PROPOSITION III: $\mathrm{SL}_2/\mathrm{SO}_2$, BOTH LEVELS FAIL

Both conditions of the pages' Proposition III fail for $X = \mathrm{SL}_2/\mathrm{SO}_2$: the stabilizers are disconnected and the leading-term map doubles. We decide both levels directly. The sheaf statement is proved in the Betti category over $\mathbb{C}$ with no transfer from $\overline{\mathbb{F}}_q$, and the function statement over every non-archimedean local field.

1. Setting

Space. $k = \mathbb{C}$, coefficients $\mathbb{C}$. $G = \mathrm{SL}_2$, $\theta(g) = (g^{\mathsf{T}})^{-1}$, $G^\theta = \mathrm{SO}_2$ (connected). The Cartan embedding $\tau(gG^\theta) = g\theta(g)^{-1} = gg^{\mathsf{T}}$ identifies

$$X = G/G^\theta \xrightarrow{\sim} \{A \in \mathrm{Mat}_2 : \ A = A^{\mathsf{T}},\ \det A = 1\}, \qquad g \cdot A = gAg^{\mathsf{T}}, \qquad x_0 = I.$$

Labels. $S = T$ is the diagonal torus, θ inverts it, $\Lambda_T = \Lambda_S = \mathbb{Z}\alpha^\vee$, and $\Lambda_T^+ = \Lambda_S^+ = \mathbb{Z}_{\geq 0}\alpha^\vee$. With $F = k(\!(t)\!)$, $LX = X(F)$, $L^+G = \mathrm{SL}_2(k[\![t]\!])$:

$$x_m = \mathrm{diag}(t^m, t^{-m}) = t^{m\alpha^\vee}, \qquad \mathcal{O}_m = L^+G \cdot x_m,$$
$$\varphi(n\alpha^\vee) = n\alpha^\vee - \theta(n\alpha^\vee) = 2n\alpha^\vee, \qquad t^{n\alpha^\vee} \cdot x_0 = x_{2n}.$$

Categories. $\mathcal{C}_K = \mathrm{Shv}_{\mathrm{cons}}(L^+G\backslash LX)$ is the pages' small, idempotent-complete stable ∞-category of bounded L^+G-equivariant constructible complexes on LX, each supported on finitely many orbit closures; $a_m \colon \mathcal{O}_m \hookrightarrow LX$; $\mathrm{IC}_0 = a_{0!}\underline{k}_{\mathcal{O}_0}$; $\mathcal{H} = \mathrm{Shv}_{\mathrm{cons}}(L^+G\backslash \mathrm{Gr})$, $\mathrm{Gr} = LG/L^+G$, acts by $\mathcal{A} * \mathcal{F} = m_!(\mathcal{A} \mathbin{\widetilde{\boxtimes}} \mathcal{F})$, $m \colon LG \times^{L^+G} LX \to LX$; $V_\lambda = \mathrm{IC}(\mathrm{Gr}^{\leq\lambda})$ and $\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}_K}\{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\}$.

Functions. F is a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field $\mathbb{F}_q$; $X = \mathrm{SL}_2/T$ with T the diagonal torus; $K = \mathrm{SL}_2(\mathcal{O})$, $K_{\mathrm{GL}} = \mathrm{GL}_2(\mathcal{O})$, $\mathcal{H}_K = C_c(K\backslash\mathrm{SL}_2(F)/K)$ with $\mathrm{vol}\, K = 1$, $\varphi_0 = \mathbf{1}_{X(\mathcal{O})}$, $C_K = C_c^\infty(X(F))^K$. For $x = gT \in X(F)$ let $d(x)$ be the distance of the vertex $g^{-1}[\mathcal{O}^2]$ of the tree of $\mathrm{PGL}_2(F)$ from the apartment of T, and $\mathbf{1}_d = \mathbf{1}_{\{d(x)=d\}}$. $T_1 = \mathbf{1}_{K_{\mathrm{GL}}\,\mathrm{diag}(\varpi,1)K_{\mathrm{GL}}}$. Over $\mathbb{C}$ the forms $\mathrm{SL}_2/\mathrm{SO}_2$ and SL_2/T are isomorphic; over $\mathbb{F}_q$ they are different forms.

