PROPOSITION VI: SAKELLARIDIS'S THEOREM IN BOTH CHARACTERISTICS

Part (a) states Theorem 8.0.2 and Corollary 8.0.4 of Sakellaridis [Sak] with every hypothesis and outlines his proof; it quotes [Sak], arXiv:0905.4244v3, whose printed and PDF page numbers agree, and re-proves nothing. Parts (b)–(d) are proved by the run *sakellaridis-char-p* [Run]: (b) moves the theorem to $\mathbb{F}_q(\!(t)\!)$ for symmetric varieties by comparison with the totally ramified fields $\mathbb{Q}_q(p^{1/e})$, (c) verifies every hypothesis for E_6/F_4 in both characteristics, including Axiom 2.2.1, and (d) is the resulting criterion. Node labels such as A4-5 refer to the run's audited nodes; S4-n to the earlier salvage audit.

1. Setting and invariants

Field and group. F is a non-archimedean local field (the paper's k), of characteristic 0 in part (a) and equal to $\mathbb{F}_q(\!(t)\!)$ in part (b), with ring of integers $\mathcal{O}$ (the paper's $\mathfrak{o}$), uniformizer ϖ and residue field $\mathbb{F}_q$. G is a reductive group over $\mathcal{O}$, $B \supseteq A$ a Borel subgroup and a maximal split torus, U the unipotent radical of B, W the Weyl group, $K = G(\mathcal{O})$, and $J \subseteq K$ the Iwahori subgroup, the inverse image of $B(\mathbb{F}_q)$. As in the paper, G, B, A also denote their groups of F-points.

The variety. X is a G-scheme over $\mathcal{O}$ with a right action, an open B_F-orbit $\mathring{X}_F$, and $\mathring{X} := X \smallsetminus \overline{X \smallsetminus \mathring{X}_F}$. A point $x_0 \in \mathring{X}(\mathcal{O})$ is fixed, H is its stabilizer and $X = H\backslash G$ [Sak, §2.1, p. 8]. The other pages use left actions, $X = G/G^\theta$; read $x \cdot g$ as $g^{-1}x$, so that $xJ \subseteq xB(\mathcal{O})$ becomes $Jx \subseteq B(\mathcal{O})x$.

Invariants [Sak, §1.7, p. 7; §2.1–2.2, pp. 8–9].

- $P(X) = \{p \in G : \mathring{X}p = \mathring{X}\}$, a standard parabolic; $L(X)$ a Levi subgroup of $P(X)$ with $[L(X), L(X)] \subseteq H$; $A \subseteq B \cap L(X)$; $\rho_{P(X)}$ the half-sum of the roots of the unipotent radical of $P(X)$, $\delta_{P(X)}$ its modular character.
- $A_X := L(X)/(L(X) \cap H)$, a torus and a quotient of A, identified with the $L(X)$-orbit of x_0.
- $\mathcal{X}(X)$ is the group of characters of the B-eigenfunctions in $F(X)$; $\Lambda_X = X_*(A_X) = \mathrm{Hom}(\mathcal{X}(X), \mathbb{Z})$; Λ_X^+ is the monoid of G-invariant $\mathbb{Z}$-valued valuations of $F(X)$ trivial on $F^\times$, restricted to B-eigenfunctions, and $\mathcal{V}$ its cone in $\mathrm{Hom}(\mathcal{X}(X), \mathbb{Q})$. Through $\Lambda_X \simeq A_X(F)/A_X(\mathcal{O})$, $\check{\lambda} \mapsto e^{\check{\lambda}}(\varpi)$, the preimage of Λ_X^+ is the set A_X^+ of X-anti-dominant points; $x_{\check{\lambda}} \in A_X^+$ denotes a point of label $\check{\lambda}$.
- W_X, the little Weyl group: Knop's action of W on the $B_{\bar{F}}$-orbits of maximal rank is transitive, and the stabilizer of $\mathring{X}$ is $W(X) = W_X \ltimes W_{P(X)}$. W_X acts faithfully on $\mathcal{X}(X) \otimes \mathbb{Q}$ by reflections, with fundamental domain $\mathcal{V}$; it is the Weyl group of a root system Φ_X whose simple roots are the normalized spherical roots Δ_X [Sak, pp. 41–42].
- $d(X)$ is the modular character of $L(X) \cap B$ and $\delta(X) = |d(X)|$. A^* and A_X^* are the complex tori of unramified characters of A and A_X; $A \twoheadrightarrow A_X$ gives $A_X^* \to A^*$, whose image is again written A_X^*. $\delta(X)^{1/2}$ is a point of A^*, and $\delta(X)^{1/2}A_X^* \subseteq A^*$ is the translate, with W_X acting by transport from A_X^* [Sak, p. 3].

Hecke algebra, restriction map, basic function. $\mathcal{H}(G,K)$ is the algebra of compactly supported K-bi-invariant functions on $G(F)$ under convolution, and $\mathcal{S}:\mathcal{H}(G,K)\xrightarrow{\sim}\mathbb{C}[A^*]^W$ the Satake isomorphism, so $\operatorname{specM}\mathcal{H}(G,K)=A^*/W$. The restriction map (8.2) of [Sak, p. 65] and the algebra $\mathcal{H}_X$ [Sak, p. 63] are

$$r:\mathbb{C}[A^*]^W\longrightarrow\mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X},\quad f\longmapsto f|_{\delta(X)^{1/2}A_X^*},\qquad \mathcal{H}_X:=\operatorname{im}\big(r\circ\mathcal{S}\big).$$

$C_c^\infty(X)^K$ is the space of K-invariant, compactly supported, locally constant functions on $X(F)$, a module over $\mathcal{H}(G,K)$; the basic function is $\varphi_0:=\mathbf{1}_{X(\mathcal{O})}$, and spherical generation is

$$\text{(SG)}\qquad \mathcal{H}(G,K)\cdot\mathbf{1}_{X(\mathcal{O})}=C_c^\infty(X)^K.$$

