PROPOSITION II: FUNCTIONS IMPLY SHEAVES OVER $\mathbb{F}_q(\!(t)\!)$

Over $F = \mathbb{F}_q(\!(t)\!)$ the trace of Frobenius turns Satake objects into Hecke translates of the basic function. We show that spherical generation of functions forces every intersection complex to be a summand of a Satake object, hence thick generation of sheaves over $\overline{\mathbb{F}}_q$. The converse is not given by this argument. Section 3 proves part (a), Section 4 part (b) and Section 5 part (c).

1. Setting

Group data. p is an odd prime and q a power of p. G is a connected reductive group over $\mathbb{F}_q$, θ an involution of G over $\mathbb{F}_q$, $X = G/G^\theta$, $x_0 = eG^\theta$; S is a maximal θ-split torus and $T \supseteq S$ a θ-stable maximal torus, with $\Lambda_T^+ \subseteq X_*(T)$ and $\Lambda_S^+ = X_*(S) \cap \Lambda_T^+$ as on the pages, ordered by dominance, and $\varphi(\lambda) = \lambda - \theta(\lambda)$. Lower sets $\{\nu \le \mu\}$ are finite.

(R) Rationality. T is split over $\mathbb{F}_q$, and G has a θ-stable Borel pair (B_0, T_0) over $\mathbb{F}_q$ with T_0 split [CY, §8.2].

(C) Connectedness. G^θ and $Z_{G^\theta}(S)$ are connected.

Sheaves. $k = \overline{\mathbb{F}}_q$; LX, L^+G are the loop space and arc group over k, with Frobenius Fr. $\mathcal{C}_{K,k}$ is the category of bounded L^+G-equivariant constructible $\overline{\mathbb{Q}}_\ell$-complexes on LX supported on finitely many orbit closures [CY, App. A.5]; IC_ν is the intersection complex of $\overline{\mathcal{O}_\nu}$ with constant coefficients, $\mathrm{IC}_\nu|_{\mathcal{O}_\nu} \simeq \overline{\mathbb{Q}}_\ell[\delta(\nu)]$; V_λ is the Satake sheaf of $\mathrm{Gr}_{\le\lambda}$, and

$$\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}_{K,k}}\{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\}.$$

Functions over F. $F = \mathbb{F}_q(\!(t)\!)$, $\mathcal{O} = \mathbb{F}_q[\![t]\!]$, and[1]

$$K = G(\mathcal{O}), \qquad \mathcal{H}_K = C_c(K\backslash G(F)/K), \qquad T_\lambda = \mathbf{1}_{Kt^\lambda K},$$

$$C_K = C_c^\infty(X(F))^K, \qquad \varphi_0 = \mathbf{1}_{X(\mathcal{O})}, \qquad (f * \varphi)(x) = \sum_{gK \in G(F)/K} f(g)\,\varphi(g^{-1}x),$$

so $\mathrm{vol}\, K = 1$. Spherical generation is

$$\text{(SG)} \qquad \mathcal{H}_K * \varphi_0 = C_K.$$

For a field E of characteristic 0 write $\mathcal{H}_K * \varphi_0 \otimes E$ for the E-span of the $\mathbb{Z}$-valued functions $T_\lambda * \varphi_0$ and $C_K \otimes E$ for the K-invariant compactly supported E-valued functions.

Trace functions. A Weil object is a pair (M, Φ), $M \in \mathcal{C}_{K,k}$, $\Phi\colon \mathrm{Fr}^* M \xrightarrow{\sim} M$. For $x \in LX(\mathbb{F}_q) = X(F)$ and $\bar{x}$ a geometric point over it, $\mathrm{Fr}(\bar{x}) = \bar{x}$, so $\Phi_{\bar{x}}$ is an endomorphism of each stalk cohomology $\mathcal{H}^i(M)_{\bar{x}}$, and

$$\mathrm{tr}(M)(x) = \sum_i (-1)^i \,\mathrm{tr}\left(\Phi_{\bar{x}},\, \mathcal{H}^i(M)_{\bar{x}}\right) \in \overline{\mathbb{Q}}_\ell.$$

[1] Nothing in Sections 2–5 needs more than one field. Since $\overline{\mathbb{F}}_{q^n} = \overline{\mathbb{F}}_q$, replacing q by q^n changes neither $\mathcal{C}_{K,k}$ nor the objects $V_\lambda * \mathrm{IC}_0$, IC_ν, and every statement below holds verbatim over $F_n = \mathbb{F}_{q^n}(\!(t)\!)$, with $K_n = G(\mathbb{F}_{q^n}[\![t]\!])$, the trace tr_n of the n-th iterate $\Phi^{(n)} = \Phi \circ \mathrm{Fr}^*\Phi \circ \cdots \circ (\mathrm{Fr}^{n-1})^*\Phi$, and $A_{\nu,i}(\lambda)^n$ in place of $A_{\nu,i}(\lambda)$, which gives the matrix $m^{(n)}$. Only Remark 6.1 uses several n at once.

V_λ, IC_0 and IC_ν carry canonical Weil structures (eigenvalue 1 on the lowest stalk cohomology of the open stratum), and $V_\lambda * \mathrm{IC}_0$ the induced one. The function $\mathrm{tr}(M)$ is K-invariant; it is additive along Fr-compatible distinguished triangles, and $\mathrm{tr}(M[1]) = -\mathrm{tr}(M)$ [Lau, (1.1.1.1)].

Inputs (verdict §6).

