PROPOSITION IV: THE SYMPLECTIC FAMILY — BOTH LEVELS HOLD ALTHOUGH H5 FAILS

For $\mathrm{GL}_{2m}/\mathrm{Sp}_{2m}$ and $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ we prove spherical generation of functions over every non-archimedean local field, without Sakellaridis's hypotheses (A1)–(A5), deduce categorical generation over $\overline{\mathbb{F}}_q$ from Proposition II, and compute the decomposition that produces the missing intersection complex for $\mathrm{GL}_4/\mathrm{Sp}_4$ directly. Source: the verdict of the run *Satake generation at two levels*, §3, Theorem 2, the symplectic part of Theorem 4, Corollary 5, with the audit corrections C-2, C-3, C-6 and A7-23.

1. Setting and statement

The family. For $m \geq 1$ let $G = \mathrm{GL}_{2m}$, $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $J = \mathrm{diag}(J_2, \dots, J_2)$, $\theta(g) = J(g^T)^{-1}J^{-1}$, $G^\theta = \mathrm{Sp}_{2m}$ and $X = G/G^\theta$. For $n \geq 1$ let $X^{\mathrm{SL}} = \mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$. T is the diagonal torus, $\Lambda_T = \mathbb{Z}^{2m}$, $\Lambda_T^+ = \{\lambda_1 \geq \cdots \geq \lambda_{2m}\}$, and

$$\Lambda_S^+ = \{\mu_a = (a_1, a_1, \dots, a_m, a_m) : a_1 \geq \cdots \geq a_m\}, \qquad \varphi(\lambda) = \mu_c, \quad c_i = \lambda_{2i-1} + \lambda_{2i}.$$

We write a label as $a \in \mathbb{Z}^m$, $a_1 \geq \cdots \geq a_m$, put $|a| = \sum_j a_j$, and order labels by

$$b \preceq a \iff |a| = |b| \text{ and } \textstyle\sum_{j \leq i} b_j \leq \sum_{j \leq i} a_j \ (1 \leq i \leq m),$$

which is the dominance order $\mu_b \leq \mu_a$ of Λ_T (Lemma 2.1). A label is *covered* if $\mu_a \in \varphi(\Lambda_T^+)$.
Function level. F is a non-archimedean local field of any characteristic, $\mathcal{O}$ its ring of integers, t a uniformizer, $\mathbb{F}_q$ the residue field; $K = G(\mathcal{O})$, $\mathrm{vol}\, K = 1$, $\mathcal{H}_K = C_c(K\backslash G(F)/K)$ with basis $T_\lambda = \mathbf{1}_{Kt^\lambda K}$ $(\lambda \in \Lambda_T^+)$, $(f * \phi)(x) = \sum_{gK} f(g)\phi(g^{-1}x)$, $C_K = C_c^\infty(X(F))^K$, $\varphi_0 = \mathbf{1}_{X(\mathcal{O})}$, and

$$\text{(SG)} \qquad \mathcal{H}_K \cdot \varphi_0 = C_K.$$

Sheaf level. $k = \overline{\mathbb{F}}_q$ with $p \neq 2$ and ℓ-adic coefficients (or $k = \mathbb{C}$, Betti sheaves); $LX = X(k(\!(t)\!))$ with the L^+G-orbits $\mathcal{O}_\mu$; $\mathcal{C}_K$, IC_0, V_λ, the action $*$ and $\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}_K}\{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\}$ are as in the verdict. IC_μ is the intersection complex of $\overline{\mathcal{O}_\mu}$ with constant coefficients.
Inputs. (verdict, §6). **D0** Chen–Yi's orbit geometry over $\overline{\mathbb{F}}_q$, $p \neq 2$ (orbit parametrization, closure order, support bound, connected stabilizers); **D1** proper base change, Grothendieck–Lefschetz and Verdier duality on finite-type models, the perverse t-structure; **D2** the decomposition of $V_\lambda * \mathrm{IC}_0$ into shifted IC_ν on finite-type equivariant models [Sun, Thm. 1.2], [CY, Prop. 70]; **D3** Lang's theorem for L^+G and the stabilizers; **D5** the comparison $\overline{\mathbb{F}}_q \leftrightarrow \mathbb{C}$ of "IC_μ is a summand of $V_\lambda * \mathrm{IC}_0$" for p outside a finite set [BF, Prop. 5], [BBD, §6.1].