Proposition III. Sheaves, over $k = \mathbb{C}$.

(a) $\mathcal{O}_m = \{A \in X(F) : \min \mathrm{val}(\textit{entries of } A) = -m\}$. *For* $m \geq 1$, $\pi_0\,\mathrm{Stab}_{L^+G}(x_m) = \{\pm I\}$; $\mathrm{Stab}_{L^+G}(x_0)$ *is connected. Let* $\mathcal{L}_m^-$ *be the rank-one equivariant local system on* $\mathcal{O}_m$ *on which* $-I$ *acts by* -1.

(b) The central element $-I \in L^+G$ *acts trivially on* LX *and on* Gr *and defines a natural automorphism* ε *of* $\mathrm{id}_{\mathcal{C}_K}$ *with* $\varepsilon^2 = \mathrm{id}$. *Put* $\mathcal{C}_K^\pm = \{F : \varepsilon_F = \pm\mathrm{id}\}$. *Then* $\mathcal{C}_K = \mathcal{C}_K^+ \oplus \mathcal{C}_K^-$, $\mathcal{C}_K^+$ *and* $\mathcal{C}_K^-$ *are thick and mutually orthogonal, and* $p^* \colon \mathrm{Shv}_{\mathrm{cons}}(L^+\mathrm{PGL}_2\backslash LX) \xrightarrow{\sim} \mathcal{C}_K^+$.

(c) $\mathcal{C}_K^{\mathrm{Sat}} \subseteq \mathcal{C}_K^+$ *and* $0 \neq a_{m!}\mathcal{L}_m^- \in \mathcal{C}_K^-$ *for* $m \geq 1$. *Hence* $\mathcal{C}_K^{\mathrm{Sat}} \neq \mathcal{C}_K$.

(d) $V_{n\alpha^\vee} * \mathrm{IC}_0 \simeq \mathrm{IC}(\overline{\mathcal{O}_{2n}})$ *up to shift, for* $n \geq 0$*: the convolution map is finite and birational.*

(e) Under the perverse input (P) *of §5,* $\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}\{\mathrm{IC}(\overline{\mathcal{O}_{2n}}) : n \geq 0\} \subsetneq \mathcal{C}_K^+ \subsetneq \mathcal{C}_K$, *with witnesses* $\mathrm{IC}(\overline{\mathcal{O}_1}, \underline{k}) \in \mathcal{C}_K^+ \setminus \mathcal{C}_K^{\mathrm{Sat}}$ *and* $\mathrm{IC}(\overline{\mathcal{O}_1}, \mathcal{L}_1^-) \in \mathcal{C}_K \setminus \mathcal{C}_K^+$.

Functions, over every F.

(f) $\mathcal{H}_K\varphi_0 = \mathbb{C}[T_1^2]\varphi_0 \subsetneq \mathbb{C}[T_1]\varphi_0 = C_{K_{\mathrm{GL}}} \subseteq C_K$, *and* $\mathbf{1}_1 \notin \mathcal{H}_K\varphi_0$. *So* $\mathcal{H}_K \cdot \varphi_0 \neq C_K$.

(g) *If* q *is odd, a non-square unit* u *gives* $\sigma(gT) = sgs^{-1}T$, $s = \mathrm{diag}(u, 1)$, *which fixes* $\mathcal{H}_K\varphi_0$ *pointwise and exchanges the two* K*-orbits in each* $\{d = m\}$, $m \geq 1$*; so* $\mathcal{H}_K\varphi_0 \subseteq C_K^\sigma \subsetneq C_K$.

Parts (a)–(d), (f), (g) are proved below with no hypothesis beyond the pages' realization of $\mathcal{C}_K$; (e) uses (P). Remark 6.4 identifies σ in (g) with the Frobenius trace of ε, under D0–D3.

2. Orbits and stabilizers

Lemma 2.1 (Step 1)**.** *Part (a) holds; for* $m \geq 1$, $\mathrm{Stab}_{L^+G}(x_m) \cong \mu_2 \times t^m k[\![t]\!]$ *as schemes, and* $\mathcal{C}_m^\heartsuit \simeq \mathrm{Rep}(\mathbb{Z}/2)$.

