftarrow{p} K\backslash LG \times^K \mathrm{Gr} \xrightarrow{q} K\backslash \mathrm{Gr},$$

where p is the natural projection morphism which exhibits $K\backslash LG \times^K \mathrm{Gr}$ as a K-torsor over $K\backslash \mathrm{Gr} \times K\backslash \mathrm{Gr}$, and q is the ind-proper convolution morphism. Thus, θ_2 is a monoidal functor by [HL22, Lemma 4.1.17]. In addition, θ_1, resp. θ_3 is functor of right, resp. left

$\mathcal{H}_K^{\mathrm{ren}}$-modules by *loc. cit.*. By the above argument and the functoriality of Lurie relative tensor product [Lur09a, Proposition 4.4.2], θ_i's induce the desired functor Θ. ☐

We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{ren}}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K^{\mathrm{ren}}} \mathcal{D}^{\mathrm{ren}}(L^+G\backslash LX) \arrow[r, "\Theta"] & \mathcal{D}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K} \mathcal{D}(L^+G\backslash LX) \\
\mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(K\backslash LX), \arrow[u, "i"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

where functor i is the natural embedding, and $\iota := \Theta \circ i$ is the desired functor (4.2). We will prove that ι is a fully faithful embedding.

4.2. **Compact generations.** In this section, we find generators of the small relative tensor product $\mathcal{D}_c(I\backslash \mathrm{Gr})^{\mathrm{ren}} \otimes_{\mathcal{H}^{\mathrm{ren}}} \mathcal{D}_c(K\backslash LX)^{\mathrm{ren}}$ such that the mapping spaces between them is easier to compute.

Lemma 4.3. The category $\mathcal{D}^{\mathrm{ren}}(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_K^{\mathrm{ren}}} \mathcal{D}^{\mathrm{ren}}(K\backslash LX)$ is compactly generated by objects of the form $\mathcal{F} \otimes \mathcal{G} \in \mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes \mathcal{D}_c(K\backslash LX)$.

Proof. We know by [Cam+24] that $\mathcal{H}_K^{\mathrm{ren}}$ is a rigid monoidal category. The statement follows directly from Corollary 3.16. ☐

Since $\mathrm{ins}^{\mathrm{ren}}$ is a continuous functor, it suffices to show that ι induces equivalences of mapping spaces between objects in $\mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes \mathcal{D}_c(K\backslash LX)$ to conclude ι is a fully faithful embedding.

4.3. **Mapping spaces.** To streamline notation,

we adopt the following conventions till the end of this section:

- $i := \mathrm{ins}$, $i^{\mathrm{r}} := \mathrm{ins}^{\mathrm{ren}}$,
- $i^R := \mathrm{ins}^R$, $i^{r,R} := \mathrm{ins}^{\mathrm{ren},R}$ the right adjoint functors,
- $\mathcal{S}_I := \mathcal{D}(I\backslash \mathrm{Gr}), \mathcal{S}_{c,I} := \mathcal{D}_c(I\backslash \mathrm{Gr}), \mathcal{S}_I^{\mathrm{ren}} := \mathcal{D}^{\mathrm{ren}}(I\backslash \mathrm{Gr})$,
- $\mathcal{T}_H := \mathcal{D}(H\backslash LX), \mathcal{T}_{c,H} := \mathcal{D}_c(H\backslash LX), \mathcal{T}_H^{\mathrm{ren}} := \mathcal{D}^{\mathrm{ren}}(H\backslash LX)$ for $H = I, K$.

Since the bar complex constructing Lurie's relative tensor product is functorial, Θ and $\theta_1 \otimes \theta_3$ are intertwined by the insertion functors

$$i : \mathcal{S}_I \otimes \mathcal{T}_K \to \mathcal{S}_I \otimes_{\mathcal{H}_K} \mathcal{T}_K,$$
$$i^{\mathrm{r}} : \mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}} \to \mathcal{S}_I^{\mathrm{ren}} \otimes_{\mathcal{H}_K^{\mathrm{ren}}} \mathcal{T}_K^{\mathrm{ren}}$$

i.e. the following diagram commutes.