Proposition IV. *Let $X = \mathrm{GL}_{2m}/\mathrm{Sp}_{2m}$ $(m \geq 1)$ or $X^{\mathrm{SL}} = \mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ $(n \geq 1)$.*

(a) (Functions.) *For every non-archimedean local field F,* (SG) *holds for X and for X^{SL}; no use of (A1)–(A5).*

(b) (Sheaves.) *Over $\overline{\mathbb{F}}_q$, $p \neq 2$, $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ for X and X^{SL}, under D0–D3; in the Betti category over $\mathbb{C}$ under D5.*

(c) ($\mathrm{GL}_4/\mathrm{Sp}_4$.) *Let* $\lambda = (1,1,0,0)$ *and let* P *be* $V_\lambda * \mathrm{IC}_0$ *on the finite-type model, shifted by* $[D-4]$ *(*$D = \dim$ *of the model). Under D0–D2,*

$$P \simeq \mathrm{IC}_{(2,2,0,0)} \oplus \mathrm{IC}_{(1,1,1,1)} \otimes \big(k[2] \oplus k \oplus k[-2]\big),$$

and over every F*,*

$$T_{(1,1,0,0)} * \varphi_0 = \mathbf{1}_{\mathcal{O}_{(2,2,0,0)}} + (1+q)(1+q^2)\,\mathbf{1}_{\mathcal{O}_{(1,1,1,1)}}, \qquad T_{(2,0,0,0)} * \varphi_0 = q(q+1)\,\mathbf{1}_{\mathcal{O}_{(2,2,0,0)}}.$$

(d) (H5.) *H5 fails for* X *when* $m \geq 2$ *and for* X^{SL} *when* $n \geq 3$.

Part (a) is proved here unconditionally (§3). Part (b) is Proposition II applied to (a) (§4). Part (c) is §5 and Example 4.2. Part (d) is the coverage lemma of the pages with Proposition III (§2).

2. The forms model

Lemma 2.1. *Let* R *be* F *or* $k(\!(t)\!)$ *with valuation ring* $\mathcal{O}$.

(i) $x \mapsto \omega_x := \tau(x)J$, $\tau(gG^\theta) = g\theta(g)^{-1}$, *identifies* $X(R)$ *with the nondegenerate alternating* $2m \times 2m$ *matrices over* R, *with* $g \cdot \omega = g\omega g^T$ *and* $\omega_{x_0} = J$; $X(\mathcal{O})$ *is the set of forms unimodular on* $\Lambda_0 = \mathcal{O}^{2m}$.

(ii) Every ω *is* $k\,D_a\,k^T$ *with* $k \in G(\mathcal{O})$, $D_a = \bigoplus_j t^{a_j} J_2$, *for a unique label* a; *so the orbits are* $\mathcal{O}_a := G(\mathcal{O}) \cdot D_a$, *and* $\mathcal{O}_a = \mathcal{O}_{\mu_a}$ *because* $\tau(x_{\mu_a}) = t^{\mu_a}$ *and* $t^{\mu_a} J = D_a$.

(iii) $\operatorname{val}\mathrm{Pf}$ *is* $G(\mathcal{O})$*-invariant and equals* $|a|$ *on* $\mathcal{O}_a$; $\mu_b \leq \mu_a$ *iff* $b \preceq a$.

(iv) $\sigma(\omega) = t\omega$ *is* $G(R)$*-equivariant with* $\sigma(\mathcal{O}_a) = \mathcal{O}_{a+(1^m)}$; *the central element* $t^c \cdot 1$ *acts by* $\omega \mapsto t^{2c}\omega$.

(v) $X^{\mathrm{SL}}(R) = \{\omega : \mathrm{Pf}\,\omega = 1\}$.