(D0) Chen–Yi's orbit geometry over k, $p \neq 2$: $LX(k) = \bigsqcup_{\mu\in\Lambda_S^+} \mathcal{O}_\mu(k)$ with $\tau(x_\mu) = t^\mu$ [CY, Prop. 17]; $\overline{\mathcal{O}_\mu} = \bigsqcup_{\nu\le\mu} \mathcal{O}_\nu$ [CY, Thm. 18]; $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{\mathcal{O}_{\varphi(\lambda)}}$; and $\mathrm{Stab}_{L^+G}(x_\mu) = Z_{G^\theta}(\mu) \ltimes U_\mu$ with U_μ pro-unipotent, $\pi_0 Z_{G^\theta}(\mu) = 1$ for all μ under (C).

(D1) Proper base change and the Grothendieck–Lefschetz trace formula on the fibres of the convolution map [Lau, (1.1.1.3)]; Chen–Yi's sheaf theory on $L^+G\backslash LX$ [CY, App. A.5], with a Fr-compatible perverse t-structure whose heart has finite length and simple objects IC_ν, $\mathrm{End}(\mathrm{IC}_\nu) = \overline{\mathbb{Q}}_\ell$.

(D2) Decomposition: $V_\lambda * \mathrm{IC}_0$ is pure and $V_\lambda * \mathrm{IC}_0 \simeq \bigoplus_{\nu,i} \mathrm{IC}_\nu[-i] \otimes E_{\nu,i}(\lambda)$ over k, with finite-dimensional $E_{\nu,i}(\lambda) = \mathrm{Hom}\big(\mathrm{IC}_\nu, {}^p\mathcal{H}^i(V_\lambda * \mathrm{IC}_0)\big)$ [Sun, Thm. 1.2], [CY, Prop. 70 and proof of Thm. 93].

(D3) Lang's theorem [Mil, Cor. 27.56] for $L^+G = \varprojlim G(\mathbb{F}_q[t]/t^m)$ and for $\mathrm{Stab}_{L^+G}(x_\mu)$:

$$H^1(\mathbb{F}_q, L^+G) = 1, \qquad H^1\big(\mathbb{F}_q, \mathrm{Stab}_{L^+G}(x_\mu)\big) = H^1\big(\mathbb{F}_q, \pi_0\,\mathrm{Stab}_{L^+G}(x_\mu)\big).$$

(D5) is the comparison of the statement "IC_μ is a summand of $V_\lambda * \mathrm{IC}_0$" between k and $\mathbb{C}$ [BF, Prop. 5]; it is used only for the last sentence of (c).

Proposition I (summand criterion). Assume H1–H3 and that every $V_\lambda * \mathrm{IC}_0$ is a finite sum of shifted IC_ν with constant coefficients. Put $\mathcal{M} = \{\mu : \mathrm{IC}_\mu[s] \text{ is a direct summand of some } V_\lambda * \mathrm{IC}_0,\ s \in \mathbb{Z}\}$. Then $\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}\{\mathrm{IC}_\nu : \nu \in \mathcal{M}\}$, and $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K \iff \mathcal{M} = \Lambda_S^+$. For $\mathcal{C}_{K,k}$, H1 follows from (D0), H2 from (D1), and H3 from (C) and (D0).

2. Statement

Recall from Proposition I the set $\mathcal{M}$ of labels whose intersection complex occurs, up to shift, as a direct summand of a Satake object. Since $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathrm{Thick}\{\mathrm{IC}_\nu : \nu \in \mathcal{M}\}$, the sheaf statement $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$ is the statement $\mathcal{M} = \Lambda_S^+$, and that is what (c) proves.

Proposition II. *Assume (R), (C) and* (D0)–(D3). *The trace* tr *of Frobenius sends Weil objects of $\mathcal{C}_{K,k}$ to $C_K \otimes \overline{\mathbb{Q}}_\ell$.*

(a) *For every $\lambda \in \Lambda_T^+$, $\mathrm{tr}(V_\lambda * \mathrm{IC}_0) = \mathrm{tr}(V_\lambda) * \varphi_0$, and $\mathcal{H}_K * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell = \mathrm{span}\{\mathrm{tr}(V_\lambda * \mathrm{IC}_0) : \lambda \in \Lambda_T^+\}$.*

(b) (i) *$\{\mathrm{tr}\,\mathrm{IC}_\nu\}_{\nu\in\Lambda_S^+}$ is a basis of $C_K \otimes \overline{\mathbb{Q}}_\ell$.*

(ii) *With $A_{\nu,i}(\lambda)$ the Frobenius of the multiplicity space $E_{\nu,i}(\lambda)$ (Definition 4.3),*

$$\mathrm{tr}(V_\lambda * \mathrm{IC}_0) = \sum_\nu m_{\nu\lambda}\,\mathrm{tr}\,\mathrm{IC}_\nu, \qquad m_{\nu\lambda} = \sum_i (-1)^i\,\mathrm{tr}\,A_{\nu,i}(\lambda);$$

in particular $m_{\nu\lambda} = 0$ when IC_ν is not a shifted summand of $V_\lambda * \mathrm{IC}_0$.

(iii) (SG) *holds if and only if the columns $(m_{\nu\lambda})_{\nu\in\Lambda_S^+}$, $\lambda \in \Lambda_T^+$, span $\bigoplus_{\Lambda_S^+} \overline{\mathbb{Q}}_\ell$.*

(c) *If* (SG) *holds, then $\mathcal{M} = \Lambda_S^+$: every IC_μ is a shifted direct summand of some $V_\lambda * \mathrm{IC}_0$. Hence*

$$\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}.$$

Under (D5) *the same holds for Betti sheaves over* $\mathbb{C}$.