Proof. Orbits. Write $A = t^{-m}A'$ with A' integral and some entry a unit. A binary symmetric form over $k[\![t]\!]$ with a unit entry represents a unit (evaluate at $(1,1)$ if only the off-diagonal entry is a unit), hence $A' \simeq \mathrm{diag}(u_1, u_2t^{2m})$ under $\mathrm{GL}_2(k[\![t]\!])$-congruence. Units of $k[\![t]\!]$ are squares and $\det A' = t^{2m}$, so the conjugating matrix can be taken of determinant 1 (correct by $\mathrm{diag}(1,-1)$), and $A \simeq \mathrm{diag}(t^{-m}, t^m) = w \cdot x_m$. The minimal valuation of the entries is a congruence invariant. The closures are $\overline{\mathcal{O}_m} = \bigsqcup_{m' \leq m} \mathcal{O}_{m'}$ by [CY, Thm. 18].

Stabilizers. $g\,x_m g^{\mathsf{T}} = x_m$ with $g \in \mathrm{SL}_2(F)$ means $g = g_{a,b} = \begin{pmatrix} a & bt^m \\ -bt^{-m} & a \end{pmatrix}$ with $a^2+b^2 = 1$. Integrality forces $a \in k[\![t]\!]$, $b \in t^m k[\![t]\!]$; for $m \geq 1$, $a^2 \in 1 + t^{2m}k[\![t]\!]$, so $a = \pm\sqrt{1-b^2}$. The two components are $a \equiv \pm 1$, the identity component is pro-unipotent, and $-I = g_{-1,0}$ lies in the other. For $m = 0$ the stabilizer is $\mathrm{SO}_2(k[\![t]\!]) \cong L^+\mathbb{G}_m$, connected. Equivariant local systems on one orbit are the representations of the component group of the stabilizer [CY, Ex. 108], so $\mathcal{C}_m^\heartsuit \simeq \mathrm{Rep}(\mathbb{Z}/2)$, with simples $\underline{k}$ and $\mathcal{L}_m^-$. □

3. Central inertia

Construction 3.1 (Step 2)**.** For a quotient stack $H\backslash Y$ and a central $z \in H$ acting trivially on Y, multiplication by z is an automorphism of every H-torsor commuting with its map to Y: a 2-automorphism of $\mathrm{id}_{H\backslash Y}$. Applied to $-I \in L^+G$ on the finite-type models $G(k[t]/t^N)\backslash Z^{(N)}$ of $\mathcal{C}_K$, where $-I$ is still central and acts trivially, it gives $\varepsilon \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}_K})$ with:

(E1) naturality, $\varepsilon_G \circ f = f \circ \varepsilon_F$ for $f\colon F \to G$; and $\varepsilon^2 = \mathrm{id}$;

(E2) $\varepsilon_{F[1]} = \varepsilon_F[1]$, and ε commutes with ordinary and perverse truncations;

(E3) $f^*, f_!, f_*$ commute with ε for L^+G-equivariant maps f;

(E4) $\varepsilon = \mathrm{id}$ on a sheaf pulled back from a point;

(E5) on an equivariant local system with monodromy ρ of $\pi_0\,\mathrm{Stab}(y)$ on an orbit $L^+G \cdot y$, $\varepsilon = \rho(-I)$.

Lemma 3.2 (Step 3)**.** $\mathrm{Hom}(F, G[n]) = 0$ *for* $F \in \mathcal{C}_K^+$, $G \in \mathcal{C}_K^-$, $n \in \mathbb{Z}$*; every* F *is* $F^+ \oplus F^-$ *with* $F^\pm \in \mathcal{C}_K^\pm$*;* $\mathcal{C}_K^+ = \{F : \mathrm{Hom}(F, G[n]) = 0\ \forall G \in \mathcal{C}_K^-, n\}$ *and symmetrically. Hence* $\mathcal{C}_K^+, \mathcal{C}_K^-$ *are thick and* $\mathcal{C}_K^+ \cap \mathcal{C}_K^- = 0$.