Spherical roots and colours [Sak, pp. 9, 41–44, 58–60]. For a $B_{\bar F}$-orbit Y of maximal rank and a simple root α, $(YP_\alpha/R(P_\alpha))_{\bar F}$ is a homogeneous spherical variety for PGL_2 of type G ($\mathrm{PGL}_2\setminus\mathrm{PGL}_2$), U ($SU\setminus\mathrm{PGL}_2$, U maximal unipotent, $S\subseteq N(U)$), T ($T\setminus\mathrm{PGL}_2$, T a non-trivial torus) or N ($N(T)\setminus\mathrm{PGL}_2$). The spherical roots generate $\mathcal{X}(X)$ on the extremal rays of the cone $\{\chi:\langle v,\chi\rangle\le 0\ \forall v\in\mathcal{V}\}$; the normalized spherical root γ is the positive multiple that is a root of G or a sum of two orthogonal roots simple for some basis; Δ_X is their set, w_γ the simple reflection of W_X, and L_γ the standard Levi subgroup of $\operatorname{supp}\gamma$. γ has type G if it is not proportional to a root of G, and otherwise type T or N according as the proportional root lies in $\mathcal{X}(X)$ or not. A Brion path for w_γ is a reduced path of B-orbits realizing w_γ (p. 43). The cocycles $B_w:W_X\to\mathbb{C}(A_X^*)$ are defined in (6.1) from the functional equations of the Mackey morphisms. Colours are the $B_{\bar F}$-stable prime divisors D of $X_{\bar F}$, with valuations $\check v_D\in\operatorname{Hom}(\mathcal{X}(X),\mathbb{Z})$; D belongs to γ if it is the codimension-one orbit of a Brion path of w_γ. A virtual weighted colour is a triple $(\check\theta,\sigma,r)$ with $\check\theta=\check v_D$, $\sigma=+$ unless D is not defined over F, and $r=\langle\check\theta,\rho_{P(X)}\rangle$, together with the additions for type T (p. 59). Θ is the smallest W_X-stable, negation-closed multiset of triples containing them, $\mathcal{T}$ the cone spanned by the colour valuations, $\Theta^+=\{\check\theta\in\Theta:\check\theta\in\mathcal{T}\}$, and $\check\theta>0$ means $\check\theta\in\Theta^+$.

E_6/F_4 and its model $\mathcal{M}_\theta$. $G_\mathbb{Z}$ is the split simply connected Chevalley group of type E_6 with pinning $(T,B,\{x_{\alpha_i}\})$ (Bourbaki labels), C the Chevalley involution ($C(t)=t^{-1}$ on T, $C(x_{\pm\alpha_i}(c))=x_{\mp\alpha_i}(-c)$), w_L the longest element of $W(D_4)$ on the black nodes $2,3,4,5$, and σ_L its Tits lift. Over every DVR $\mathcal{O}$ with $2\in\mathcal{O}^\times$,

$$\theta=\theta_0:=\operatorname{Int}(\sigma_L)\circ C,\qquad \theta|_T=-w_L,$$
$$X_\mathcal{O}:=\big(G_\mathbb{Z}/G_\mathbb{Z}^{\theta_0}\big)\otimes\mathcal{O},\qquad H_\mathcal{O}:=G_\mathcal{O}^{\theta_0},\qquad x_0=eH;$$

this is the model $\mathcal{M}_\theta$. θ_0 is an involution of type EIV defined over $\mathbb{Z}$, and the standard parabolic $P=U_P\rtimes L$ of the black nodes is minimal θ_0-split, with $L=Z_G(S)$ and $S=(T^{-\theta_0})^\circ$ of rank 2 (S4-11). With $\varphi=1-\theta$ and $e_i=\varphi(\alpha_i^\vee)$,

$$\Lambda_X=\varphi(\Lambda_T)=\Lambda_S=\mathbb{Z}e_1\oplus\mathbb{Z}e_6,\qquad W_X=S_3,\qquad \langle\rho_L,\alpha_j^\vee\rangle_{j=1,\dots,6}=(-3,1,1,1,1,-3),$$

ρ_L the half-sum of the positive roots of the black D_4; the restricted root system is A_2 with all multiplicities 8. In the coordinates $u=y(e_1)$, $v=y(e_6)$ of A_X^* put $y_1=(q^{-5},q)$, and let σ be the diagram automorphism $1\leftrightarrow 6$, $3\leftrightarrow 5$.

Close fields and symmetric varieties. $\Lambda=W(\mathbb{F}_q)$ and $\mathbb{Q}_q=\Lambda[1/p]$. The test fields are $F=\mathbb{F}_q(\!(t)\!)$, with $\mathcal{O}=\mathbb{F}_q[\![t]\!]$, and $F_e'=\mathbb{Q}_q(p^{1/e})$, with $\mathcal{O}_e'=\Lambda[\varpi']$, $\varpi'=p^{1/e}$. Since $p=\varpi'^e$, $\mathcal{O}_e'/\varpi'^m\cong\mathbb{F}_q[t]/(t^m)$ for $m\le e$: the fields are e-close [Del, Kaz]. Over a test field E, $K=G(\mathcal{O}_E)$ and $t^\lambda=\lambda(\varpi_E)$; $c\preceq d$ means $\langle d-c,\omega_j\rangle\ge 0$ for all j. For the transfer, G is split

reductive over Λ with connected fibres and simply connected derived group, θ an involution of G_Λ stabilizing a split maximal torus T that contains a maximal θ-split torus S (split over Λ), $H = G^\theta$, $X = G/H$ with left action, $x_0 = eH$, and

$$\tau(gx_0) = g\theta(g)^{-1}, \qquad \Lambda_S^+ = X_*(S) \cap \Lambda_T^+, \qquad x_\kappa = \kappa(\sqrt{\varpi})\, x_0, \quad \tau(x_\kappa) = t^\kappa;$$

x_κ is rational since $S[2] \subseteq H$ and p is odd, and for E_6/F_4 it equals $t^\mu x_0$ with $(1-\theta)\mu = \kappa$. $T_\lambda = \mathbf{1}_{Kt^\lambda K}$ for $\lambda \in \Lambda_T^+$.