Corollary 2.1. *Under (R), (C) and* (D0)–(D3), *"(SG) holds over* $\mathbb{F}_q(\!(t)\!)$ *but* $\mathcal{C}_{K,k}^{\mathrm{Sat}} \neq \mathcal{C}_{K,k}$*" does not occur.*

3. Proof of (a): traces of Satake objects

Lemma 3.1 (Convolution)**.** $\mathrm{tr}(V_\lambda * \mathrm{IC}_0) = \mathrm{tr}(V_\lambda) * \varphi_0$.

Proof. For $x \in X(F)$ the fibre of $m\colon \mathrm{Gr}_{\leq\lambda} \mathbin{\widetilde{\times}} L^+X \to LX$ over x is $\{gK \in \mathrm{Gr}_{\leq\lambda} : g^{-1}x \in L^+X\}$, a closed subscheme of the projective $\mathrm{Gr}_{\leq\lambda}$, and $(V_\lambda \mathbin{\widetilde{\boxtimes}} \mathrm{IC}_0)_{(g,g^{-1}x)} = (V_\lambda)_g \otimes (\mathrm{IC}_0)_{g^{-1}x}$ compatibly with Frobenius. Proper base change and the Grothendieck–Lefschetz formula (D1), with $\mathrm{Gr}(\mathbb{F}_q) = G(F)/K$ and $L^+X(\mathbb{F}_q) = X(\mathcal{O})$ (D3), give

$$\mathrm{tr}(V_\lambda * \mathrm{IC}_0)(x) = \sum_{gK \in G(F)/K} \mathrm{tr}(V_\lambda)(g)\, \mathbf{1}_{X(\mathcal{O})}(g^{-1}x) = (\mathrm{tr}\, V_\lambda * \varphi_0)(x),$$

as in [Zhu, Lemma 5.6.1] with LX in place of Gr. □

Lemma 3.2 (Hecke basis)**.** $\mathrm{tr}\, V_\lambda = (-1)^{\langle 2\rho,\lambda\rangle} T_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} T_\mu$*, and* $\{\mathrm{tr}\, V_\lambda\}_{\lambda\in\Lambda_T^+}$ *is a basis of* $\mathcal{H}_K \otimes \overline{\mathbb{Q}}_\ell$.

Proof. $\mathrm{tr}\, V_\lambda$ is K-bi-invariant and supported on $\mathrm{Gr}_{\leq\lambda}(\mathbb{F}_q) = \bigsqcup_{\mu\leq\lambda} Kt^\mu K/K$, with value $(-1)^{\langle 2\rho,\lambda\rangle}$ on the open cell, where V_λ is $\overline{\mathbb{Q}}_\ell[\langle 2\rho,\lambda\rangle]$ with Frobenius 1. The transition matrix to $\{T_\mu\}$ is unitriangular for the dominance order, and lower sets are finite, so it is invertible. □

Proof of (a). By Lemma 3.2, $\mathcal{H}_K * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell$ is spanned by the $\mathrm{tr}\, V_\lambda * \varphi_0$, which are the $\mathrm{tr}(V_\lambda * \mathrm{IC}_0)$ by Lemma 3.1. □

4. Proof of (b): the IC basis and the matrix m

Lemma 4.1 (Orbits)**.** $X(F) = \bigsqcup_{\mu\in\Lambda_S^+} \mathcal{O}_\mu(\mathbb{F}_q)$*, each* $\mathcal{O}_\mu(\mathbb{F}_q) = K \cdot x_\mu$ *is one compact open* K*-orbit, and* C_K *has the basis* $\{\mathbf{1}_{\mathcal{O}_\mu(\mathbb{F}_q)}\}_{\mu\in\Lambda_S^+}$.

Proof. By (D0), $X(F) = LX(\mathbb{F}_q) \subseteq \bigsqcup_\mu \mathcal{O}_\mu(k)$. Since S is split, $t^\mu \in S(F)$, and τ is injective on points, x_μ is $\mathbb{F}_q$-rational; so each $\mathcal{O}_\mu$ is Fr-stable and $X(F) = \bigsqcup_\mu \mathcal{O}_\mu(\mathbb{F}_q)$. For $y = gx_\mu \in \mathcal{O}_\mu(\mathbb{F}_q)$, $g \in L^+G(k)$, the class of $g^{-1}\mathrm{Fr}(g) \in \mathrm{Stab}(x_\mu)(k)$ in $H^1(\mathbb{F}_q, \mathrm{Stab}(x_\mu))$ depends only on y, and two points are K-conjugate if and only if their classes agree, because $H^1(\mathbb{F}_q, L^+G) = 1$ (D3). By (D0) and (D3), $H^1(\mathbb{F}_q, \mathrm{Stab}(x_\mu)) = H^1(\mathbb{F}_q, \pi_0 Z_{G^\theta}(\mu)) = 1$. Hence $\mathcal{O}_\mu(\mathbb{F}_q) = K \cdot x_\mu$. It is compact because K is, and open because $G \to X$ is smooth, so $G(F) \to X(F)$ is open, and K is open in $G(F)$. A K-invariant compactly supported function is a finite combination of these indicators. □

Lemma 4.2 (IC basis: (b)(i))**.** $\mathrm{tr}\,\mathrm{IC}_\nu = (-1)^{\delta(\nu)} \mathbf{1}_{\mathcal{O}_\nu(\mathbb{F}_q)} + \sum_{\nu'<\nu} c_{\nu'\nu} \mathbf{1}_{\mathcal{O}_{\nu'}(\mathbb{F}_q)}$ *with* $c_{\nu'\nu} = \mathrm{tr}\,\mathrm{IC}_\nu(x_{\nu'})$*, and* $\{\mathrm{tr}\,\mathrm{IC}_\nu\}_{\nu\in\Lambda_S^+}$ *is a basis of* $C_K \otimes \overline{\mathbb{Q}}_\ell$.