Proof. Orthogonality. For $f\colon F \to G[n]$, (E1)–(E2) give $f = f\varepsilon_F = \varepsilon_{G[n]}f = -f$, so $f = 0$ as $2 \neq 0$ in k. Splitting. $e = (1 + \varepsilon_F)/2$ is an idempotent of F in $h\mathcal{C}_K$; $\mathcal{C}_K$ is idempotent complete, hence so is $h\mathcal{C}_K$ [HA, Lemma 1.2.4.6], so $F \simeq F_1 \oplus F_2$ with $e = i_1p_1$. By (E1),

$\varepsilon_{F_1} = p_1\varepsilon_F i_1 = p_1(2e-1)i_1 = \mathrm{id}$ and $\varepsilon_{F_2} = p_2(2e-1)i_2 = -\mathrm{id}$. Orthogonal description. If $\mathrm{Hom}(F, \mathcal{C}_K^-[n]) = 0$, the projection $F \to F_2$ vanishes, so $F_2 = 0$ and $F \in \mathcal{C}_K^+$. A left orthogonal is thick [Stacks, Lemma 13.6.3]. □

This avoids the cofibre step of the run's first version, which needed $\varepsilon^2 = \mathrm{id}$ in addition.

Lemma 3.3 (Step 4)**.** PGL_2 *acts on* X *by* $g \cdot A = gAg^{\mathsf{T}}/\det g$ *with stabilizer* PSO_2 *at* I*, connected. Since units of* $k[\![t]\!]$ *are squares,* $L^+\mathrm{PGL}_2 = L^+\mathrm{SL}_2/\mu_2$*, the orbits are the* $\mathcal{O}_m$ *with connected stabilizers, and* $p\colon L^+\mathrm{SL}_2\backslash LX \to L^+\mathrm{PGL}_2\backslash LX$ *is a* μ_2*-gerbe. In characteristic* 0*,* $p_*p^* = (-)\otimes H^*(B\mu_2, k) = \mathrm{id}$*, so* p^* *is fully faithful with essential image the objects on which the* μ_2*-inertia acts trivially, namely* $\mathcal{C}_K^+$*.*

Here $X = \mathrm{PGL}_2/\mathrm{PSO}_2$, the double cover of the pages' row $\mathrm{PGL}_2/\mathrm{PO}_2$. Steps 5–6 do not use Lemma 3.3.

4. The Satake span misses the odd part

Lemma 4.1 (Step 5)**.** $\mathrm{IC}_0 \in \mathcal{C}_K^+$*; every* $V \in \mathcal{H}$ *has trivial inertia;* $\varepsilon_{V*F} = \varepsilon_V * \varepsilon_F$*. Hence* $\mathcal{H} * \mathcal{C}_K^+ \subseteq \mathcal{C}_K^+$ *and* $\mathcal{C}_K^{\mathrm{Sat}} \subseteq \mathcal{C}_K^+$*.*

Proof. $\underline{k}_{\mathcal{O}_0}$ is pulled back from a point and a_0 is equivariant, so $\mathrm{IC}_0 \in \mathcal{C}_K^+$ by (E3)–(E4). The stabilizers $L^+G \cap t^\lambda L^+G t^{-\lambda}$ of the orbits in Gr are connected [Zhu, §5.1], so every equivariant local system on an orbit of Gr has $\varepsilon = \mathrm{id}$ by (E5); splitting $V = V^+ \oplus V^-$ as in Lemma 3.2, the cohomology sheaves of V^- on orbits are equivariant local systems with $\varepsilon = -1$ (E2–E3), hence 0, so $V^- = 0$. On $LG \times^{L^+G} LX$, $(-I)[g,x] = [g(-I),x] = [g,(-I)x] = [g,x]$; tracing this through the descent defining $V \mathbin{\widetilde{\boxtimes}} F$ [Zhu, §A.1.2], the left inertia acts on the stalk $V_g \otimes F_x$ by $\varepsilon_V \otimes \varepsilon_F$ (the right inertia of V, pulled back from Gr, is trivial). The map m is L^+G-equivariant, so $m_!$ commutes with ε (E3). As $\mathcal{C}_K^+$ is thick and contains every $V_\lambda * \mathrm{IC}_0$, it contains $\mathcal{C}_K^{\mathrm{Sat}}$. □