(4.3)
$$\begin{tikzcd}
\mathcal{S}_I^{\mathrm{ren}} \otimes_{\mathcal{H}_K^{\mathrm{ren}}} \mathcal{T}_K^{\mathrm{ren}} \arrow[r, "\Theta"] & \mathcal{S}_I \otimes_{\mathcal{H}_K} \mathcal{T}_K \\
\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}} \arrow[r, "\theta"] \arrow[u, "i^{\mathrm{r}}"] & \mathcal{S}_I \otimes \mathcal{T}_K \arrow[u, "i"']
\end{tikzcd}$$

where $\theta := \theta_1 \otimes \theta_3$.

By Proposition 3.15 (see also Corollary 3.16), the insertion functors both admit continuous right adjoints, which we denote by i^R and $i^{\mathrm{r},R}$.

Lemma 4.4. For any $\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}' \in \mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$, there exists a morphism

$$\vartheta : \mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{H}_K^{\mathrm{ren}}}(\mathcal{F} \otimes \mathcal{G}, i^{\mathrm{r},R} \circ i^{\mathrm{r}}(\mathcal{F}' \otimes \mathcal{G}')) \to \mathrm{Maps}_{\mathcal{S}_I \otimes \mathcal{S}_K}(\theta(\mathcal{F} \otimes \mathcal{G}), i^R \circ i \circ \theta(\mathcal{F}' \otimes \mathcal{G}'))$$

making the following diagram commutes

$$\begin{tikzcd}
\mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes_{\mathcal{H}_K^{\mathrm{ren}}} \mathcal{T}_K^{\mathrm{ren}}}(i^{\mathrm{r}}(\mathcal{F} \otimes \mathcal{G}), i^{\mathrm{r}}(\mathcal{F}' \otimes \mathcal{G}')) \arrow[r, "\Theta"] \arrow[d, "\simeq"'] & \mathrm{Maps}_{\mathcal{S}_I \otimes_{\mathcal{H}_K} \mathcal{T}_K}(\Theta(i^{\mathrm{r}}(\mathcal{F} \otimes \mathcal{G}))), \Theta(i^{\mathrm{r}}(\mathcal{F}' \otimes \mathcal{G}'))) \arrow[d, "\simeq"] \\
\mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}}(\mathcal{F} \otimes \mathcal{G}, i^{\mathrm{r},R} \circ i^{\mathrm{r}}(\mathcal{F}' \otimes \mathcal{G}')) \arrow[r, "\vartheta"] & \mathrm{Maps}_{\mathcal{S}_I \otimes \mathcal{T}_K}(\theta(\mathcal{F} \otimes \mathcal{G}), i^R \circ i \circ \theta(\mathcal{F}' \otimes \mathcal{G}')),
\end{tikzcd}$$

where

1. the top arrow is induced by Θ,
2. the left vertical equivalence are given by the adjunction $(i^{\mathrm{r}}, i^{\mathrm{r},R})$,
3. the right vertical equivalence is given by the commutative diagram (4.3) and the adjoint pair$(i^{\mathrm{r}}, i^{\mathrm{r},R})$.

Proof. The construction of ϑ is given by the Beck-Chevalley correspondence via a formal argument. □

Thus, to prove the full faithfulness of ι, it suffices to prove ϑ is an equivalence. This will follow from two technical ingredients: a Künneth type formula in the next section and calculation of monads $i^{\mathrm{r},R} \circ i^{\mathrm{r}}$ and $i^R \circ i$ in subsection 4.5.

4.4. **A Künneth-type formula.**

In this section, we prove the final piece of ingredients establishing (4.2).

Lemma 4.5. Let $\mathcal{F}, \mathcal{F}' \in \mathcal{S}_{c,I}$ and $\mathcal{G}, \mathcal{G}' \in \mathcal{T}_{c,K}$, there is an equivalence

$$\underline{\mathrm{Maps}}_{\mathcal{S}_I \otimes \mathcal{T}_K}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}') \simeq \underline{\mathrm{Maps}}_{\mathcal{S}_I}(\mathcal{F}, \mathcal{F}') \otimes \underline{\mathrm{Maps}}_{\mathcal{T}_K}(\mathcal{G}, \mathcal{G}')$$

where $\underline{\mathrm{Map}}$ denotes the Vect enriched mapping space.