2. The hypotheses

(A1) Field and group. [Sak, §1.1, p. 2; §1.7, p. 6]: F is a local non-archimedean field of characteristic zero; "to avoid complicated formulas" F is assumed unramified over $\mathbb{Q}_p$, "the modifications needed to remove this assumption are trivial". G is reductive, split and with connected fibres over $\operatorname{Spec}\mathcal{O}$, with simply connected derived group; $B \supseteq A$ are fixed. The simplification only fixes (ψ, dx), which §1.7 already fixes with conductor $\mathcal{O}$ (Step 11). For E_6/F_4 it holds for every odd p and every p-adic F.
(A1$_p$) Field and group in characteristic p. $F = \mathbb{F}_q((t))$, and G is as in (A1) over $\mathcal{O} = \mathbb{F}_q[\![t]\!]$. Not in the paper, which works in characteristic 0 ([Sak, §1.1], [Sa08, §1.7]). For E_6/F_4 it holds for every odd p (A3-2).
(A2) The variety over $\mathcal{O}$. [Sak, §1.7, p. 7; §1.1, p. 2; §2.1, p. 8]: X is a quasi-affine G-scheme over $\mathcal{O}$ with an open B_F-orbit $\mathring{X}_F$ on X_F such that the complement $\mathring{X}$ of the closure in X of the complement of $\mathring{X}_F$ is smooth and surjective over $\operatorname{Spec}\mathcal{O}$, and its special fibre is homogeneous under the special fibre of B; X admits a G-eigenmeasure and $\mathring{X}$ a B-invariant measure. X is homogeneous, $x_0 \in \mathring{X}(\mathcal{O})$ and $X = H\backslash G$. The model fixes $K\backslash X(F)$ and $\mathbf{1}_{X(\mathcal{O})}$. For $\mathcal{M}_\theta$ it holds over every complete DVR with 2 invertible (S4-11, S4-4; A3-2; Step 14).
(A3a) Rank-one models over $\mathcal{O}$ (Axiom 2.2.1). [Sak, p. 9]: for every B_F-orbit $Y_F \subseteq X_F$ of maximal rank and every simple root α, the $\mathcal{O}$-scheme $YP_\alpha/R(P_\alpha)$ is isomorphic to one of $\mathrm{PGL}_2 \setminus \mathrm{PGL}_2$, $SU\backslash \mathrm{PGL}_2$ ($S \subseteq N(U)$), $T\backslash \mathrm{PGL}_2$ (T a non-trivial torus over $\mathcal{O}$) or $N(T)\backslash \mathrm{PGL}_2$; moreover the complement of YP_α is the closure of the complement of $(YP_\alpha)_F$. Here Y is the complement of the closure of the complement of Y_F in its closure. Both complements are read inside the closure of $(YP_\alpha)_F$; read inside X the clause fails for every non-open saturation (A6-7). For $\mathcal{M}_\theta$ it holds over every DVR with 2 invertible (A2-8, audited A6-9; Step 17); before the run it was the open item (I).
(A3b) Cartan decomposition (Axiom 2.4.1). [Sak, p. 10]: the set A_X^+ contains a complete set of representatives for the K-orbits on X; elements of A_X^+ which map to distinct elements of Λ_X^+ belong to different K-orbits. For $\mathcal{M}_\theta$, p odd, it holds in both characteristics (Steps 9 and 16).
(A3c) Iwahori orbits (Axiom 2.4.2). [Sak, p. 11]: for $x \in A_X^+$, $xJ \subseteq xB(\mathcal{O})$. The three axioms hold at almost every place when X is defined over a global field (pp. 7, 9, 11). For $\mathcal{M}_\theta$, p odd, it holds in both characteristics (Step 15).
(A4) Spherical roots and colours. [Sak, §6, p. 41; §7, p. 58; pp. 44–45, 60]. (i) No type N (opening of §6, p. 41): for no orbit Y of maximal rank and simple root α is (Y, α) of type N. This is stronger in form than "no spherical root of type N", the wording of Theorem 6.3.2 and of §7 (p. 58). The same opening assumes that the character $\delta(X)^{-1/2}\tilde{\chi}$ of A_X is unramified, a condition on the characters used rather than on X. (ii) For every $\gamma \in \Delta_X$ the Levi subgroup

L_γ is a classical group. (iii) *Statement 6.3.1*: for every $\gamma \in \Delta_X$ and every Brion path of w_γ, with $\check\theta$ the valuation of the codimension-one orbit of the path: if γ has type G then $\check\theta = \check\gamma$ and

$$B_{w_\gamma}(\tilde\chi) = -e^{\check\gamma}\,\frac{1 - q^{-\langle\check\gamma,\rho_{P(X)}\rangle}e^{-\check\gamma}}{1 - q^{-\langle\check\gamma,\rho_{P(X)}\rangle}e^{\check\gamma}}(\chi);$$