Theorem 4.2 (Step 6)**.** *For* $m \geq 1$*,* $\Delta_m^- := a_{m!}\mathcal{L}_m^-$ *is a nonzero object of* $\mathcal{C}_K^-$*; hence* $\Delta_m^- \notin \mathcal{C}_K^{\mathrm{Sat}}$ *and* $\mathcal{C}_K^{\mathrm{Sat}} \neq \mathcal{C}_K$*.*

Proof. Δ_m^- is bounded, constructible, equivariant and supported on $\overline{\mathcal{O}_m}$, so it lies in $\mathcal{C}_K$; it is nonzero since $a_m^*\Delta_m^- = \mathcal{L}_m^-$. By (E5) and (E3), $\varepsilon_{\Delta_m^-} = a_{m!}(\rho(-I)) = -\mathrm{id}$, so $\Delta_m^- \in \mathcal{C}_K^-$. By Lemma 4.1 $\mathcal{C}_K^{\mathrm{Sat}} \subseteq \mathcal{C}_K^+$, and $\mathcal{C}_K^+ \cap \mathcal{C}_K^- = 0$ by Lemma 3.2. □

No weights, decomposition theorem, H1–H5 or comparison with $\overline{\mathbb{F}}_q$ enter: only $2 \neq 0$ in k.

5. The exact Satake span

Lemma 5.1 (Step 7)**.** $m_n\colon Z_n = \mathrm{Gr}^{\leq n\alpha^\vee} \mathbin{\widetilde{\times}} L^+X \to LX$ *has image* $\overline{\mathcal{O}_{2n}}$*; the fibre over a point at tree distance* m' *from the apartment has* $s_n(m') = \#\{j : m' \leq 2j \leq 2n\} + \#\{j : m' < 2j \leq 2n\}$ *points;* m_n *is finite on finite-type slice models and an isomorphism over* $\mathcal{O}_{2n}$*.*

Proof. Over $\mathbb{C}$, $X \simeq \mathrm{SL}_2/T$, and L^+G-orbits on LX are $T(F)$-orbits of tree vertices, classified by the distance d from the apartment A_T [Ca1, Prop. 7.4]. The fibre over $y = y_0T$ consists of the vertices w with $\mathrm{dist}(w, v_0) \leq 2n$, even, on the line y_0A_T; the line has one vertex at distance m' from v_0 and two at each larger distance, which gives s_n, and $s_n(2n) = 1$. m_n is proper with finite fibres, hence finite on the slice models $Z_S \to S^\mu \cap \overline{\mathcal{O}_{2n}}$ of finite type [CY, Lemma 45, Prop. 47]; in characteristic 0 the generic fibre is reduced, so m_n is an isomorphism over the dense orbit $\mathcal{O}_{2n}$. □

Lemma 5.2 (Step 8)**.** *$V_{n\alpha^\vee} * \mathrm{IC}_0 \simeq \mathrm{IC}(\overline{\mathcal{O}_{2n}}) =: \mathrm{IC}_{2n}$, up to shift.*

Proof. Z_n is locally $\mathrm{Gr}^{\leq n\alpha^\vee} \times L^+X$, so $V_{n\alpha^\vee} \mathbin{\widetilde{\boxtimes}} \mathrm{IC}_0 = \mathrm{IC}(Z_n)$ and $V_{n\alpha^\vee} * \mathrm{IC}_0 = m_{n*}\mathrm{IC}(Z_n)$. Let $\nu\colon Z \to Y$ be finite and an isomorphism over a dense open $j\colon U \hookrightarrow Y$ with complement i. Finite pushforward is t-exact and commutes with $i^*, i^!$ by base change, so $i^*\nu_*\mathrm{IC}_Z$ is in perverse degrees ≤ -1 and $i^!\nu_*\mathrm{IC}_Z$ in degrees ≥ 1; by [BBD, Cor. 1.4.24], $\nu_*\mathrm{IC}_Z = j_{!*}(\mathrm{IC}_Y|_U) = \mathrm{IC}_Y$. Apply this with $\nu = m_n$, componentwise on the slice models (distinct components map birationally onto distinct components). □

(P) On each closed finite union of orbits, $\mathcal{C}_Z$ has a bounded perverse t-structure with finite-length heart whose simples are $\mathrm{IC}(\overline{\mathcal{O}_m}, \underline{k})$ and $\mathrm{IC}(\overline{\mathcal{O}_m}, \mathcal{L}_m^-)$ [BBD, Thm. 4.3.1] (standing input of Prop. III, step 6).

