PROPOSITION I: THE SUMMAND CRITERION

Proposition I of the spherical-generation workspace deduces $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ from (H1)–(H5). When every Satake object is a finite sum of shifted intersection complexes, the thick span of the Satake objects is controlled by one set of orbit labels, the labels whose intersection complex occurs as a summand. This note proves that criterion, shows that (H5) is sufficient but not necessary for it, and gives the maximal-defect lemma that detects summands where (H5) fails.

1. Setting

G is connected reductive over an algebraically closed field, θ an involution with G^θ connected, $X = G/G^\theta$, $S \subseteq T$ a maximal θ-split torus in a θ-stable maximal torus, Λ_T^+ and $\Lambda_S^+ = X_*(S) \cap \Lambda_T^+$ the dominant cocharacters, $\leq$ the dominance order, $\varphi(\lambda) = \lambda - \theta(\lambda)$. The arc group L^+G acts on LX with orbits $\mathcal{O}_\mu = L^+G \cdot x_\mu$, $\mu \in \Lambda_S^+$, and $t^\lambda x_0 = x_{\varphi(\lambda)}$ [CY, Prop. 17].

$\mathcal{C}_K = \mathrm{Shv}_{\mathrm{cons}}(L^+G\backslash LX)$ is the idempotent-complete stable category of bounded L^+G-equivariant constructible complexes on LX with coefficients in a field k (complex coefficients in the Betti setting of the pages; $\overline{\mathbb{Q}}_\ell$ over $\overline{\mathbb{F}}_q$, $p \neq 2$, in Chen–Yi's setting), each supported on finitely many orbit closures. $a_\mu \colon \mathcal{O}_\mu \hookrightarrow LX$ and $i_\mu \colon \overline{\mathcal{O}_\mu} \setminus \mathcal{O}_\mu \hookrightarrow LX$ are the inclusions, $\Delta_\mu = a_{\mu!}\underline{k}_{\mathcal{O}_\mu}$ the standard objects, $\mathrm{IC}_0 = \Delta_0$, and

$$\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}_K}\{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\},$$

where Thick is the smallest full subcategory containing 0 and closed under isomorphisms, shifts, cofibres and retracts. δ is the dimension function of the orbits [CY, Prop. 41], and

$$\mathrm{IC}_\nu := \mathrm{IC}(\overline{\mathcal{O}_\nu}, k), \qquad a_\nu^* \mathrm{IC}_\nu \simeq \underline{k}_{\mathcal{O}_\nu}[\delta(\nu)], \qquad \operatorname{supp} \mathrm{IC}_\nu = \overline{\mathcal{O}_\nu}.$$

Hypotheses. (H1)–(H3) are those of Proposition I: finite orbit filtration, equivariant recollement, and constant coefficients on each orbit (connected stabilizers). In addition:

- (PH) *Perverse heart.* $\mathcal{C}_K$ carries a bounded t-structure whose heart $\mathcal{P}$ has finite length, whose simple objects are the IC_ν, $\nu \in \Lambda_S^+$, pairwise non-isomorphic, and whose cohomology functors ${}^pH^i \colon \mathcal{C}_K \to \mathcal{P}$ are cohomological.
- (DT) *Decomposition.* For each $\lambda \in \Lambda_T^+$ there are finitely many $(\nu_b, s_b) \in \Lambda_S^+ \times \mathbb{Z}$ with $V_\lambda * \mathrm{IC}_0 \simeq \bigoplus_b \mathrm{IC}_{\nu_b}[s_b]$.

For X with $Z_{G^\theta}(S)$ connected, over $\overline{\mathbb{F}}_q$ ($p \neq 2$), these hold: (H1)–(H2) by [CY, Thm. 18, Thm. 36, App. A], (H3) by Proposition III of the workspace, (PH) by [CY, A.7], (DT) by the decomposition theorem [Sun, Thm. 1.2], [BBD, 5.4.5–5.4.6] as in [CY, Prop. 70, proof of Thm. 93]. Over $\mathbb{C}$ they are the workspace's open item 1 (D5 of the run).

Proposition I (summand criterion)**.** *Assume* (H1)–(H3)*,* (PH) *and* (DT)*. Put*

$$\mathcal{M} := \{\mu \in \Lambda_S^+ : \ \mathrm{IC}_\mu[s] \text{ is a direct summand of } V_\lambda * \mathrm{IC}_0 \text{ for some } \lambda \in \Lambda_T^+, \ s \in \mathbb{Z}\}.$$

(a) $\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}_K}\{\mathrm{IC}_\nu : \nu \in \mathcal{M}\}$*, and for* $\mu \in \Lambda_S^+$*:* $\mathrm{IC}_\mu \in \mathcal{C}_K^{\mathrm{Sat}} \iff \mu \in \mathcal{M}$*.*
(b) $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K \iff \mathcal{M} = \Lambda_S^+$*.*

(c) Under the support bound and the top-stalk computation of Proposition III (steps 2 and 6), $\varphi(\Lambda_T^+) \subseteq \mathcal{M}$.