Proof. $\mathrm{tr}\,\mathrm{IC}_\nu$ is not an indicator. It is K-invariant, because the Weil structure is L^+G-equivariant, and supported on $\overline{\mathcal{O}_\nu}(\mathbb{F}_q) = \bigsqcup_{\nu'\leq\nu} \mathcal{O}_{\nu'}(\mathbb{F}_q)$ (D0), a finite union of compact open orbits (Lemma 4.1); so it is constant on each $\mathcal{O}_{\nu'}(\mathbb{F}_q) = K \cdot x_{\nu'}$, with value the stalk trace $c_{\nu'\nu} = \mathrm{tr}\,\mathrm{IC}_\nu(x_{\nu'})$. On $\mathcal{O}_\nu$, $\mathrm{IC}_\nu = \overline{\mathbb{Q}}_\ell[\delta(\nu)]$ with Frobenius 1, so the value there is $(-1)^{\delta(\nu)}$. On a finite lower set L the families $\{\mathrm{tr}\,\mathrm{IC}_\nu\}_{\nu\in L}$ and $\{\mathbf{1}_{\mathcal{O}_\nu}\}_{\nu\in L}$ are related by a triangular matrix

with diagonal ± 1, so they span the same space; every finite set of labels lies in a finite lower set, so the IC traces are a basis [CY, §8.2.3]. See Figure 1. □

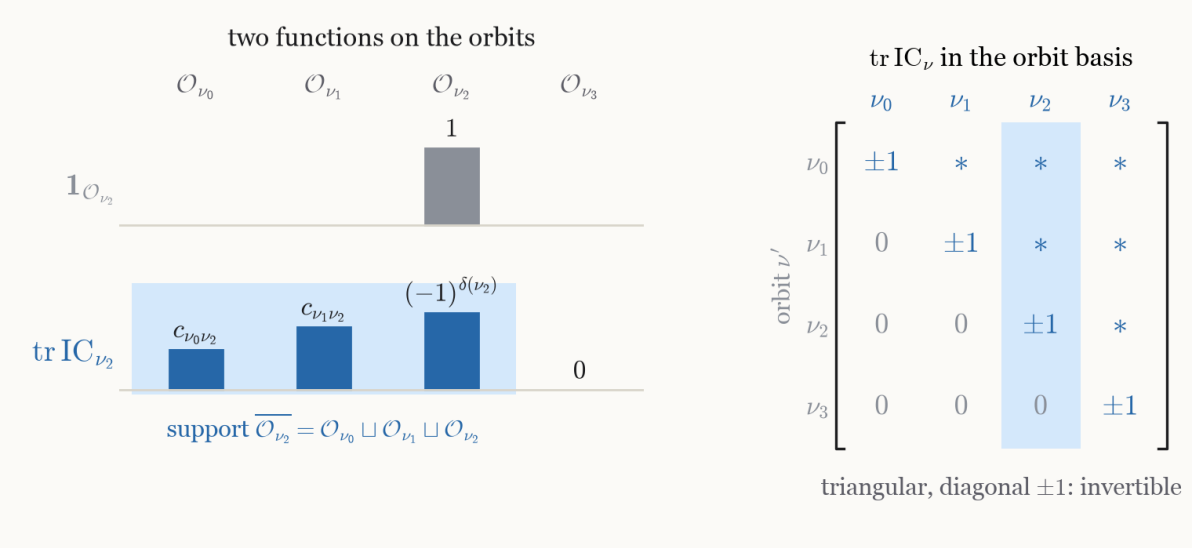


FIGURE 1. $\operatorname{tr} \mathrm{IC}_{\nu_2}$ is ± 1 on its own orbit and generally nonzero below it, so the IC traces form a second, triangular basis (schematic: orbits drawn as a chain, heights illustrative).

Part (b)(ii) expands $\operatorname{tr}(V_\lambda * \mathrm{IC}_0)$ in this basis. Three objects enter, all attached to one λ.

Definition 4.3 (Multiplicity spaces and their Frobenius). (1) $P_i = {}^p\mathcal{H}^i(V_\lambda * \mathrm{IC}_0)$ is the i-th perverse cohomology sheaf. By (D2), over k,

$$V_\lambda * \mathrm{IC}_0 \simeq \bigoplus_i P_i[-i], \qquad P_i \simeq \bigoplus_\nu \mathrm{IC}_\nu \otimes E_{\nu,i}(\lambda).$$

(2) $E_{\nu,i}(\lambda) = \operatorname{Hom}(\mathrm{IC}_\nu, P_i)$ is the *multiplicity space*: $\dim E_{\nu,i}(\lambda)$ is the number of copies of $\mathrm{IC}_\nu[-i]$ in $V_\lambda * \mathrm{IC}_0$.