if γ has type T split then $\langle\check\theta,\gamma\rangle = 1$ and (6.4) holds; if type T non-split then $\check\theta = \check\gamma/2$ and (6.5) holds; type (U,ψ), (6.6), concerns twisted spaces only. (iv) *Statement 7.1.5*: every $(\check\theta,\sigma,r) \in \Theta$ has $\check\theta > 0$ or $\check\theta < 0$, and for every $\gamma \in \Delta_X$ the set $\{(\check\theta,\sigma,r) \in \Theta^+ : w_\gamma\check\theta < 0\}$ consists precisely of the virtual weighted colours belonging to γ. By Theorem 6.3.2 (p. 45), (i) and (ii) imply (iii); the hypotheses of Theorem 7.2.1 proper are (iii) and (iv) (p. 60), and (i), (ii) make the types of colours and Θ well defined (Lemma 7.1.4, p. 59). Sakellaridis conjectures (iv) for affine homogeneous X and calls it very easy to check in each case; for affine X every colour is relevant (Lemma 7.1.1, p. 58). In plain words: these compute the cocycles of the simple spherical reflections and package all of them into one product of L-factor shapes. For E_6/F_4, (i), (ii), (iv) hold in every characteristic $\neq 2$ and (iii) in characteristic 0 (Step 13).
(A5) Homogeneous, affine, one open orbit. [Sak, §8, p. 63]: "From now on we let X be a homogeneous, affine spherical variety which satisfies the assumptions of Theorem 7.2.1. Assume also that $\mathring{X}$ has a unique B-orbit. In particular, the strong form of the Cartan decomposition holds: $X/K \simeq \Lambda_X^+$ and $A_X^* \subset A^*$." So H is reductive and $\mathring{X}(F)$ is a single $B(F)$-orbit; it fails for SL_2/T. For E_6/F_4 it holds for every odd p, in both characteristics (S4-4, item 7; A3-3).
The earlier summary. The labels group the paper's standing assumptions: (A1) is §1.1 and the group part of §1.7, (A2) the $\mathcal{O}$-scheme of §1.7, (A3a)–(A3c) the three axioms (formerly one label (A3); (I) was the earlier name of (A3a)), (A4) items (i)–(iv), (A5) §8. The audit S4-2 recorded six points lost by the earlier summary, restored above: the simplification "$F/\mathbb{Q}_p$ unramified"; the $\mathcal{O}$-model; the second clause of Axiom 2.2.1 and the $\mathcal{O}$-structure on Y; that Statement 6.3.1 follows from (i) and (ii); that "(SG) iff r onto" combines Theorem 8.0.2 with Corollary 8.0.4(1); and that the hypotheses are now verified for E_6/F_4.
(T1)–(T5) The hypotheses of the transfer. For a symmetric $X = G/G^\theta$ over Λ, p odd, as in Section 1, with test fields $F = \mathbb{F}_q(\!(t)\!)$ and F'_e (A4-9, audited A5-7):

- (T1) H has connected fibres and is split over Λ;
- (T2) $X(E) = G(E)x_0$ for $E = F$ and for $E = F'_e$, $e \gg 0$;
- (T3) $X(\mathcal{O}_E) = \{y \in X(E) : \tau(y) \in G(\mathcal{O}_E)\}$ for these E;
- (T4) $X(F'_e) = \bigsqcup_{\kappa\in\Lambda_S^+} Kx_\kappa$ for $e \gg 0$;
- (T5) Theorem 8.0.2 holds over F'_e for $e \gg 0$ (for instance under (A1)–(A5) over $\mathcal{O}'_e$, with ψ' of conductor $\mathcal{O}'_e$), with Sakellaridis's label of Kx_κ independent of e.

(T4) and (T5) live at p-adic places; in characteristic p only (T1)–(T3), Cartan and Iwasawa decompositions, Hensel and Lang are used.

3. Statement

Proposition VI (Sakellaridis's theorem in both characteristics)**.** *Let p be odd, and let F, G, X, x_0 be as in Section 1.*

(a) ***Characteristic*** 0 [Sak, Thm. 8.0.2, Cor. 8.0.4]. *Let* $\operatorname{char} F = 0$ *and assume* (A1)–(A5). *There is a canonical isomorphism of* $\mathcal{H}(G,K)$*-modules*

$$C_c^\infty(X)^K \xrightarrow{\sim} \mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}, \qquad \mathbf{1}_{X(\mathcal{O})} \longmapsto 1,$$

under which $f \in \mathcal{H}(G,K)$ *acts by multiplication by* $r(\mathcal{S}f)$*; hence* $\mathcal{H}(G,K)\cdot\mathbf{1}_{X(\mathcal{O})} = C_c^\infty(X)^K$ *if and only if* r *is onto. If* r *is onto, Corollary 8.0.4 adds multiplicity one; by Example 8.0.1,* r *need not be onto. This holds for* F *ramified over* $\mathbb{Q}_p$*: the unramified simplification only fixes* (ψ, dx)*, and the results taken from* [Sa08] *are proved there for every local field of characteristic* 0*. Status: proved, cited.*

(b) ***Characteristic*** p***.*** *Let* $F = \mathbb{F}_q(\!(t)\!)$ *and let* $X = G/G^\theta$ *be symmetric over* $W(\mathbb{F}_q)$ *with* (T1)–(T5)*. Then* $K\backslash X(F) \simeq \Lambda_S^+$*,* $\kappa \mapsto Kx_\kappa$*, and the conclusion of (a) holds:* $C_c^\infty(X)^K \simeq \mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}$ *compatibly with* $r\circ\mathcal{S}$*, with* $\mathbf{1}_{Kx_\kappa} \mapsto P_\kappa$*, the image of* $\mathbf{1}_{Kx_\kappa}$ *over* F'_e*,* $e \gg 0$*, which depends only on* q*, and* $\mathbf{1}_{X(\mathcal{O})} \mapsto 1$*; hence (SG) holds iff* r *is onto. Status: proved by the run, audited (A4-9 with A4-5, A4-6, A4-8; A5-3, A5-4, A5-6, A5-7).*

(c) E_6/F_4 ***satisfies the hypotheses.*** *For* $\mathcal{M}_\theta$ *and every DVR* $\mathcal{O}$ *with* $2 \in \mathcal{O}^\times$ *(every complete DVR of odd residue characteristic, mixed or equal characteristic,* $p = 3$ *included): (A3a) holds;* (A1)–(A5) *hold over every p-adic* $\mathcal{O}$*, and* (A1$_p$)*,* (A2)*, (A3b), (A3c),* (A4)*(i), (ii), (iv) and* (A5) *hold over* $\mathbb{F}_q[\![t]\!]$*;* (T1)–(T5) *hold. Status: proved by the run, audited (A2-8, A6-9; A3, A5-8 to A5-12; A4-9).*