Theorem 5.3 (Step 9)**.** *Under* (P)*, part (e) holds.*

Proof. For a set Σ of simples, $\mathcal{D}_\Sigma = \{F : \text{all composition factors of all } {}^pH^i(F) \text{ lie in } \Sigma\}$ is thick and contains Σ; conversely each $F \in \mathcal{D}_\Sigma$ is an iterated extension of shifted objects of Σ. So $\mathcal{D}_\Sigma = \mathrm{Thick}(\Sigma)$, and a simple outside Σ is not in $\mathrm{Thick}(\Sigma)$. By Lemma 5.2, $\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}\{\mathrm{IC}_{2n}\}$. By (E2) ε commutes with ${}^pH^i$; subobjects and quotients of a perverse sheaf with $\varepsilon = \mathrm{id}$ have $\varepsilon = \mathrm{id}$; on a simple, $\varepsilon = \mathrm{id}$ exactly when the coefficient system is constant. So $\mathcal{C}_K^+ = \mathrm{Thick}\{\mathrm{IC}(\overline{\mathcal{O}_m}, \underline{k}) : m \geq 0\}$, and the witnesses are simples outside the respective sets. □

The gap $\mathcal{C}_K^+ \setminus \mathcal{C}_K^{\mathrm{Sat}}$ is the coverage failure $\varphi(\Lambda_T^+) = 2\Lambda_S^+$; the gap $\mathcal{C}_K \setminus \mathcal{C}_K^+$ is the failure of (H3), $Z_{G^\theta}(S) = \{\pm I\}$. The trivial-coefficient ICs on the labels $\varphi(\Lambda_T)$ are the ones in $\mathrm{Rep}(\check{G}_X)$ [CY, proof of Thm. 87]; Remark 89 there names $\mathrm{SL}_2/\mathrm{SO}_2$ as the example where the hypotheses fail.

6. Function level

Lemma 6.1 (Step 10)**.** *$X(F) = \mathrm{SL}_2(F)/T(F) = \mathrm{GL}_2(F)/T_{\mathrm{GL}}(F)$, $K_{\mathrm{GL}}\backslash X(F) = \{d \geq 0\}$, and on $C_{K_{\mathrm{GL}}}$*

$$T_1\mathbf{1}_0 = 2\,\mathbf{1}_0 + \mathbf{1}_1, \qquad T_1\mathbf{1}_1 = (q-1)\mathbf{1}_0 + \mathbf{1}_2, \qquad T_1\mathbf{1}_d = q\,\mathbf{1}_{d-1} + \mathbf{1}_{d+1}\ (d \geq 2).$$

So $T_1^d\varphi_0 = \mathbf{1}_d + (\text{terms with smaller } d)$ and $p \mapsto p(T_1)\varphi_0$ is an isomorphism $\mathbb{C}[y] \to C_{K_{\mathrm{GL}}}$.

Proof. Hilbert 90 for T. $T(F)$ translates the apartment and $T(\mathcal{O})$ acts on the vertices above an apartment vertex by $x \mapsto a_1a_2^{-1}x$ [Ca1, Prop. 7.4]; T_1 is the adjacency operator, and a vertex on the apartment has two neighbours on it and $q-1$ at distance 1, a vertex at distance $d \geq 1$ has one neighbour nearer and q farther. □