(3) P_i inherits a Weil structure $\Phi_i \colon \mathrm{Fr}^* P_i \to P_i$, because the perverse t-structure is Fr-stable (D1); IC_ν has its canonical one, Φ_ν. The Frobenius of $E_{\nu,i}(\lambda)$ is

$$A_{\nu,i}(\lambda) \colon E_{\nu,i}(\lambda) \to E_{\nu,i}(\lambda), \qquad f \longmapsto \Phi_i \circ \mathrm{Fr}^*(f) \circ \Phi_\nu^{-1} \quad \big(\mathrm{IC}_\nu \xrightarrow{\Phi_\nu^{-1}} \mathrm{Fr}^*\mathrm{IC}_\nu \xrightarrow{\mathrm{Fr}^* f} \mathrm{Fr}^* P_i \xrightarrow{\Phi_i} P_i\big).$$

Example. If $E_{\nu,i}(\lambda)$ is a line, $A_{\nu,i}(\lambda)$ is a scalar α: that copy of IC_ν in P_i is IC_ν with its canonical Frobenius multiplied by α (for a Tate twist $\mathrm{IC}_\nu(-j)$, $\alpha = q^j$), and its trace is $\alpha \operatorname{tr} \mathrm{IC}_\nu$.

Lemma 4.4 (Frobenius on the summands). *Under the evaluation isomorphism* $\mathrm{ev} \colon \bigoplus_\nu \mathrm{IC}_\nu \otimes E_{\nu,i}(\lambda) \to P_i$, $x \otimes f \mapsto f(x)$, *the Weil structure* Φ_i *is* $\bigoplus_\nu \Phi_\nu \otimes A_{\nu,i}(\lambda)$. *Hence* $\operatorname{tr} P_i = \sum_\nu \operatorname{tr} A_{\nu,i}(\lambda) \cdot \operatorname{tr} \mathrm{IC}_\nu$.

Proof. ev is an isomorphism because P_i is semisimple (D2), $\operatorname{End}(\mathrm{IC}_\nu) = \overline{\mathbb{Q}}_\ell$ and $\operatorname{Hom}(\mathrm{IC}_\nu, \mathrm{IC}_{\nu'}) = 0$ for $\nu \neq \nu'$ (D1). On the summand $\mathrm{Fr}^*\mathrm{IC}_\nu \otimes f$, $\mathrm{ev} \circ (\Phi_\nu \otimes A_{\nu,i}(\lambda))$ is $(A_{\nu,i}(\lambda) f) \circ \Phi_\nu = \Phi_i \circ \mathrm{Fr}^*(f) = \Phi_i \circ \mathrm{Fr}^*(\mathrm{ev})$. On stalks, the trace of $\Phi_\nu \otimes A$ on $\mathcal{H}^j(\mathrm{IC}_\nu)_{\bar{x}} \otimes E$ is $\operatorname{tr}(\Phi_\nu) \cdot \operatorname{tr}(A)$. □

Lemma 4.5 (Expansion: (b)(ii))**.** *For every $\lambda \in \Lambda_T^+$,*

$$\operatorname{tr}(V_\lambda * \mathrm{IC}_0) = \sum_i (-1)^i \operatorname{tr} P_i = \sum_\nu m_{\nu\lambda} \operatorname{tr} \mathrm{IC}_\nu, \qquad m_{\nu\lambda} = \sum_i (-1)^i \operatorname{tr} A_{\nu,i}(\lambda).$$

Each column $(m_{\nu\lambda})_\nu$ has finitely many nonzero entries, and $m_{\nu\lambda} = 0$ when every $E_{\nu,i}(\lambda)$ is zero.

Proof. tr is additive along the Fr-compatible perverse truncation triangles of $V_\lambda * \mathrm{IC}_0$, and $\operatorname{tr}(P[-i]) = (-1)^i \operatorname{tr}(P)$ [Lau, (1.1.1.1)]; this is the first equality. Lemma 4.4 gives the second. The sum in (D2) is finite, so only finitely many $E_{\nu,i}(\lambda)$ are nonzero. See Figure 2. □