(d) ***The criterion for*** E_6/F_4***.*** *For* $\mathcal{M}_\theta$*, over* $\mathbb{F}_q(\!(t)\!)$ *and over every p-adic* F *of any ramification,* $C_c^\infty(X)^K \simeq \mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}$ *with* $\varphi_0 \mapsto 1$*, so (SG) holds iff* r *is onto. Status: proved, from (a)–(c). Since* r *is not onto (Proposition VII), (SG) fails and* $C_K = \mathcal{H}_K\varphi_0 \oplus \mathbb{C}\,\mathbf{1}_{Kx_{3\omega_1^\vee}}$*.*

Remark 3.1 (two gaps in the printed proof, both closed). Sakellaridis derives the inequality (7.5) from $|\Theta^+| - w|\Theta^+| = \check\rho - w\check\rho$, which is false by a factor 2 when $\Theta^+ = \check\Phi_X^+$ with multiplicity one. (7.5) holds whenever the virtual colours of each simple spherical root sum to its coroot (S1-1), which his Lemma 7.1.4 gives in characteristic 0; with a doubled coroot, which that lemma excludes, the printed conclusion fails (A5-20). His last step also needs the lowest coefficient of each $P_{\check\lambda}$ to be non-zero, which the paper does not prove; positivity of the open-orbit integrals gives it (S1-3). So (a) holds as stated (S1-4); parts (c) and (d) are not affected.

Remark 3.2 (the direct form of (b)). Rerunning Sakellaridis's proof over $\mathbb{F}_q(\!(t)\!)$ (A1-9, A5-20; complete list in S1-5) gives the conclusion of (b) for symmetric X under (A2), (A5), (A3b) with anti-dominant labels, (A3c), Statements 6.3.1 and 7.1.5 read in characteristic p, $\operatorname{cone}(\Theta^+) \cap C^- = \{0\}$, and three hypotheses the audits found tacit: (H1) the virtual colours of each simple spherical root sum to its coroot, equivalently (7.5) for the actual Θ of X (S1-1); (H2) F-rational adapted data $T \supseteq S$, B; (H3) no spherical root of type N. The non-vanishing of the leading coefficients is automatic (S1-3), and (A3a) is not needed in this form.

(a) char 0 + (A1)–(A5), (b) $\mathbb{F}_q(\!(t)\!)$ + (T1)–(T5) :
$C_c^\infty(X)^K \simeq \mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}$, (SG) $\iff$ r onto;
(c), (d) E_6/F_4, $\mathcal{M}_\theta$, both characteristics :
the hypotheses hold, and (SG) fails

4. Part (a): how Sakellaridis proves it

Step 1 (Mackey morphisms). For an unramified character χ of A let $I(\chi) = \mathrm{Ind}_B^G(\chi\delta^{1/2})$. Mackey theory on the open orbit gives morphisms $S_\chi : C_c^\infty(X) \to I(\chi)$ and adjoints Δ_χ (§3); with $\Phi_K \in I(\chi)^K$ (Proposition 4.2.1, p. 26) the Hecke eigenfunctions are $\Omega_{\tilde\chi} = \Delta_{\tilde\chi}(\Phi_K)$. The eigenmeasures are normalized integrally on every orbit of maximal rank (§3.3, using (A2) and (A3a)) and by $\mu(x_0J) = 1$, which uses $x_0J \subseteq x_0B(\mathcal{O})$ ((A3c), p. 17).
Step 2 (sum formula). Theorem 4.2.2 (p. 26), in the form (6.2) (p. 44): for almost every $\tilde\chi$ and $x \in A_X^+$,

$$\Omega_{\delta(X)^{1/2}\tilde\chi}(x) = \delta_{P(X)}^{-1/2}(x) \sum_{w\in W_X} B_w(\tilde\chi)\, {}^w\tilde\chi(x).$$

(A3c) enters through Lemma 3.9.1 (p. 24), and (A3b) makes A_X^+ meet every K-orbit, so these values determine K-invariant functions. The functional equations behind B_w are [Sa08, Thm. 5.2.1], quoted as Theorem 3.7.1 (p. 21); [Sa08] proves them over every local field of characteristic 0 (§1.7, p. 7; §4.1, p. 17).
Step 3 (simple spherical reflections). Statement 6.3.1 evaluates B_{w_γ} for $\gamma \in \Delta_X$. Theorem 6.3.2 (p. 45) proves it under (A4)(i), (ii), by reducing to the spherical varieties of rank at most two of Theorem 6.11.1 (pp. 51–52) and computing them for classical groups.
Step 4 (formula with L-values). Theorem 7.2.1 (pp. 60–62): under Statements 6.3.1 and 7.1.5 the cocycle is a coboundary,

$$B_w(\tilde\chi) = \frac{\beta(\tilde\chi)}{\beta({}^w\tilde\chi)}, \qquad \beta(\tilde\chi) = \frac{\prod_{\check\gamma\in\check\Phi_X^+}(1-e^{\tilde\gamma})}{\prod_{\check\theta\in\Theta^+}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{\check\theta})}(\tilde\chi),$$