(d) If (H5) *holds then* $\mathcal{M} = \Lambda_S^+$*; the converse fails: for* $\mathrm{GL}_4/\mathrm{Sp}_4$ *over* $\overline{\mathbb{F}}_q$*,* (H5) *fails at* $(1,1,1,1)$ *while* $\mathcal{M} = \Lambda_S^+$.

(e) (Maximal defect.) *Let* $m\colon Y \to Z$ *be a proper* H*-equivariant morphism of varieties of finite type,* H *an algebraic group,* Y *smooth irreducible of dimension* D*, and* $Z = \bigsqcup_\nu \mathcal{O}_\nu$ *finitely many* H*-orbits with connected stabilizers. Put* $f_\nu = \dim m^{-1}(x_\nu)$*,* $c_\nu = D - \dim \mathcal{O}_\nu$*,* $\mathrm{def}(\nu) = 2f_\nu - c_\nu$*,* $r = \max_\nu \mathrm{def}(\nu)$*. If* $\mathrm{def}(\mu) = r$*, then* $\mathrm{IC}_\mu[-r]$ *(perverse on* Z*) is a direct summand of* $K = m_!\underline{k}_Y[D]$*, with multiplicity the number of irreducible components of* $m^{-1}(x_\mu)$ *of dimension* f_μ.

2. Proof

Step 1 (reduction to intersection complexes). The thick span contains retracts and shifts, so every IC_ν with $\nu \in \mathcal{M}$ lies in $\mathcal{C}_K^{\mathrm{Sat}}$. The proof compares $\mathcal{C}_K^{\mathrm{Sat}}$ with the subcategories generated by these IC_ν.

Step 2 (span). If $\nu \in \mathcal{M}$, then $\mathrm{IC}_\nu[s]$ is a retract of some $V_\lambda * \mathrm{IC}_0 \in \mathcal{C}_K^{\mathrm{Sat}}$, so $\mathrm{IC}_\nu \in \mathcal{C}_K^{\mathrm{Sat}}$; hence $\mathrm{Thick}\{\mathrm{IC}_\nu : \nu \in \mathcal{M}\} \subseteq \mathcal{C}_K^{\mathrm{Sat}}$. Conversely, by (DT) each generator is $\bigoplus_b \mathrm{IC}_{\nu_b}[s_b]$; each summand has $\nu_b \in \mathcal{M}$ by definition, and a finite direct sum is an iterated cofibre of split triangles. So every generator lies in $\mathrm{Thick}\{\mathrm{IC}_\nu : \nu \in \mathcal{M}\}$, and minimality of $\mathcal{C}_K^{\mathrm{Sat}}$ gives equality.

Step 3 (constituent subcategories are thick). For $\Sigma \subseteq \Lambda_S^+$ let $\mathcal{P}_\Sigma \subseteq \mathcal{P}$ be the objects whose Jordan–Hölder constituents are among $\{\mathrm{IC}_\nu : \nu \in \Sigma\}$, a Serre subcategory, and

$$\mathcal{D}_\Sigma = \{M \in \mathcal{C}_K : \ {}^pH^i(M) \in \mathcal{P}_\Sigma \text{ for all } i\}.$$

With $q\colon \mathcal{P} \to \mathcal{P}/\mathcal{P}_\Sigma$ the exact Serre quotient, $h = q \circ {}^pH^0$ is cohomological and $\mathcal{D}_\Sigma = \{M : h(M[i]) = 0\ \forall i\}$. By [Stacks, Lemma 13.6.3, Tag 05RD] it is a saturated triangulated subcategory: closed under shifts, cofibres and direct summands, i.e. thick.

Step 4 (only summands are reached). Take $\Sigma = \mathcal{M}$. By (DT), ${}^pH^i(V_\lambda * \mathrm{IC}_0) = \bigoplus_{s_b = -i} \mathrm{IC}_{\nu_b} \in \mathcal{P}_\mathcal{M}$, so every Satake object lies in $\mathcal{D}_\mathcal{M}$ and $\mathcal{C}_K^{\mathrm{Sat}} \subseteq \mathcal{D}_\mathcal{M}$. If $\mu \notin \mathcal{M}$, then ${}^pH^0(\mathrm{IC}_\mu) = \mathrm{IC}_\mu$ is simple and not isomorphic to any IC_ν, $\nu \in \mathcal{M}$, so $\mathrm{IC}_\mu \notin \mathcal{D}_\mathcal{M} \supseteq \mathcal{C}_K^{\mathrm{Sat}}$. With Step 2 this proves (a), and $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K \Rightarrow \mathcal{M} = \Lambda_S^+$.