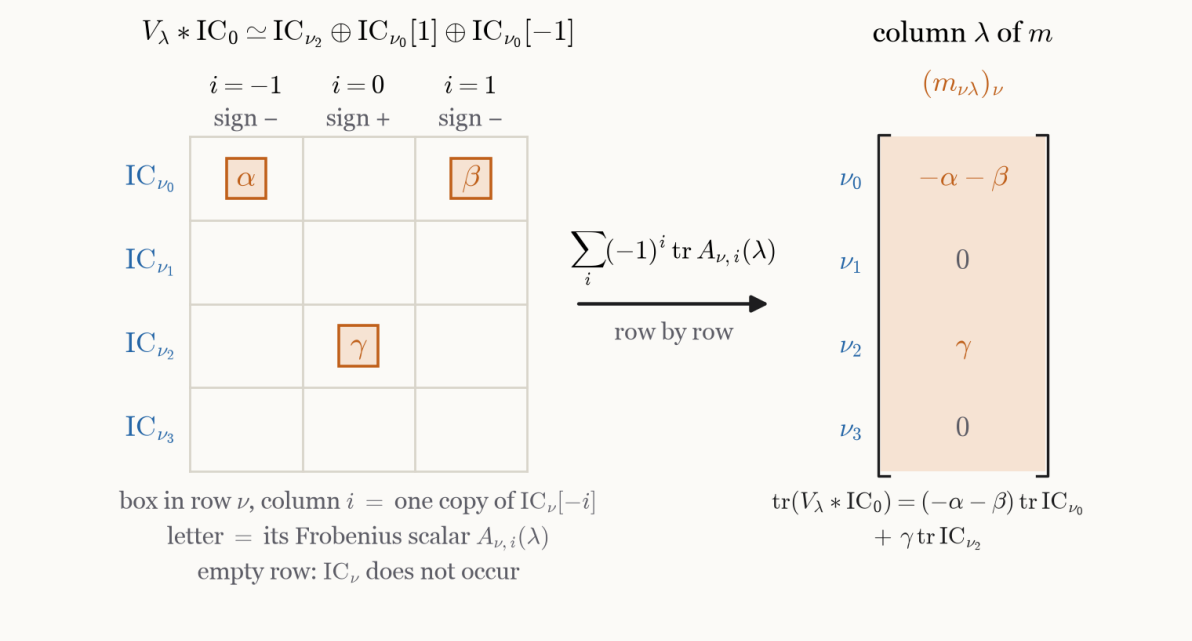


FIGURE 2. One Satake object, $V_\lambda * \mathrm{IC}_0 \simeq \mathrm{IC}_{\nu_2} \oplus \mathrm{IC}_{\nu_0}[1] \oplus \mathrm{IC}_{\nu_0}[-1]$: each copy of $\mathrm{IC}_\nu[-i]$ is a box in row ν, column i, carrying its Frobenius scalar; summing each row with signs $(-1)^i$ gives the column $(m_{\nu\lambda})_\nu$ (schematic).

The converse of the last sentence of Lemma 4.5 can fail: eigenvalues α and $-\alpha$ on one $E_{\nu,i}(\lambda)$ cancel. It is never used.

Lemma 4.6 (Spans in coordinates)**.** *Let U be a vector space over a field E with basis $(b_\nu)_{\nu\in I}$. The coordinate map*

$$\kappa\colon U \longrightarrow \bigoplus_{\nu\in I} E, \qquad \sum_\nu x_\nu b_\nu \longmapsto (x_\nu)_{\nu\in I},$$

is a linear isomorphism, with inverse $(x_\nu)_{\nu\in I} \mapsto \sum_\nu x_\nu b_\nu$. Let $m = (m_{\nu\lambda})_{\nu\in I,\lambda\in J}$ have finitely supported columns and $w_\lambda = \sum_\nu m_{\nu\lambda} b_\nu$, so that $\kappa(w_\lambda) = (m_{\nu\lambda})_{\nu\in I}$ is the λ-th column of m. Then

$$\kappa\big(\operatorname{span}_\lambda w_\lambda\big) = \operatorname{span}_\lambda (m_{\nu\lambda})_{\nu\in I}, \qquad \operatorname{span}_\lambda w_\lambda = U \iff \operatorname{span}_\lambda (m_{\nu\lambda})_{\nu\in I} = \bigoplus_I E.$$

If the columns span, every row $(m_{\mu\lambda})_{\lambda\in J}$ is nonzero; the converse fails.

Proof. *κ is an isomorphism.* Every $u \in U$ is a finite combination of the b_ν because they span, and the coefficients are unique because they are independent; so κ is well defined, and it is linear. Let $\iota(x_\nu)_\nu = \sum_\nu x_\nu b_\nu$, a finite sum. Then $\iota\kappa(u) = u$ by definition of the coordinates, and $\kappa\iota(x) = x$ by their uniqueness. *Spans.* $\kappa(\sum_\lambda a_\lambda w_\lambda) = \sum_\lambda a_\lambda (m_{\nu\lambda})_\nu$ gives the first equality. ($\Rightarrow$) If $\operatorname{span} w_\lambda = U$, the column span is $\kappa(U) = \bigoplus_I E$. ($\Leftarrow$) If the columns span, $\operatorname{span} w_\lambda = \iota(\bigoplus_I E) = U$. *Rows.* The coordinate form $\pi_\mu(x) = x_\mu$ is nonzero; if row μ were zero it would vanish on every column, hence on their span. For the converse take $m = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ (Figure 3, bottom). □