Proof. (i) $\tau(gG^\theta)J = gJg^TJ^{-1}J = gJg^T$; over a field all nondegenerate alternating forms of rank $2m$ are congruent ($H^1(R, \mathrm{Sp}_{2m}) = 0$). (ii) is the elementary-divisor theorem for alternating matrices over the principal ideal domain $\mathcal{O}$. (iii) $\mathrm{Pf}(g\omega g^T) = \det g\ \mathrm{Pf}\,\omega$. The positive coroots are $e_i - e_j$, $i < j$; the partial sums of $\mu_a - \mu_b$ are $2S_i$ at position $2i$ and $S_{i-1} + S_i$ at position $2i-1$, where $S_i = \sum_{j \leq i}(a_j - b_j)$; they are all ≥ 0 with total 0 iff $b \preceq a$. (iv) t is central in G and $\mathrm{Pf}(t\omega) = t^m\,\mathrm{Pf}\,\omega$. (v) $\mathrm{Pf}(gJg^T) = \det g$. □

3. H5 fails

Lemma 3.1 (pages, *symplectic family*, §5; Proposition III)**.** μ_a *is covered iff no* i *has* $a_i = a_{i+1}$ *odd. Hence* $(1,1,0,\ldots,0)$ *is uncovered for* GL_{2m}, $m \geq 2$, *and* $(1,1,0,\ldots,0,-2)$ *for* SL_{2n}, $n \geq 3$. *Since* $Z_{G^\theta}(S) = \mathrm{SL}_2^m$ *is connected, Proposition III says that H1–H5 hold iff* $\varphi(\Lambda_T^+) = \Lambda_S^+$; *so H5 fails in these cases.*

At an uncovered μ no Satake object qualifies: $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{\mathcal{O}_\mu}$ forces $\varphi(\lambda) \leq \mu$ and $a_\mu^*(V_\lambda * \mathrm{IC}_0) \neq 0$ forces $\mu \leq \varphi(\lambda)$. For GL_4 the uncovered labels are exactly (c,c), c odd, and $(1,1)$ is minimal in its component.

4. Function level, every F

Lemma 4.1 (Hecke values as lattice counts)**.** *For $\lambda \in \Lambda_T^+$ with $\lambda_{2m} \geq 0$ and a label b,*

$$N(\lambda, b) := (T_\lambda * \varphi_0)(D_b) = \#\{\Lambda'' \supseteq \Lambda_0 : \ \Lambda''/\Lambda_0 \simeq \textstyle\bigoplus_i \mathcal{O}/t^{\lambda_i},\ D_b \text{ unimodular on } \Lambda''\}.$$

$N(\lambda, b) \neq 0$ implies $b_m \geq 0$ and $|b| = |\lambda|$, and then "integral on Λ''" is equivalent to "unimodular on Λ''". Moreover $T_{c\cdot 1} * \mathbf{1}_{\mathcal{O}_b} = \mathbf{1}_{\mathcal{O}_{b+2c(1^m)}}$ and $T_{c\cdot 1} * T_\lambda = T_{\lambda + c\cdot 1}$.

Proof. $(T_\lambda * \varphi_0)(x) = \#\{gK \subseteq Kt^\lambda K : g^{-1}x \in X(\mathcal{O})\}$. Now $g^{-1}\cdot\omega = g^{-1}\omega g^{-T}$ is unimodular iff ω is unimodular on $\Lambda'' = g^{-T}\Lambda_0$, and $gK \mapsto g^{-T}\Lambda_0$ is a bijection from $Kt^\lambda K/K$ onto the lattices $\Lambda'' \supseteq \Lambda_0$ with Λ''/Λ_0 of type λ [Zhu, §1.1]. Integrality on $\Lambda_0 \subseteq \Lambda''$ gives $b_m \geq 0$; the Pfaffian valuation of D_b on a basis of Λ'' is $|b| - |\lambda|$, which must vanish. Finally $(T_{c\cdot 1} * f)(x) = f(t^{-c}x)$ and $t^{-c}\cdot\omega = t^{-2c}\omega$. □

Example 4.2 ($\mathrm{GL}_4/\mathrm{Sp}_4$)**.** Let $m = 2$. By Lemma 4.1 both translates below live on $\{b \geq 0, |b| = 2\} = \{(2,0),(1,1)\}$, i.e. on $\mathcal{O}_{(2,2,0,0)} \cup \mathcal{O}_{(1,1,1,1)}$.