Step 5 (intersection complexes and standards). For every μ: $\mathrm{Thick}\{\mathrm{IC}_\nu : \nu \le \mu\} = \mathrm{Thick}\{\Delta_\nu : \nu \le \mu\}$. Induct on $h(\mu) = \#\{\nu \le \mu\}$, finite by (H1)(a). The localization triangle of (H2)(b), pushed to LX by (H2)(c), reads

$$\Delta_\mu[\delta(\mu)] \longrightarrow \mathrm{IC}_\mu \longrightarrow i_{\mu *} i_\mu^* \mathrm{IC}_\mu \xrightarrow{+1} .$$

Standard generation on the boundary (Proposition I, step 1(a), from (H1)–(H3)) puts the right term in $\mathrm{Thick}\{\Delta_\nu : \nu < \mu\}$, which equals $\mathrm{Thick}\{\mathrm{IC}_\nu : \nu < \mu\}$ by induction: the claim at every $\nu < \mu$, since $\{\nu < \mu\} = \bigcup_{\nu' < \mu}\{\nu' \le \nu\}$. This is used only for $\Delta_\mu \in \mathrm{Thick}\{\mathrm{IC}_\nu : \nu \le \mu\}$. Two out of three gives both inclusions.

Step 6 (every summand present gives everything). If $\mathcal{M} = \Lambda_S^+$, Step 2 puts every IC_ν in $\mathcal{C}_K^{\mathrm{Sat}}$, Step 5 every Δ_ν, and standard generation $\mathcal{C}_K = \mathrm{Thick}\{\Delta_\mu\}$ gives $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$. With Step 4 this proves (b). (Alternatively, finite length and boundedness in (PH) give $\mathcal{C}_K = \mathrm{Thick}\{\mathrm{IC}_\nu\}$ directly, without (H1)–(H3); cf. [Stacks, Lemma 13.35.7, Tag 0FX7].)

Step 7 (covered labels). Let $\lambda \in \Lambda_T^+$ and $V_\lambda * \mathrm{IC}_0 \simeq \bigoplus_b \mathrm{IC}_{\nu_b}[s_b]$. Proposition III gives $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{\mathcal{O}_{\varphi(\lambda)}}$ (step 2) and a nonzero stalk at $x_{\varphi(\lambda)} = t^\lambda x_0$ (step 6). The first gives $\nu_b \le \varphi(\lambda)$ for all b, since $\mathcal{O}_{\nu_b} \subseteq \mathrm{supp}$. A summand with nonzero stalk at $x_{\varphi(\lambda)}$ has $x_{\varphi(\lambda)} \in \overline{\mathcal{O}_{\nu_b}}$, i.e. $\varphi(\lambda) \le \nu_b$ [CY, Thm. 18]. So $\nu_b = \varphi(\lambda)$ for that b, and $\varphi(\lambda) \in \mathcal{M}$. This is (c).

Step 8 ((H5) is sufficient). Let λ satisfy (H5) at μ. (H5)(a) gives $\nu_b \le \mu$ for all b. (H5)(b) gives $a_\mu^* \mathrm{IC}_{\nu_b} \neq 0$ for some b, hence $\mu \le \nu_b$. So $\nu_b = \mu \in \mathcal{M}$. Only the nonvanishing in (H5)(b) is used; under (DT) the splitting is automatic.

Remark 2.1 (a weaker (H5) is ruled out). Asking only $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{\mathcal{O}_\mu}$ and $a_\mu^*(V_\lambda * \mathrm{IC}_0) \neq 0$ does not extend generation beyond covered labels. (i) By Step 7's two inputs the first condition gives $\varphi(\lambda) \le \mu$ and the second $\mu \le \varphi(\lambda)$, so $\mu = \varphi(\lambda)$: at an uncovered label no Satake object satisfies it. (ii) Without (DT), nonvanishing does not produce the standard object. On an orbit with stabilizer $\mathbb{G}_m$, $\mathcal{C}_\mu \simeq D^{\mathrm{fg}}(k[u])$, $\deg u = 2$, with $\underline{k}_{\mathcal{O}_\mu} \mapsto k[u]$ [BY, §0.3]. The object $E = \mathrm{Cone}(u\colon \underline{k} \to \underline{k}[2])$ corresponds to $k[u]/(u)$; its cohomology sheaves are constant and nonzero, but the objects with finite-dimensional total cohomology form a thick subcategory containing E and not $k[u]$, so $\Delta_\mu \notin \mathrm{Thick}(E)$.