Theorem 6.2 (Step 11)**.** *Part (f) holds, for every q and every F.*

Proof. $\mathcal{H}_K = \mathbb{C}[S]$ with $S = \mathbf{1}_{K\,\mathrm{diag}(\varpi,\varpi^{-1})K}$ [Ca2, Cor. 1.9]. The diagonal units normalize K and commute with $t = \mathrm{diag}(\varpi, \varpi^{-1})$, and $K_{\mathrm{GL}} \cap \mathrm{SL}_2(F) = K$, so $KtK/K \to K_{\mathrm{GL}}tK_{\mathrm{GL}}/K_{\mathrm{GL}}$ is a bijection; the centre acts trivially on X, so on $C_{K_{\mathrm{GL}}}$ the operator S is $T_{(2,0)} = T_1^2 - (q+1)$ [Ca2, Lemma 1.6]. Hence $\mathcal{H}_K\varphi_0 = \mathbb{C}[T_1^2]\varphi_0$, which corresponds to $\mathbb{C}[y^2]$ under Lemma 6.1, while $\mathbf{1}_1 = (T_1 - 2)\varphi_0$ corresponds to $y - 2 \notin \mathbb{C}[y^2]$. Equivalently, $p(T_1^2)\varphi_0$ is supported in $\{d \leq 2\deg p\}$ and its support contains $2\deg p$, so $\dim\big(\mathcal{H}_K\varphi_0 \cap C_c(\{d \leq 2N\})\big) = N + 1 < 2N + 1$. □

Lemma 6.3 (Step 12)**.** *Part (g) holds.*

Proof. For $d(x) \geq 1$ the K_{GL}-stabilizer of x has determinants $\mathcal{O}^{\times 2}(1+\varpi^d\mathcal{O}) = \mathcal{O}^{\times 2}$ (q odd), so $\{d\}$ is two K-orbits; $\{d=0\} = K \cdot x_0$ since $T(\mathcal{O})$ has all determinants. s normalizes K, T and every $Kt^{\lambda}K$ and fixes x_0, so σ commutes with $\mathcal{H}_K$ and fixes φ_0; it exchanges the two K-orbits because $K_{\mathrm{GL}} = K \cdot \{\mathrm{diag}(u,1)\}$ (Proposition J(2) of the pages computes $d=1$). □

Remark 6.4 (Trace of ε). Over $\mathbb{F}_{q^n}(\!(t)\!)$, q odd, σ_n is the action of $h = \mathrm{diag}(\sqrt{u}, \sqrt{u}^{-1}) \in L^+\mathrm{SL}_2(\overline{\mathbb{F}}_q)$, whose Lang cocycle is $h^{-1}\mathrm{Fr}^n(h) = -I$. Hence, under D0–D3 with disconnected stabilizers, $\mathrm{tr}_n(\mathrm{IC}_{\chi,\nu}) \circ \sigma_n = \chi(-1)\,\mathrm{tr}_n(\mathrm{IC}_{\chi,\nu})$: the geometric isomorphism $h^*M \simeq M$ gives a second Weil structure, $\mathrm{End}\,\mathrm{IC}_{\chi,\nu}$ is scalar, and the scalar is read at x_ν. So $\mathcal{C}_K^+$ and $\mathcal{C}_K^-$ trace to the σ-even and σ-odd functions. This repairs the run's first ℓ-adic proof, whose step asserting the σ-invariance of $\mathrm{tr}_n\,\mathrm{IC}_{\mathrm{triv},\mu}$ restated its claim.

7. Corrections carried from the audit

- For GL_2/T, $V_{(1,0)} * \mathrm{IC}_0 \simeq \mathrm{IC}(\overline{\mathcal{O}_1})$ is simple: the convolution map is finite, two-to-one over $\mathcal{O}_0$ and an isomorphism over $\mathcal{O}_1$. The run's remark that IC_0 is a summand of it is false; nothing here uses it.
- "$p(T_1^2)\varphi_0$ has support exactly $\{0,\dots,2\deg p\}$" is false, e.g. $(T_1^2-(q+3))\varphi_0 = 2\,\mathbf{1}_1 + \mathbf{1}_2$; Theorem 6.2 uses only that the support contains $2\deg p$.
- The ℓ-adic proof of non-generation by traces has a circular step; Remark 6.4 supplies it, and Theorem 4.2 does not depend on it.

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