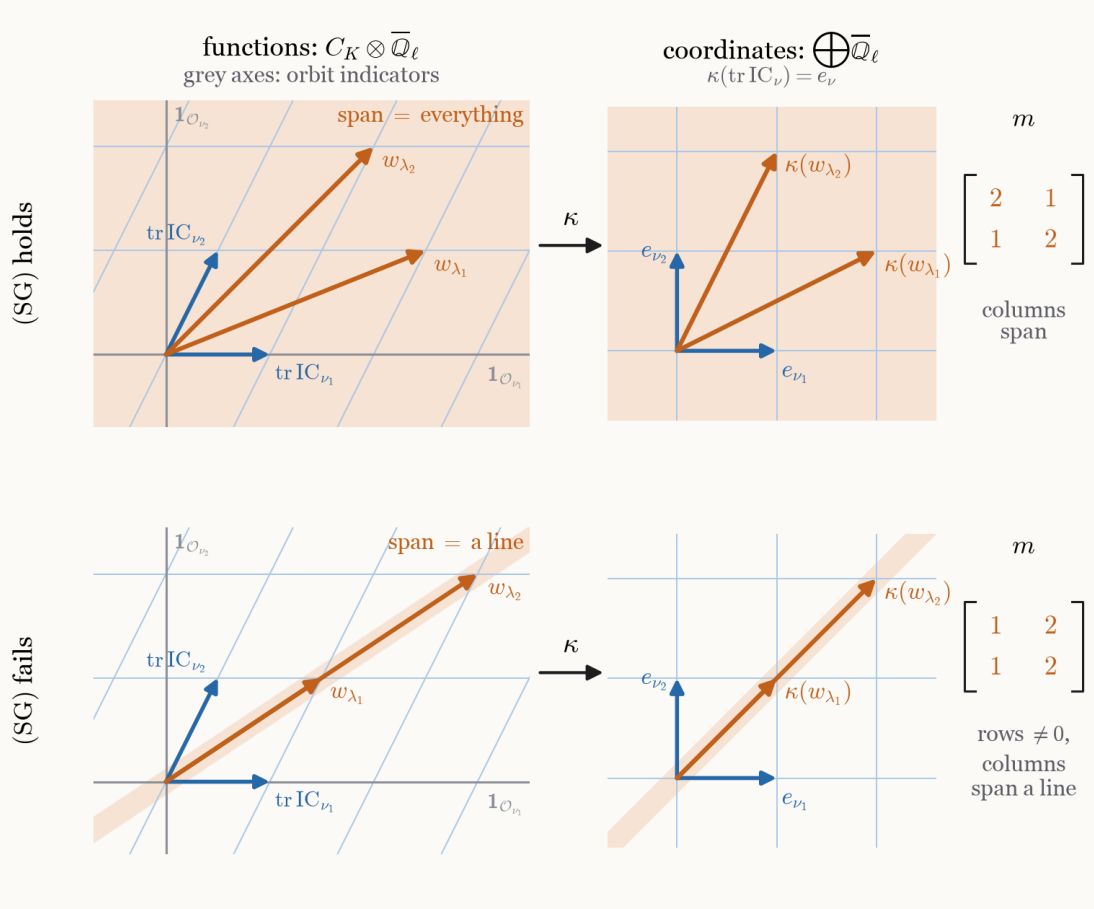


FIGURE 3. κ straightens the IC grid and sends the Satake traces w_λ to the columns $\kappa(w_\lambda)$, so their span is everything exactly when the column span is (two orbits; the bottom m is hypothetical).

Lemma 4.7 (Columns: (b)(iii))**.** (SG) *holds if and only if the columns of* $m = (m_{\nu\lambda})$ *span* $\bigoplus_{\Lambda_S^+} \overline{\mathbb{Q}}_\ell$.

Proof. Apply Lemma 4.6 with $E = \overline{\mathbb{Q}}_\ell$, $I = \Lambda_S^+$, $J = \Lambda_T^+$, $U = C_K \otimes \overline{\mathbb{Q}}_\ell$, $b_\nu = \operatorname{tr} \mathrm{IC}_\nu$ (a basis by Lemma 4.2) and $w_\lambda = \operatorname{tr}(V_\lambda * \mathrm{IC}_0)$, whose coordinates are the columns of m by Lemma 4.5 and whose span is $\mathcal{H}_K * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell$ by (a). It remains to compare coefficients. The functions $f_\lambda = T_\lambda * \varphi_0$ and $\mathbf{1}_\mu = \mathbf{1}_{\mathcal{O}_\mu(\mathbb{F}_q)}$ are $\mathbb{Z}$-valued; for fixed μ and a finite set of λ, writing $\mathbf{1}_\mu$ as a

combination of the f_λ is a linear system with an integer matrix, solvable over a field $E \supseteq \mathbb{Q}$ exactly when the matrix and the augmented matrix have the same rank, which does not depend on E.

($\Rightarrow$) Under (SG) each system is solvable over $\mathbb{C}$, hence over $\mathbb{Q} \subseteq \overline{\mathbb{Q}}_\ell$, so $\mathcal{H}_K * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell = C_K \otimes \overline{\mathbb{Q}}_\ell$; by Lemma 4.6 ($\Rightarrow$) the columns span.

($\Leftarrow$) If the columns span, Lemma 4.6 ($\Leftarrow$) gives $\mathcal{H}_K * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell = C_K \otimes \overline{\mathbb{Q}}_\ell$; each system is solvable over $\overline{\mathbb{Q}}_\ell$, hence over $\mathbb{Q} \subseteq \mathbb{C}$, which is (SG).

□

5. Proof of (c): $\mathcal{M} = \Lambda_S^+$

The proof is the chain

$$\text{(SG)} \xrightarrow{5.1} \forall \mu\ \exists \lambda :\ m_{\mu\lambda} \neq 0 \xrightarrow{5.2} \mathcal{M} = \Lambda_S^+ \xrightarrow{\text{Prop. I}} \mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}.$$

Lemma 5.1 (Rows)**.** (SG) *implies that every row of m is nonzero: for each μ there is λ with $m_{\mu\lambda} \neq 0$.*

Proof. By Lemma 4.7 ($\Rightarrow$) the columns span, so every row is nonzero by Lemma 4.6. □

Lemma 5.2 (Extraction)**.** *If $m_{\mu\lambda} \neq 0$, then $\mathrm{IC}_\mu[-i]$ is a direct summand of $V_\lambda * \mathrm{IC}_0$ for some i, so $\mu \in \mathcal{M}$.*

Proof. If every $E_{\mu,i}(\lambda)$ vanished, so would $m_{\mu\lambda}$ by Lemma 4.5. So some $E_{\mu,i}(\lambda) \neq 0$, and $\mathrm{IC}_\mu[-i]$ is a direct summand by (D2). Only "zero multiplicity space $\Rightarrow$ zero coefficient" is used: no positivity, and no purity beyond (D2). □