- $\lambda = (1,1,0,0)$: $\Lambda'' = \Lambda_0 + t^{-1}\widetilde{W}$, $W \in \mathrm{Gr}(2, \mathbb{F}_q^4)$. For $\omega = tJ$, integrality is $\bar{J}(W,W) = 0$: W is Lagrangian, $(1+q)(1+q^2)$ choices. For $\omega = D_{(2,0)}$, pairing $t^{-1}w$ with Λ_0 forces the e_3, e_4-coordinates of w to vanish, so $W = \langle \bar{e}_1, \bar{e}_2\rangle$: one choice.
- $\lambda = (2,0,0,0)$: $\Lambda'' = \Lambda_0 + \mathcal{O}t^{-2}v$, v primitive modulo t^2 up to units. For tJ, integrality forces $v \in t\Lambda_0$: none. For $D_{(2,0)}$ it forces the e_3, e_4-coordinates of v into $t^2\mathcal{O}$: the points of $\mathbb{P}^1(\mathcal{O}/t^2)$, $q(q+1)$ of them.

This gives Proposition IV(c) at function level, and

$$\mathbf{1}_{\mathcal{O}_{(1,1,1,1)}} = \frac{1}{(1+q)(1+q^2)}\Big(T_{(1,1,0,0)} - \frac{1}{q(q+1)}T_{(2,0,0,0)}\Big) * \varphi_0.$$

The translates nonzero on $\mathcal{O}_{(1,1,1,1)}$ are exactly $T_{(1+a,1+b,-b,-a)}$, $a \geq b \geq 0$ (correction C-3); $T_{(2,0,0,0)}$ is not among them, and only its value $q(q+1)$ on $\mathcal{O}_{(2,2,0,0)}$ is used.

Lemma 4.3 (the row $\lambda^*(b)$)**.** *For a label b with $b_m \geq 0$ put $\lambda^*(b) = (b_1, \ldots, b_m, 0, \ldots, 0) \in \Lambda_T^+$. Then*

(i) $N(\lambda^*(b), b) \geq 1$;

(ii) $N(\lambda^*(b), b') \neq 0 \implies b'_m \geq 0,\ |b'| = |b|,\ b' \succeq b$.

Proof. (i) $\Lambda'' = \bigoplus_i (t^{-b_i}\mathcal{O}e_{2i-1} \oplus \mathcal{O}e_{2i})$ has $\Lambda''/\Lambda_0 \simeq \bigoplus_i \mathcal{O}/t^{b_i}$ and $D_b(t^{-b_i}e_{2i-1}, e_{2i}) = 1$.

(ii) Let $\Lambda'' \supseteq \Lambda_0$ be of type $\lambda^*(b)$ with ω unimodular on Λ'' and $\omega|_{\Lambda_0} \simeq D_{b'}$. Put $\Lambda_0^\# = \{v : \omega(v, \Lambda_0) \subseteq \mathcal{O}\}$. Then $\Lambda'' = \Lambda''^\# \subseteq \Lambda_0^\#$ and

$$0 \longrightarrow M = \Lambda''/\Lambda_0 \longrightarrow M' = \Lambda_0^\#/\Lambda_0 \longrightarrow N = \Lambda_0^\#/\Lambda'' \longrightarrow 0,$$

where M has type b, $M' \simeq \bigoplus_i (\mathcal{O}/t^{b'_i})^2$ has type $b'^{(2)} = (b'_1, b'_1, \ldots, b'_m, b'_m)$, and ω induces a perfect pairing $N \times M \to F/\mathcal{O}$, so $N \simeq \mathrm{Hom}(M, F/\mathcal{O})$ has type b. By the sublemma below, choose k-generated $S_1 \subseteq M$, $S_2 \subseteq N$ of orders $q^{b_1+\cdots+b_k}$; then $S = S_1 + \langle$lifts of generators of $S_2\rangle$ is $2k$-generated with $|S| \geq |S_1||S_2|$, so $2(b'_1 + \cdots + b'_k) \geq 2(b_1 + \cdots + b_k)$. Integrality and the Pfaffian give $b'_m \geq 0$ and $|b'| = |b|$. □