so $\Omega_{\delta(X)^{1/2}\tilde\chi}(x_{\check\lambda})/\beta(\tilde\chi) = \delta_{P(X)}^{-1/2}(x_{\check\lambda})\prod_{\Theta^+}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}T_{\check\theta})s_{\check\lambda}(\tilde\chi)$ (7.4), with $s_{\check\lambda}$ the Schur polynomial of lowest weight $\check\lambda$ and $T_{\check\theta}s_{\check\lambda} = s_{\check\theta+\check\lambda}$. For affine X the value at x_0 is $c\,\beta$, $c = \beta(\delta_{P(X)}^{1/2})^{-1}$ (7.3). Affineness enters through Knop's criterion [Sak, p. 61]: X is affine iff $\mathcal{V}$ and the colour valuations are separated by a hyperplane, whence $\mathcal{T}\cap\mathcal{V} = \{0\}$ (p. 62). The proof ends with the inequality (7.5), $\check\rho - w\check\rho + w|I| \succ_2 0$ for $w \in W_X$, $I \subseteq \Theta^+$, derived from $|\Theta^+| - w|\Theta^+| = \check\rho - w\check\rho$; this identity is false by a factor 2 when $\Theta^+ = \check\Phi_X^+$ with multiplicity one. When the virtual colours of each simple spherical root sum to its coroot, the correct identity is $|\Theta^+| - w|\Theta^+| = 2(\check\rho - w\check\rho)$ and (7.5) holds with $\mathcal{T} = \mathrm{cone}(\Theta^+)$ (S1-1); in characteristic 0 that is Lemma 7.1.4. For E_6/F_4 an inversion-set argument gives the same (A1-7, A5-17). With a doubled coroot the printed conclusion fails (A5-20).
Step 5 (Theorem 8.0.2). Proof on p. 64. Put $P_{\check\lambda}(\chi\delta(X)^{1/2}) = \Omega_{\chi\delta(X)^{1/2}}(x_{\check\lambda})/L_X^{1/2}(\chi)$ (8.1), where $L_X^{1/2} = c\beta$ (Definition 7.2.3). Then $P_0 = 1$, $P_{\check\lambda}$ is c^{-1} times the right side of (7.4), and $\mathbf{1}_{x_{\check\lambda}K} \mapsto P_{\check\lambda}$ is $\mathcal{H}(G,K)$-equivariant for the Satake isomorphism. By the Remark after Theorem 7.2.1 (p. 62) the other weights are $\succ_2 \check\lambda$, where $\check\lambda \succ_2 \check\mu$ means $\check\lambda - \check\mu \in \mathcal{T}$; on

p. 64 $\check\lambda$ is the lowest weight of $P_{\check\lambda}$ (the Remark rests on (7.5), see Step 4). The Remark says nothing about the coefficient of $e^{\check\lambda}$, which must be non-zero; the paper does not prove this, and comparing coefficients of the open-orbit integrals, which have non-negative terms, gives it (S1-3, S1-4). Induction, started by $\mathcal{T}\cap\mathcal{V}=\{0\}$, shows that the $P_{\check\lambda}$ generate $\mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}$; the indicators $\mathbf{1}_{x_{\check\lambda}K}$ form a basis of $C_c^\infty(X)^K$ by $X/K\simeq\Lambda_X^+$ (A5). The conclusion of the theorem is the module isomorphism with $\mathbf{1}_{X(\mathcal{O})}\mapsto 1$; the image of each orbit indicator, with its lowest weight, comes from the proof.

Step 6 (the criterion). Under (a), $\mathcal{H}(G,K)$ acts by multiplication through $r\circ\mathcal{S}$ and $\mathbf{1}_{X(\mathcal{O})}\mapsto 1$, so $\mathcal{H}(G,K)\cdot\mathbf{1}_{X(\mathcal{O})}$ corresponds to $(\operatorname{im} r)\cdot 1=\operatorname{im} r$. ($\Rightarrow$) If (SG) holds, $\operatorname{im} r$ is the whole target. ($\Leftarrow$) If r is onto, the module is generated by $\mathbf{1}_{X(\mathcal{O})}$: Corollary 8.0.4(1). Parts (2) and (3) of the corollary are "straightforward" and their proof "will be omitted" (p. 65).

Step 7 (Example 8.0.1). Two points of $\delta(X)^{1/2}A_X^*$ that are not W_X-conjugate but are W-conjugate are separated by $\mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}$, since W_X is finite, while every element of $\operatorname{im} r$ takes equal values at them. In Example 8.0.1 this happens in the diagonal SO_4 (p. 64). The Remark on p. 65 records the resulting correction to [Sa08, Theorem 6.3.2], and the remark after Corollary 8.0.4 (pp. 65–66) notes that at such points the multiplicity of the fibre exceeds one although the generic multiplicity is one. The paper does not check (A1)–(A5) for this X, so "(SG) fails" there is conditional.

5. Part (b): the transfer to $\mathbb{F}_q(\!(t)\!)$

Step 8 (close fields and finite determinacy). F and F_e' are e-close. For a smooth affine Λ-scheme Y (G,H,X), $Y(\mathcal{O}_E)\to Y(\mathcal{O}_E/\varpi^m)$ is onto and $Y(\mathcal{O}_E/\varpi^m)$ depends only on the ring $\mathcal{O}_E/\varpi^m$. With V_i the Weyl module of highest weight ω_i over Λ, ρ_i the action and ℓ_i the lowest-weight coordinate,

$$g\in Kt^\nu K\iff \operatorname{val}\rho_i(g)=\langle\nu,w_0\omega_i\rangle\ \forall i,\qquad g\in U(E)t^\nu K\iff \operatorname{val}\ell_i\rho_i(g)=\langle\nu,w_0\omega_i\rangle\ \forall i.$$

So a finite Boolean combination of valuation conditions on matrix coefficients of points of $G(\mathcal{O}_E)$, $H(\mathcal{O}_E)$ is decided modulo ϖ^D, D independent of E, and counts and existence statements agree over D-close fields (A4-2; audit A5-1).

Step 9 (the Cartan decomposition over $\mathbb{F}_q(\!(t)\!)$). Under (T1)–(T4), $X(F)=\bigsqcup_{\kappa\in\Lambda_S^+}Kx_\kappa$, i.e. $G(F)=K\,T(F)\,H(F)$. For a test field E and $\lambda\in\Lambda_T^+$ put $Z_\lambda(E)=Kt^\lambda X(\mathcal{O}_E)$; $X(E)=\bigcup_\lambda Z_\lambda(E)$ by (T2). Distinct labels give distinct orbits because $K\tau(x)K$ depends only on Kx and $\tau(x_\kappa)=t^\kappa$. The number of K-orbits in $Z_\lambda(E)$ and the labels it contains are computed in $G(\mathcal{O}/\varpi^D)$, $H(\mathcal{O}/\varpi^D)$ with $D=D(\lambda)$ independent of E ((T1), (T3), Step 8). Over F_e', $e\ge D(\lambda)$, every orbit in Z_λ contains some x_κ by (T4); the counts agree over F (A4-5; audit A5-3, which notes that matching the labels with Λ_X^+ needs the valuation cone in characteristic p, A3-7). For E_6/F_4 the decomposition also follows from Delorme–Sécherre in characteristic p (A3-4, A3-5, A3-8; audit A5-9).