Proof. We employ a similar strategy as in [Gai11, Proposition 4.6.2][6]. By Theorem 3.22, the colimit of the K-orbit closures $\overline{LX}_\beta$ in LX provides a K-placid ind-scheme presentation for LX, and we have $\mathcal{D}(LX) \simeq \varinjlim_\beta \mathcal{D}(\overline{LX}_\beta)$. Since $\mathcal{D}(\mathrm{Gr}) \simeq \varinjlim_{\alpha \in \mathbb{X}_\bullet^+} \mathcal{D}(\mathrm{Gr}_{\leq\alpha})$, external tensor product induces an equivalence

$$\mathcal{D}(\mathrm{Gr}_{\leq\alpha}) \otimes \mathcal{D}(\overline{LX}_\beta) \simeq \mathcal{D}(\mathrm{Gr}_{\leq\alpha} \times \overline{LX}_\beta)$$

by [Ras, Lemma 6.9.2]. Taking colimits, we obtain an equivalence

$$\mathcal{D}(\mathrm{Gr}) \otimes \mathcal{D}(LX) \simeq \mathcal{D}(\mathrm{Gr} \times LX).$$

It follows from [Ras, Prop. 6.7.1] that, there is an equivalence

$$\mathcal{S}_I \otimes \mathcal{T}_K \simeq \mathcal{D}(I\backslash \mathrm{Gr} \times K\backslash LX).$$

We may assume $\mathcal{F} = \iota_* c$ and $\mathcal{G} = i_* d$ where $\iota : \mathrm{Gr}_{\leq\alpha} \hookrightarrow \mathrm{Gr}$, $c \in \mathcal{D}_c(\mathrm{Gr}_{\leq\alpha})^I$ for some $\alpha \in \mathbb{X}_*(T)^+$; and $i : \overline{LX}_\beta \hookrightarrow LX$, $d \in \mathcal{D}_c(\overline{LX}_\beta)^K$ for some $\beta \in W_{X,\mathrm{ext}}$.

It suffices to prove

$$\underline{\mathrm{Maps}}(\mathcal{F} \boxtimes \mathcal{G}, \iota^! \mathcal{F}' \boxtimes i^! \mathcal{G}') \simeq \underline{\mathrm{Maps}}(c, \iota^! \mathcal{F}') \otimes \underline{\mathrm{Maps}}(d, i^! \mathcal{G}').$$

where we used the fact that i and ι are closed embeddings and $(\iota \times i)_* := \iota_* \otimes i_*$ has the right adjoint $(\iota \otimes i)^! := \iota^! \otimes i^!$. Moreover, since $\overline{LX}_\beta$ is K-placid by [CY25, Prop. 29.(ii)], we may further assume that $d \in \mathcal{D}_c(K_i\backslash\overline{LX}_j)$ where $\overline{LX}_j$ is a constituent of the placid presentation of $\overline{LX}_\beta$ and K_i is a finite-type quotient of K through which K acts on $\overline{LX}_j$.

Assume that we have an approximation for both c and d

$$c_0 \to c \to c_1, \ d_0 \to d \to d_1,$$

where c_0, d_0 are compact objects and the cones c_1, d_1 are concentrated in negative enough cohomological degrees. For m large enough, we have isomorphisms

$$\tau^{\leq -m}(\underline{\mathrm{Map}}(c, \iota^! \mathcal{F}')) \to \tau^{\leq -m}(\underline{\mathrm{Map}}(c_0, \iota^! \mathcal{F}')),$$
$$\tau^{\leq -m}(\underline{\mathrm{Map}}(d, i^! \mathcal{G}')) \to \tau^{\leq -m}(\underline{\mathrm{Map}}(d_0, i^! \mathcal{G}')),$$

which induce equivalences

$$\tau^{\leq -m}(\underline{\mathrm{Map}}(c, \iota^! \mathcal{F}') \otimes \underline{\mathrm{Map}}_{\mathcal{T}_K}(d, i^! \mathcal{G}')) \to \tau^{\leq -m}(\underline{\mathrm{Map}}(c_0, \iota^! \mathcal{F}') \otimes \underline{\mathrm{Map}}(d_0, i^! \mathcal{G}')),$$
$$\tau^{\leq -m}(\underline{\mathrm{Maps}}(c \boxtimes d, \iota^! \mathcal{F}' \boxtimes i^! \mathcal{G}')) \to \tau^{\leq -m}(\underline{\mathrm{Maps}}(c_0 \boxtimes d_0, \iota^! \mathcal{F}' \boxtimes i^! \mathcal{G}')).$$

We have by [GR17, Proposition 10.5.8] an equivalence

$$\underline{\mathrm{Map}}(c_0, \iota^! \mathcal{F}') \otimes \underline{\mathrm{Map}}(d_0, i^! \mathcal{G}') \longrightarrow \underline{\mathrm{Map}}(c_0 \boxtimes d_0, \iota^! \mathcal{F} \ \boxtimes i^! \mathcal{G}'),$$

which gives the desired equivalence.