Claim 4.4 (sublemma)**.** *For a finite $\mathcal{O}$-module M' of type $t_1 \geq t_2 \geq \cdots$ and $k \geq 1$, $\max\{\log_q |S| : S \subseteq M' \text{ generated by } k \text{ elements}\} = t_1 + \cdots + t_k$.*

Proof. Multiplication by t^{j-1} maps $S[t^j]/S[t^{j-1}]$ injectively into $S[t] \cap t^{j-1}M'$, whose dimension is at most $\min(k, \#\{i : t_i \geq j\})$ since $\dim S[t] \leq k$. Summing over j gives $\log_q |S| \leq \sum_j \min(k, \#\{i : t_i \geq j\}) = t_1 + \cdots + t_k$; the top k cyclic summands attain it. □

Theorem 4.5. (SG) *holds for* $X = \mathrm{GL}_{2m}/\mathrm{Sp}_{2m}$, *every* $m \geq 1$, *over every non-archimedean local field* F.

Proof. Orbits. Each $\mathcal{O}_b$ is compact (image of K) and open (a form congruent to D_b modulo a high power of t has the same elementary divisors, after a central translate making $b_m \geq 0$); so the indicators $\mathbf{1}_{\mathcal{O}_b}$ form a basis of C_K.

Slices. For $d, c \in \mathbb{Z}$ the set $S_{d,c} = \{b : \ b_m \geq 2c, \ |b| = d\}$ is finite (its entries lie in $[2c, d - 2c(m-1)]$), and every b lies in $S_{|b|,\lfloor b_m/2 \rfloor}$. For $b \in S_{d,c}$ put $f_b = T_{c\cdot 1} * T_{\lambda^*(b-2c(1^m))}$. Lemmas 4.1 and 4.3 give

$$f_b * \varphi_0 = \sum_{b' \in S_{d,c},\ b' \succeq b} N\big(\lambda^*(b-2c),\, b'-2c\big)\, \mathbf{1}_{\mathcal{O}_{b'}}, \qquad N\big(\lambda^*(b-2c), b-2c\big) \geq 1 :$$

a translate of f_b never leaves $S_{d,c}$.

Induction. Induct on $\#\{b' \in S_{d,c} : b' \succ b\}$, which is finite. If it is 0, $f_b * \varphi_0$ is a nonzero multiple of $\mathbf{1}_{\mathcal{O}_b}$. Otherwise the terms with $b' \succ b$ lie in $\mathcal{H}_K\varphi_0$ by induction, so $\mathbf{1}_{\mathcal{O}_b} \in \mathcal{H}_K\varphi_0$. Hence every orbit indicator, and so C_K, lies in $\mathcal{H}_K\varphi_0$. □

Remark 4.6 (correction A7-23). Proposition B of the classical page needs one order with finite lower sets. The reversed dominance order on all labels of a fixed degree has infinite lower sets: for $m = 2$, $(k, -k) \succeq (0, 0)$ for every $k \geq 0$. The induction above runs inside the finite slices $S_{d,c}$, which the translates $f_b * \varphi_0$ do not leave; no global order is needed.

Corollary 4.7. (SG) *holds for* $X^{\mathrm{SL}} = \mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$, *every* $n \geq 1$, *over every* F.