Step 10 (structure constants agree). For $\lambda\in\Lambda_T^+$, $\mu,\nu\in\Lambda_S^+$ put $c^\nu_{\lambda\mu}(E)=(T_\lambda*\mathbf{1}_{Kx_\mu})(x_\nu)$ and $\mathcal{S}(T_\lambda)=\sum_\nu q^{-\langle\rho,\nu\rangle}n_{\lambda\nu}(E)e^\nu$, with $n_{\lambda\nu}(E)=\#(U(E)t^\nu K\cap Kt^\lambda K)/K$. The support $\{\nu\preceq\lambda+\mu+(-\theta\lambda)^+\}$ is finite and field-independent; with the Cartan decomposition on both fields, $c^\nu_{\lambda\mu}$ counts the $\bar k\in G(\mathcal{O}/\varpi^D)$ for which $t^{-\lambda}k^{-1}t^\nu\theta(k)t^{\theta\lambda}$ has prescribed Weyl-module valuations, and $n_{\lambda\nu}$ is an Iwasawa count. Both agree over F and F_e' for $e\ge D(\lambda,\mu)$ (A4-6; audit A5-4 supplies the bound $\langle\lambda,w_0\omega_i\rangle\le\langle\nu,w_0\omega_i\rangle\le\langle\lambda,\omega_i\rangle$).

Step 11 (Theorem 8.0.2 over ramified p-adic fields). Let $F' \supseteq \mathbb{Q}_q$ have residue field $\mathbb{F}_q$, of any ramification. "Unramified" enters the proof of (a) only in §3.8 (the measure on k^2, "the correct choice only under our simplifying assumption", p. 23) and §5.3 (Tate's formulas, "we are using our simplifying assumption", p. 32), to make ψ of conductor $\mathcal{O}$ compatible with $dx(\mathcal{O}) = 1$; §1.7 already fixes ψ with conductor $\mathcal{O}$. With $\psi' = \psi_{\mathbb{Q}_p}(\mathrm{Tr}(c\,\cdot))$, $c\mathcal{O}' = \mathfrak{d}^{-1}_{F'/\mathbb{Q}_p}$, of conductor $\mathcal{O}'$ and $dx(\mathcal{O}') = 1$, every formula holds as printed (A4-7; audit A5-5). The morphisms and functional equations taken from [Sa08] are proved there for every local field of characteristic 0 ([Sa08, §1.7, §4.1]). So Theorem 8.0.2 holds over F' under (A1)–(A5), and $\mathbf{1}_{Kx_\kappa} \mapsto P_\kappa = c^{-1}\delta_{P(X)}^{-1/2}(x_\kappa)\prod_{\Theta^+}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}T_{\check\theta})s_{\check\lambda(\kappa)}$ depends only on q.
Step 12 (the characteristic-p theorem). Under (T1)–(T5), over $F = \mathbb{F}_q(\!(t)\!)$, define $\Psi(\mathbf{1}_{Kx_\kappa}) = P_\kappa$. By Step 9 and (T4) the indicators form bases over F and F'_e, and over F'_e their images form a basis of the target, so Ψ is bijective. For λ, μ take $e \geq D(\lambda,\mu)$; by Step 10 and Theorem 8.0.2 over F'_e,

$$\Psi(T_\lambda * \mathbf{1}_{Kx_\mu}) = \sum_\nu c^\nu_{\lambda\mu}(F)P_\nu = \sum_\nu c^\nu_{\lambda\mu}(F'_e)P_\nu = r(\mathcal{S}T_\lambda)P_\mu.$$

$X(\mathcal{O}) = Kx_0$ by (T2), (T3), and $P_0 = 1$. For $\mathcal{M}_\theta$ the isomorphism is unique, since $\operatorname{im} r$ has codimension one. The target is read over F'_e; identifying it with the invariants of X_F uses structure theory over $\bar F$ (A5-6), done for E_6/F_4 (A3-7, A3-10). (A4-8, A4-9; audits A5-6, A5-7.) A second route, which holds for E_6/F_4 (audited, A5-21), reruns Sakellaridis's proof over $\mathbb{F}_q(\!(t)\!)$ (A1): its open-orbit half uses characteristic-$\neq 2$ inputs only (A1-5, A1-6, A1-2, A1-7; A5-14 to A5-17), and its one remaining input, Statement 6.3.1 in characteristic p, follows from (A3a) through Theorem 8.0.2 at F'_e (A1-8; A5-19). So both routes rest on (A3a).

6. Part (c): the hypotheses for E_6/F_4

Step 13 (roots and colours). For E_6/F_4 (every characteristic $\neq 2$; statements over $\bar F$; S4-3, and A3-6, A3-7, A3-10 with audits A5-10 to A5-12): $\mathcal{X}(X) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_6$ and

$$\gamma_1 = 2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 = \alpha_1 + w_L\alpha_1 = 2\omega_1 - \omega_6,$$
$$\gamma_6 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 = -\omega_1 + 2\omega_6,$$