Now we justify the existence of the two approximations in the above. By our discussion, $K_i\backslash\overline{LX}_j$ is a "quasi compact with affine automorphism group" (QCA) stack [DG13, Def 1.1.8]. In other words, it is quasi-compact, whose geometric points have an affine automorphism group, and its classical inertia stack is finite over itself.

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[6] We thank Justin Campbell for pointing us to this reference.

By [DG13, Lemma 9.4.7], there exists $d_0 \in \mathcal{D}(K_i\backslash\overline{LX}_j)^\omega$ such that there exists a morphism $d_0 \to d$ and its cone d_1 belongs to $\mathcal{D}(K_i\backslash\overline{LX}_j)^{\leq -N}$ some N large enough. Completely analogous argument gives a similar approximation of c cf. [AG14, Sec. 12.2.1]. We thus complete the proof. □

4.5. **Calculation of monads.** Lemma 4.4 suggests that we need to compute the actions of monads

$$i^R \circ i, \;\; i^{\mathrm{r},R} \circ i^{\mathrm{r}}$$

for objects in the full subcategory $\mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$.

Lemma 4.6. *For any $\mathcal{F} \otimes \mathcal{G} \in \mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$,*

$$\theta \circ i^{\mathrm{r},R} \circ i^{\mathrm{r}}(\mathcal{F} \otimes \mathcal{G}) \simeq i^R \circ i \circ \theta(\mathcal{F} \otimes \mathcal{G}).$$

Proof. Our strategy is completely similar to that in the proof of [CD21, Theorem 3.1.5]. We first work in the non-renormalized categories. Consider the convolution diagram

$$I\backslash \mathrm{Gr} \times K\backslash LX \xleftarrow{q} I\backslash LG \times^K LX \xrightarrow{p} I\backslash LX,$$

where p, q are the natural projection morphisms. We note that p is a K-torsor, and p^* thus admit a left adjoint p_* by [Ras, Proposition 6.18.1]. Since p is ind-proper, it follows from [Ras, Section 3.22] that p_* has a right adjoint $p^!$.

By [Gai15, Corollary C.2.3], the functor i^R is monadic and the monad $i^R \circ i$ may be identified with

$$(\mathrm{act}_{\mathcal{D}(I\backslash \mathrm{Gr}),\ \mathcal{H}_K} \otimes \mathrm{id}_{\mathcal{D}(K\backslash \mathrm{Gr})}) \otimes (\mathrm{id}_{\mathcal{D}(I\backslash \mathrm{Gr})} \otimes \mathrm{act}_{\mathcal{H}_K,\ \mathcal{D}(K\backslash LX)})^R,$$

as plain endo-functors, where $\mathrm{act}_{\bullet,\bullet}$ are the natural action functors. Spreading out the above monad as pull-push along convolution diagrams as in [CD21, Theorem 3.1.5], we have an isomorphism

$$i^R \circ i \simeq q_* p^! p_* q^*.$$

Similarly,

$$i^{\mathrm{r},R} \circ i^{\mathrm{r}} \simeq q_*^{\mathrm{ren}} p^{!\mathrm{ren}} p_*^{\mathrm{ren}} q^{*\mathrm{ren}}.$$

Under the renormalization functor $\iota' : \mathcal{S}_I \otimes \mathcal{T}_K \hookrightarrow \mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}$, we conclude that

$$i^{\mathrm{r},R} \circ i^{\mathrm{r}}(\mathcal{F} \otimes \mathcal{G}) \simeq \iota' \circ i^R \circ i(\mathcal{F} \otimes \mathcal{G})$$

since $\mathcal{F} \otimes \mathcal{G} \in \mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$. The statement follows since θ is continuous and restricts to the identity on $\mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$. □

Lemma 4.7. *For any $\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}' \in \mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$, θ induces an equivalence*

$$\mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}') \simeq \mathrm{Maps}_{\mathcal{S}_I \otimes \mathcal{T}_K}(\theta(\mathcal{F} \otimes \mathcal{G}), \theta(\mathcal{F}' \otimes \mathcal{G}'))$$