Proof. Let $G = \mathrm{GL}_{2n}$, $K_{\mathrm{SL}} = K \cap \mathrm{SL}_{2n}(F)$. By Lemma 2.1(v), $X^{\mathrm{SL}}(F) = \{\mathrm{Pf} = 1\}$, and $\mathcal{O}_b \cap X^{\mathrm{SL}}(F)$ ($|b| = 0$) is one K_{SL}-orbit: if $\mathrm{Pf}\,\omega = 1 = \mathrm{Pf}\,\omega'$ and $k\omega k^T = \omega'$ then $\det k = 1$; surjectivity uses $\mathrm{diag}(u, 1, \ldots, 1)$. For $|\lambda| = 0$ the map $K_{\mathrm{SL}} t^\lambda K_{\mathrm{SL}}/K_{\mathrm{SL}} \to K t^\lambda K/K$ is bijective ($K = K_{\mathrm{SL}} \cdot \{\mathrm{diag}(u, 1, \ldots, 1)\}$, diagonal matrices commute with t^λ, $K \cap \mathrm{SL}_{2n}(F) = K_{\mathrm{SL}}$), and for $g \in \mathrm{SL}_{2n}(F)$ and $\mathrm{Pf}\, x = 1$, $x \in gX(\mathcal{O})$ iff $x \in gX^{\mathrm{SL}}(\mathcal{O})$. So $T^{\mathrm{SL}}_\lambda * \varphi^{\mathrm{SL}}_0 = (T_\lambda * \varphi_0)|_{X^{\mathrm{SL}}(F)}$. For $|b| = 0$ the element f_b of Theorem 4.5 is $T_{\lambda'}$ with $\lambda' = \lambda^*(b-2c) + c \cdot 1$, $|\lambda'| = 0$, and the induction stays in $S_{0,c}$; restricting it to $X^{\mathrm{SL}}(F)$ proves (SG) for X^{SL}. □

5. Sheaf level through Proposition II

Theorem 5.1 (Corollary 5 of the verdict)**.** *Over* $\overline{\mathbb{F}}_q$, $p \neq 2$, $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ *for* $\mathrm{GL}_{2m}/\mathrm{Sp}_{2m}$ *and* $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$, *all* m, n, *under D0–D3; in the Betti category over* $\mathbb{C}$ *under D5.*

Proof. Proposition II (functions ⇒ sheaves over $\mathbb{F}_q(\!(t)\!)$) assumes q odd, (R) T split and contained in a θ-stable Borel subgroup over $\mathbb{F}_q$, (C) G^θ and $Z_{G^\theta}(S)$ connected, D0–D3, and (SG) over $\mathbb{F}_q(\!(t)\!)$. Here (R) holds for the diagonal torus and the upper-triangular Borel in the basis $e_1, e_3, \ldots, e_{2m-1}, e_{2m}, \ldots, e_4, e_2$, in which J is antidiagonal; (C) holds since $G^\theta = \mathrm{Sp}$ and $Z_{G^\theta}(S) = \mathrm{SL}_2^m$ (resp. SL_2^n); and Theorem 4.5 and Corollary 4.7 give (SG) over $\mathbb{F}_q(\!(t)\!)$ for every odd q. Its mechanism: the traces of $V_\lambda * \mathrm{IC}_0$ span $\mathcal{H}_K\varphi_0 = C_K$, so every $\mathrm{tr}\,\mathrm{IC}_\mu$-coordinate is reached, so every IC_μ is a summand of a Satake object, and Proposition I concludes. Since this holds for every odd p, D5 applies for all p outside its finite exceptional set. □

This closes the open SL_6 label $(1,1,-2)$ of A3-7. H5 fails for $m \geq 2$, $n \geq 3$ (Lemma 3.1), so Proposition I with H5 cannot give this.

6. $\mathrm{GL}_4/\mathrm{Sp}_4$ directly

Lemma 6.1 (codimensions, correction C-2)**.** *Over $\mathbb{C}$, for labels $b \preceq a$ of $\mathrm{GL}_{2m}/\mathrm{Sp}_{2m}$ in one component,*

$$\operatorname{codim}_{\overline{\mathcal{O}_a}} \mathcal{O}_b = \langle \rho, \mu_a - \mu_b \rangle = \textstyle\sum_{j=1}^{m} (2m+2-4j)(a_j - b_j).$$

For $m = 2$ the closed orbit $\mathcal{O}_{(1,1)}$ has codimension 4 in $Z = \overline{\mathcal{O}_{(2,0)}} = \{\omega \text{ integral}, \operatorname{val}\operatorname{Pf}\omega = 2\}$, over $\mathbb{C}$ and over $\overline{\mathbb{F}}_q$.