Proof of (c) and Corollary 2.1. By Lemmas 5.1 and 5.2, $\mathcal{M} = \Lambda_S^+$. The hypotheses of Proposition I hold for $\mathcal{C}_{K,k}$ (§1 and (D2)), so $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$. *Over* $\mathbb{C}$. Each statement "$\mathrm{IC}_\mu[-i]$ is a summand of $V_\lambda * \mathrm{IC}_0$" lives on the finite-type model $\mathrm{Gr}_{\leq\lambda} \,\widetilde{\times}\, \overline{\mathcal{O}_0} \to \overline{\mathcal{O}_{\varphi(\lambda)}}$, and (D5) says it has the same answer over $\overline{\mathbb{F}}_q$ and over $\mathbb{C}$, for p outside a finite set. So the set $\mathcal{M}$ is the same over $\mathbb{C}$, and Proposition I applies there. The corollary is the contrapositive of (c). □

Remark 5.3 (One orbit at a time). (c) has a pointwise form, not needed above: if a single indicator $\mathbf{1}_{\mathcal{O}_\mu(\mathbb{F}_q)}$ lies in $\mathcal{H}_K * \varphi_0$, then $\mu \in \mathcal{M}$. Indeed, by Lemma 5.2 $m_{\nu\lambda} = 0$ for $\nu \notin \mathcal{M}$, so every $\mathrm{tr}(V_\lambda * \mathrm{IC}_0)$, and by (a) all of $\mathcal{H}_K * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell$, lies in $\mathrm{span}\{\mathrm{tr}\,\mathrm{IC}_\nu : \nu \in \mathcal{M}\}$; the indicator lies there too (coefficients as in Lemma 4.7):

$$\mathbf{1}_{\mathcal{O}_\mu(\mathbb{F}_q)} = \sum_{\nu \in \mathcal{M}} c_\nu \,\mathrm{tr}\,\mathrm{IC}_\nu.$$

Take ν_0 maximal with $c_{\nu_0} \neq 0$ and evaluate at x_{ν_0}. $\mathrm{tr}\,\mathrm{IC}_\nu(x_{\nu_0}) \neq 0$ needs $\nu_0 \leq \nu$, so only $\nu = \nu_0$ contributes, with $c_{\nu_0}(-1)^{\delta(\nu_0)} \neq 0$. Hence $x_{\nu_0} \in \mathcal{O}_\mu$, so $\mu = \nu_0 \in \mathcal{M}$ [run, audit node A7-14].

6. What the argument does not give

Remark 6.1 (The converse). Here n varies: $m^{(n)}$ is the matrix of the footnote in §1, over $F_n = \mathbb{F}_{q^n}(\!(t)\!)$, with $m_{\nu\lambda}^{(n)} = \sum_i (-1)^i \,\mathrm{tr}\, A_{\nu,i}(\lambda)^n$ and $m^{(1)} = m$. Assume $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$. Proposition I gives, for each μ, some λ with $E_{\mu,i}(\lambda) \neq 0$ for some i. By purity (D2) the eigenvalues of $A_{\mu,i}(\lambda)$

have absolute value $q^{(w+i-w_\mu)/2}$, distinct for distinct i, so $m^{(n)}_{\mu\lambda} = \sum_\alpha c_\alpha \alpha^n$ with distinct α and all $c_\alpha \neq 0$; by Vandermonde $m^{(n)}_{\mu\lambda} \neq 0$ for some $n \leq \dim \bigoplus_i E_{\mu,i}(\lambda)$. Thus every row of $(m^{(n)})$ is nonzero for a suitable $n = n(\mu)$, whereas (SG) asks that the columns of m span. Nonzero rows do not give spanning columns, and trace functions at one n do not determine a class [Lau, (1.1.2)]. Whether $\mathcal{C}^{\mathrm{Sat}}_{K,k} = \mathcal{C}_{K,k} \Rightarrow$ (SG) fails for some symmetric X satisfying (C) is open: no example is known. At E_6/F_4 sheaves hold over $\overline{\mathbb{F}}_q$ while (SG) fails over p-adic fields under (A1)–(A5), a comparison across fields; over $\mathbb{F}_q(\!(t)\!)$ the span of the columns is open there.

Remark 6.2 (Fields). Sakellaridis's Theorem 8.0.2 (Theorem E on the pages) assumes a local field of characteristic zero [Sak, §1.1]. So a cell "(SG) under (A)" gives no input to Proposition II, which needs (SG) over $\mathbb{F}_q(\!(t)\!)$. Transporting it requires Theorem 8.0.2 in characteristic p, or a direct computation of the Hecke action over $\mathbb{F}_q(\!(t)\!)$; neither is available here. The sheaf statement of the pages is Betti over $\mathbb{C}$, reached from $\overline{\mathbb{F}}_q$ through (D5) [BF, Prop. 5]. Of the three settings (Betti over $\mathbb{C}$; p-adic functions under (A); $\mathbb{F}_q(\!(t)\!)$ against ℓ-adic sheaves over $\overline{\mathbb{F}}_q$) only the last carries a functor between the two levels, the trace of Frobenius.

Remark 6.3 (Every field, row by row). For p-adic F Corollary 2.1 does not apply; the table of the verdict (§1) excludes "functions hold, sheaves fail" row by row. In every row where (SG) holds, $\mathcal{C}^{\mathrm{Sat}}_K = \mathcal{C}_K$ is proved: by the spherical-generation pages for the group, sphere, $\mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$ and rank-one rows, and by Proposition IV of this workspace for the symplectic rows (over $\overline{\mathbb{F}}_q$, and over $\mathbb{C}$ through (D5) except in the group case). The rows $\mathrm{SL}_2/\mathrm{SO}_2$, $\mathrm{GL}_n/\mathrm{O}_n$, $\mathrm{PGL}_2/\mathrm{PO}_2$ fail at both levels, and at E_6/F_4 the sheaf statement holds.

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