Proof. By definition θ restricts to $\mathrm{id} \otimes \mathrm{id}$ on $\mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$. Thus, it suffices to prove

$$\mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}') \simeq \mathrm{Maps}_{\mathcal{S}_I \otimes \mathcal{T}_K}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}').$$

Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}') \arrow[r, "\theta"] \arrow[d, "K_1"'] & \mathrm{Maps}_{\mathcal{S}_I \otimes \mathcal{T}_K}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}') \arrow[d, "K_2"] \\
\mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}}}(\mathcal{F}, \mathcal{F}') \otimes \mathrm{Maps}_{\mathcal{T}_K^{\mathrm{ren}}}(\mathcal{G}, \mathcal{G}') \arrow[d, "\simeq"'] & \mathrm{Maps}_{\mathcal{S}_I}(\mathcal{F}, \mathcal{F}') \otimes \mathrm{Maps}_{\mathcal{T}_K}(\mathcal{G}, \mathcal{G}') \arrow[d, "="] \\
\mathrm{Maps}_{\mathcal{S}_{c,I}}(\mathcal{F}, \mathcal{F}') \otimes \mathrm{Maps}_{\mathcal{T}_{c,K}}(\mathcal{G}, \mathcal{G}') \arrow[r, "\iota_1 \otimes \iota_3"] & \mathrm{Maps}_{\mathcal{S}_I}(\mathcal{F}, \mathcal{F}') \otimes \mathrm{Maps}_{\mathcal{T}_K}(\mathcal{G}, \mathcal{G}').
\end{tikzcd}$$

The functor K_1 is an equivalence by the Künneth type formula cf. [GR17, Proposition 10.5.8]. The functor $\iota_1 \otimes \iota_3$ is clearly an equivalence. The statement follows from the fact that K_2 is an equivalence by Lemma 4.5. ☐

Proposition 4.8. The morphism

$$\vartheta : \mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}}(\mathcal{F} \otimes \mathcal{G}, i^{\mathrm{r},R} \circ i^{\mathrm{r}}(\mathcal{F}' \otimes \mathcal{G}')) \to \mathrm{Maps}_{\mathcal{S}_I \otimes \mathcal{T}_K}(\theta(\mathcal{F} \otimes \mathcal{G}), i^R \circ i \circ \theta(\mathcal{F}' \otimes \mathcal{G}'))$$

is an equivalence.

Proof. By Lemma 4.6 and that fact that $\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}$ is compactly generated by $\mathcal{S}_{c,I}^{\mathrm{ren}} \otimes \mathcal{T}_{c,K}^{\mathrm{ren}}$, it suffices to prove the that θ induces an equivalence

$$\mathrm{Maps}_{\mathcal{S}_I^{\mathrm{ren}} \otimes \mathcal{T}_K^{\mathrm{ren}}}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}') \simeq \mathrm{Maps}_{\mathcal{S}_I \otimes \mathcal{T}_K}(\mathcal{F} \otimes \mathcal{G}, \mathcal{F}' \otimes \mathcal{G}')$$

for $\mathcal{F}' \otimes \mathcal{G}' \in \mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$, which follows from Lemma 4.6. ☐

4.6. **Construction of Theorem 4.1.** We complete the proof of Theorem 4.1 in this subsection.

We have shown that

$$\iota : \mathcal{S}_{c,I} \otimes_{\mathcal{H}_{c,K}} \mathcal{T}_{c,K} \to \mathcal{S}_I \otimes_{\mathcal{H}_K} \mathcal{T}_K$$

is a fully faithful embedding. Composing with (4.1), we obtain a fully faithful functor

$$F_c : \mathcal{S}_{c,I} \otimes_{\mathcal{H}_{c,K}} \mathcal{T}_{c,K} \to \mathcal{T}_I$$

of $\mathcal{H}_{c,I}$-modules.

For any $\mathcal{F} \otimes \mathcal{G} \in \mathcal{S}_{c,I} \otimes \mathcal{T}_{c,K}$, we know that $F \circ i(\mathcal{F} \otimes \mathcal{G})$ is given by pull-push along the convolution diagram

$$I \backslash \mathrm{Gr} \times K \backslash LX \xleftarrow{p} I \backslash LG \times^K LX \xrightarrow{q} I \backslash LX.$$

Since p^* and q_* preserve coherent objects, F_c factors through a fully faithful functor

$$\mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(K\backslash LX) \to \mathcal{D}_c(I\backslash LX).$$

Thus, we obtain a fully faithful functor

$$F_c' : \mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(K\backslash LX)^{\mathrm{Sat}} \to \mathcal{D}_c(I\backslash LX).$$

The essential image of this functor is $\mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}}$, by definition Definition 3.28.