Proof. After a power of σ assume $a_m \geq 0$; put $s = |a|$ and fix $N > s$. On $Z_s = \{\omega \text{ integral}, \operatorname{val}\operatorname{Pf} = s\}$ reduction modulo t^N is a trivial bundle onto an open $Z_s^{(N)}$, equivariant for $L^+G \to G_N = \mathrm{GL}_{2m}(\mathcal{O}/t^N)$, and each orbit is the full preimage of a G_N-orbit ($\omega' \equiv D_e \bmod t^N$ gives $\omega' = D_e(1 + D_e^{-1}(\omega' - D_e))$ with the same elementary divisors). The stabilizer Lie algebra $\{X : XD_a + D_aX^T \equiv 0\}$, computed in 2×2 blocks with $YJ_2 + J_2Y^T = \operatorname{tr}(Y)J_2$, has dimension $(2m^2+m)N + \sum_j (4j-3)a_j$, so $\dim \mathcal{O}_a^{(N)} = m(2m-1)N - \sum_j (4j-3)a_j$, and the difference is the displayed formula because $\sum_j (a_j - b_j) = 0$. For $m = 2$, $N = 3$, the orbits of $Z^{(3)}$ have $\#\mathcal{O}^{(3)}_{(2,0)}(\mathbb{F}_q) = q^{10}(q-1)(q^3-1)(q^2+1)$ and $\#\mathcal{O}^{(3)}_{(1,1)}(\mathbb{F}_q) = q^8(q-1)(q^3-1)$ for every q; they are smooth and geometrically irreducible, so their dimensions are 16 and 12 in every characteristic. □

Remark 6.2 (the citation it replaces). The run cited Chen–Yi, Prop. 64 for this formula. Their §7.2 assumes LX connected, and the proof uses $\pi_0(LX) \simeq \Lambda_S/(\Lambda_S \cap Q) = 0$; here $\pi_0(LX) \simeq \mathbb{Z}$ via $\operatorname{val}\operatorname{Pf}$. The scalar shift multiplies the Pfaffian by t^m, so it carries component s to component 0 only when $m \mid s$ (audit of A7-11); the stabilizer count above covers every component.

Lemma 6.3 (fibres)**.** *Let $\lambda = (1,1,0,0)$, minuscule, so $\mathrm{Gr}^{\leq\lambda} = \mathrm{Gr}^{\lambda} \simeq \mathrm{Gr}(2,4)$ [Zhu, Lemma 2.1.13], and $m\colon Y = \mathrm{Gr}^{\lambda} \widetilde{\times} L^+X \to Z$. The fibre over ω is $\{\Lambda'' \supseteq \Lambda_0$ of type λ : ω perfect on $\Lambda''\}$: a point over $D_{(2,0)}$ and $\mathrm{LG}(2,4) \simeq Q^3$ over tJ, with $H^*(\mathrm{LG}(2,4)) = k \oplus k[-2] \oplus k[-4] \oplus k[-6]$.*

Proof. $gL^+G \in m^{-1}(\omega)$ iff $g^{-1}\cdot\omega \in L^+X$ iff ω is perfect on $g^{-T}\Lambda_0$; the two fibres are the computation of Example 4.2 over k. $\mathrm{LG}(2,4)$ is a smooth quadric threefold, paved by cells of dimensions $0,1,2,3$. □

Theorem 6.4 (Theorem 2 of the verdict)**.** *Under D0–D2, with $P = m_! k_Y[D]$ on the finite-type model, $P \simeq \mathrm{IC}_{(2,0)} \oplus \mathrm{IC}_{(1,1)}[2] \oplus \mathrm{IC}_{(1,1)} \oplus \mathrm{IC}_{(1,1)}[-2]$.*