Step 9 (maximal defect). *Amplitude.* By proper base change $\mathcal{H}^i(K)_x = H^{i+D}(m^{-1}x)$, which vanishes on $\mathcal{O}_\nu$ unless $i + D \le 2f_\nu$. For such ν, $\dim \mathcal{O}_\nu = D - c_\nu \le r - i$ because $r \ge 2f_\nu - c_\nu \ge i + D - c_\nu$. So $K \in {}^pD^{\le r}$, and $\mathbb{D}K \simeq K$ (Y smooth, m proper) gives $K \in {}^pD^{\ge -r}$.

Decomposition. By the decomposition theorem for $Y/H \to Z/H$ [Sun, Thm. 1.2], [dCM, Thm. 1.6.1] and connected stabilizers, $K \simeq \bigoplus_{\nu,j} \mathrm{IC}_\nu[-j]^{\oplus n_{\nu,j}}$ with $n_{\nu,j} = 0$ for $|j| > r$.

Stalks at x_μ. For $\nu \neq \mu$ with $x_\mu \in \overline{\mathcal{O}_\nu}$ the support condition puts $\mathrm{IC}_\nu[-j]$ at x_μ in degrees $\le -\dim \mathcal{O}_\mu - 1 + j \le -D + 2f_\mu - 1$. The summand $\mathrm{IC}_\mu[-j]$ sits in degree $-D + c_\mu + j \le -D + 2f_\mu$, with equality iff $j = r$. Hence

$$k^{\oplus n_{\mu,r}} \simeq \mathcal{H}^{-D+2f_\mu}(K)_{x_\mu} = H^{2f_\mu}(m^{-1}x_\mu) \neq 0,$$

the top cohomology of a proper variety of dimension f_μ, of dimension the number of its f_μ-dimensional components. This is (e); for $r = 0$ it is the relevant-stratum statement of [dCM, Thm. 4.2.7].

In the loop setting the lemma is applied on finite-type models. For minuscule $\omega_1, \dots, \omega_n$, the convolution space $Y = \mathrm{Gr}^{\omega_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}^{\omega_n} \tilde{\times} L^+X$ is smooth and $m_! \underline{k}_Y[D]$ is a shift of $(V_{\omega_1} * \cdots * V_{\omega_n}) * \mathrm{IC}_0$, a sum of shifted Satake objects; by Krull–Schmidt ($\mathrm{End}\,\mathrm{IC}_\nu = k$) each IC summand is a summand of one Satake object, so maximal-defect strata lie in $\mathcal{M}$.

Step 10 ((H5) is not necessary). For $X = \mathrm{GL}_4/\mathrm{Sp}_4$ and $\lambda = (1,1,0,0)$, m is an isomorphism over $\mathcal{O}_{(2,2,0,0)}$ and has fibre $\mathrm{LG}(2,4) \cong Q^3$ over the closed orbit $\mathcal{O}_{(1,1,1,1)}$, of codimension 4. So $\mathrm{def} = 0$ on top, $\mathrm{def}(1,1,1,1) = 2 \cdot 3 - 4 = 2 = r$, and Step 9 with Verdier duality gives

$$V_{(1,1,0,0)} * \mathrm{IC}_0 \simeq \mathrm{IC}_{(2,2,0,0)} \oplus \mathrm{IC}_{(1,1,1,1)} \otimes \big(k[2] \oplus k \oplus k[-2]\big).$$

The label $(1,1,1,1)$ has no dominant lift, so (H5) fails there, yet $(1,1,1,1) \in \mathcal{M}$; central translates give every (a,a,a,a), and $\mathcal{M} = \Lambda_S^+$ (Proposition IV of the run, proved over $\overline{\mathbb{F}}_q$ under D0–D2). This is (d). $\square$

References

[BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque 100 (1982).

[BY] R. Bezrukavnikov, Z. Yun, *On Koszul duality for Kac–Moody groups*, arXiv:1101.1253v2.

[CY] T.-H. Chen, L. Yi, *Singularities of orbit closures in loop spaces of symmetric varieties*, arXiv:2310.20006v4.

[dCM] M. A. de Cataldo, L. Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, arXiv:0712.0349v2.

[Stacks] The Stacks Project, Chapter *Derived Categories*.

[Sun] S. Sun, *Decomposition theorem for perverse sheaves on Artin stacks over finite fields*, arXiv:1009.4398v3.