5. Proof of the Main Theorem

Finally, we are in the place of proving the tamely ramified relative local conjecture. We pass to the spectral side by the integral transform of Ben–Zvi–Francis–Nadler [BFN10].

Theorem 5.1. Under Assumption 1.2, there is an equivalence

$$\mathbb{L}^{\mathrm{Sat}} : \mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}} \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\tilde{\check{\mathfrak{g}}}^*(2) \times_{\check{\mathfrak{g}}^*(2)} \check{M})/\check{G}), \tag{5.1}$$

which is compatible with the actions coming from

$$\mathrm{End}_{\mathcal{H}_{c,K}}(\mathcal{D}_c(I\backslash \mathrm{Gr})) \simeq \mathrm{End}_{\mathrm{Perf}(\check{\mathfrak{g}}^*/\check{G})}(\mathrm{Perf}(\tilde{\check{\mathfrak{g}}}^*[2]/\check{G}))$$

Proof. By Theorem 4.1, it suffices to identify the small relative tensor product

$$\mathcal{D}_c(I\backslash \mathrm{Gr}) \otimes_{\mathcal{H}_{c,K}} \mathcal{D}_c(K\backslash LX)^{\mathrm{Sat}}$$

with

$$\mathrm{Perf}(\mathrm{sh}^{1/2}(\tilde{\check{\mathfrak{g}}}(2) \times_{\check{\mathfrak{g}}^*(2)} \check{M})/\check{G}).$$

We have the following equivalences

$$\mathcal{D}_c(I\backslash \mathrm{Gr}) \simeq \mathrm{Perf}(\tilde{\check{\mathfrak{g}}}^*[2]/\check{G}) \text{ (see [ABG04; Gai18])},$$
$$\mathcal{H}_{c,K} \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G}) \text{ (see [BF08])},$$
$$\mathcal{D}_c(K\backslash LX)^{\mathrm{Sat}} \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\check{M})/\check{G}) \text{ (see Assumption 1.2)}.$$

Since the shearing functor $\mathrm{sh}^{1/2}$ is symmetric monoidal by Proposition 3.2,

$$\begin{aligned}\mathrm{Perf}(\mathrm{sh}^{1/2}(\tilde{\check{\mathfrak{g}}}(2) \times_{\check{\mathfrak{g}}^*(2)} \check{M})/\check{G}) &\simeq \mathrm{Perf}((\mathrm{sh}^{1/2}(\tilde{\check{\mathfrak{g}}}^*(2)) \times_{\mathrm{sh}^{1/2}(\check{\mathfrak{g}}^*(2))} \mathrm{sh}^{1/2}(\check{M}))/\check{G}),\\ &\simeq \mathrm{Perf}((\tilde{\check{\mathfrak{g}}}^*[2] \times_{\check{\mathfrak{g}}^*[2]} \mathrm{sh}^{1/2}(\check{M}))/\check{G}),\\ &\simeq \mathrm{Perf}(\tilde{\check{\mathfrak{g}}}^*[2]/\check{G}) \otimes_{\mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G})} \mathrm{Perf}(\mathrm{sh}^{1/2}(\check{M})/\check{G}),\end{aligned}$$

where the last equality follows from the integral transform [BFN10] since all stacks considered here are perfect stacks.

By the previous argument, $\mathrm{Perf}(\mathrm{sh}^{1/2}(\tilde{\check{\mathfrak{g}}}^*(2) \times_{\check{\mathfrak{g}}^*(2)} \check{M})/\check{G})$ admits a natural action by

$$\mathrm{End}_{\mathrm{Perf}(\check{\mathfrak{g}}^*/\check{G})}(\mathrm{Perf}(\tilde{\check{\mathfrak{g}}}^*[2]/\check{G})),$$

which induces a natural action of $\mathrm{End}_{\mathcal{H}_{c,K}}(\mathcal{D}_c(I\backslash \mathrm{Gr}))$ on the right hand side of (5.1). We complete the proof. ☐

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