Proof. For $N \geq 3$, Y and Z are pulled back from $Y^{(N)} \to Z^{(N)}$, a proper map of finite type with smooth source of dimension $D = \dim Z^{(N)}$ (birational by Lemma 6.3). The stabilizers are connected (D0), so the simple equivariant perverse sheaves on $Z^{(N)}$ are $\mathrm{IC}_{(2,0)}$ and $\mathrm{IC}_{(1,1)} = k_{\mathcal{O}_{(1,1)}}[D-4]$, and D2 gives $P \simeq \mathrm{IC}_{(2,0)} \otimes A \oplus \mathrm{IC}_{(1,1)} \otimes B$.

(1) Over $\mathcal{O}_{(2,0)}$ the fibre is a point, so $P = k[D]$ there and $A = k$.
(2) At $x \in \mathcal{O}_{(1,1)}$, proper base change gives $P_x = H^*(\mathrm{LG}(2,4))[D]$: rank one in degrees $-D, -D+2, -D+4, -D+6$.

(3) $\dim \mathcal{O}_{(1,1)} = D-4$ (Lemma 6.1); the support condition gives $\mathcal{H}^i(\mathrm{IC}_{(2,0)})_x = 0$ for $i \geq -D+4$, and $\mathrm{IC}_{(1,1)}[-j]$ sits in degree $-D+4+j$.

(4) Degrees $-D+4, -D+6$ force $j = 0, 2$ in B. P and $\mathrm{IC}_{(2,0)}$ are Verdier self-dual (Y smooth, m proper), so B is symmetric: $j = -2$ occurs and fills degree $-D+2$, and $j = \pm 4$ cannot occur since degree $-D+8$ is empty.

Hence $B = k[2] \oplus k \oplus k[-2]$ and $(\mathrm{IC}_{(2,0)})_x = k[D]$: Z is rationally smooth along $\mathcal{O}_{(1,1)}$. Frobenius traces check it: $1 + (q + q^2 + q^3) = (1+q)(1+q^2)$. □

Corollary 6.5. *For* $\mathrm{GL}_4/\mathrm{Sp}_4$, $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ *over* $\overline{\mathbb{F}}_q$ *under D0–D2, and over* $\mathbb{C}$ *under D5.*

Proof. By Proposition I it suffices that every IC_μ lies in $\mathcal{C}_K^{\mathrm{Sat}}$. Covered $\mu = \varphi(\lambda)$: $V_\lambda * \mathrm{IC}_0$ is supported on $\overline{\mathcal{O}_\mu}$ and nonzero at $t^\lambda x_0 \in \mathcal{O}_\mu$, and among its summands $\mathrm{IC}_\nu[j]$ ($\nu \leq \mu$) only $\nu = \mu$ is nonzero on $\mathcal{O}_\mu$; so IC_μ is a summand. $\mu = (1,1,1,1)$: Theorem 6.4. The scalar shift: $\sigma_* \mathrm{IC}_0 = \mathrm{IC}_{(1,1,1,1)}$ up to shift is a summand of $V_{(1,1,0,0)} * \mathrm{IC}_0$, and σ is LG-equivariant, so $\sigma_*(V_\lambda * \mathrm{IC}_0) \simeq V_\lambda * \sigma_* \mathrm{IC}_0$ is a retract of $V_\lambda * V_{(1,1,0,0)} * \mathrm{IC}_0$, a finite sum of shifted Satake objects; with $\sigma_*^2 = V_{(1,1,1,1)} * (-)$ invertible, $\sigma_*^{\pm 1}$ preserve $\mathcal{C}_K^{\mathrm{Sat}}$, and $\sigma_*^{c-1} \mathrm{IC}_{(1,1,1,1)} = \mathrm{IC}_{(c,c,c,c)}$ for every odd c. □

7. Proof of Proposition IV

(a) Theorem 4.5 and Corollary 4.7. (b) Theorem 5.1. (c) Theorem 6.4 with $V_{(1,1,0,0)} * \mathrm{IC}_0 = P[4-D]$, and Example 4.2. (d) Lemma 3.1. □

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