with $(\alpha_1, w_L\alpha_1) = 0$; $\Delta_X = \{\gamma_1, \gamma_6\}$, both of type G (no root is proportional to γ_i), so no type N; no pair (Y,α) of maximal rank has type T or N (S4-5, item 3), so the opening assumption of §6 holds (S4-10; Knop's transport of weight lattices, cited; in characteristic $\neq 2$ directly, A3-6). L_{γ_1}, L_{γ_6} are the standard Levi subgroups of $\{1,\dots,5\}$, $\{2,\dots,6\}$, of type D_5. There are two colours, $\check v_{D_i} = e_i = \check\gamma_i$, with $\langle\check\gamma_i, \rho_{P(X)}\rangle = 4$, so $\Theta = \check\Phi_X \times \{(+,4)\}$ and $\Theta^+ = \check\Phi_X^+$; Statement 7.1.5 follows from Lemma 7.1.2 (p. 58), and Statement 6.3.1 from Theorem 6.3.2, matching §6.19 (p. 56) for $\mathrm{Spin}_9 \setminus \mathrm{Spin}_{10}$ with $n = 5$:

$$B_{w_\gamma}(\tilde\chi) = -e^{\tilde\gamma}\frac{1-q^{-4}e^{-\tilde\gamma}}{1-q^{-4}e^{\tilde\gamma}}, \qquad \beta(\tilde\chi) = \prod_{\check\gamma\in\check\Phi_X^+}\frac{1-e^{\tilde\gamma}}{1-q^{-4}e^{\tilde\gamma}}.$$

So (A4) holds in characteristic 0; over $\mathbb{F}_q(\!(t)\!)$ (i), (ii), (iv) hold, and the analytic part of (iii) is open (A3-11) and not used by part (b).

Step 14 (the $\mathcal{O}$-model). For $\mathcal{M}_\theta$ over every complete DVR $\mathcal{O}$ with $2 \in \mathcal{O}^\times$ (S4-11, S4-4; over $\mathbb{F}_q[\![t]\!]$, A3-2 and A3-3, audited A5-8): C fixes every σ_{α_i}, so $\theta_0^2 = \mathrm{Int}(\sigma_L^2)$; an exact computation in $\mathfrak{so}_8$ shows that θ_0 is the identity on Spin_8, whence $\sigma_L^2 = 1$; θ_0 acts on $\mathcal{X}(T)$ by $-w_L \notin W(E_6)$ and $\dim \mathfrak{g}^{\theta_0} = 52$, so it has type E IV. (A1) (or (A1$_p$)) holds for every p. For $p \neq 2$: $H_{\mathcal{O}}$ is smooth with connected reductive fibres of type F_4; $X_{\mathcal{O}}$ is smooth, separated and quasi-affine of relative dimension 26; $B \cap H = B_L \cap L^\theta$ is a Borel subgroup scheme of $L^\theta = \mathrm{Spin}_8$, and $B_{\mathcal{O}}/(B \cap H) \to X_{\mathcal{O}}$ is an open immersion onto $U \simeq U_P \times A_X$; $\mathring{X} = U$ by purity of the complement of an affine open (EGA IV, 21.12.7); the invariant measures exist; $X(\mathcal{O}) = Kx_0$ (Hensel for the smooth $H_{\mathcal{O}}$, Lang for $H_{\mathbb{F}_q}$), so $\varphi_0 = \mathbf{1}_{Kx_0}$. (A5) for every p: X_F is affine and $B \cap H$ is connected solvable with split torus, so $\mathring{X}(F)$ is one $B(F)$-orbit. Cited, not re-proved: the Chevalley involution over $\mathbb{Z}$ (SGA 3, Exp. XXIII); smoothness of $G_{\mathcal{O}}^{\theta_0}$ (Edixhoven 1992, Prop. 3.4) and connectedness of its fibres (Steinberg 1968, Thm. 8.1); quasi-affineness of $G_{\mathcal{O}}/G_{\mathcal{O}}^{\theta_0}$ (Alper 2014; in characteristic p, Conrad's Theorem 2.3.1); EGA IV 17.9.1 and 21.12.7; Kneser, replaced in characteristic p by Bruhat–Tits III, Thm. 4.7.

Step 15 ((A3c), distinct labels, types). For $\mathcal{M}_\theta$, $p \neq 2$, in both characteristics (S4-5; over $\mathbb{F}_q[\![t]\!]$, A3-9, audited A5-12): distinct labels $\kappa \neq \kappa'$ in Λ_S^+ give distinct K-orbits (the Cartan decomposition of $G(F)$ through $\tau(gx_0) = g\theta(g)^{-1}$), the second clause of (A3b); $Jx_\kappa \subseteq B(\mathcal{O})x_\kappa$ for every label, (A3c), by the Iwahori factorization at labels with a dominant lift and, at the uncovered labels $3k\omega_1^\vee$, $3k\omega_6^\vee$ (k odd), by the wall parabolic with Levi D_5 and a rational central half-translation; and the 45 Borel orbits of maximal rank meet only rank-one quotients of type G or U (over $\bar{F}$; Knop 1995, or A3-6 in characteristic $\neq 2$).

Step 16 ((A3b) in full). For $\mathcal{M}_\theta$, $p \neq 2$, over every p-adic F (S4-12, S4-13) and over $\mathbb{F}_q(\!(t)\!)$ (A3-4, A3-5, A3-8, audited A5-9; or Step 9): $G(F) = K\,T(F)\,H(F)$, hence $X(F) = \bigsqcup_{\kappa \in \Lambda_S^+} Kx_\kappa$ and $K\backslash X(F) \simeq \Lambda_S^+$. Delorme–Sécherre, Pacific J. Math. 251 (2011), Theorem 1.1 (arXiv:math/0612545, Theorem 0.4 = 4.8), gives $G(F) = \bigcup_{j \in J} H(F)\, y_j\, T(F)\, K$ over the $H(F)$-classes of maximal (θ, F)-split tori, K the stabilizer of a special point in the apartment of T. Here $|J| = 1$: a maximal (θ, F)-split torus is $G(F)$-conjugate to S (Helminck–Wang, Prop. 10.3), hence hSh^{-1} with $h \in H(\bar{F})$ (Vust; Richardson), and the cocycle $h^{-1}\tau(h)$ lies in $Z_H(S) = \mathrm{Spin}_8$ with $H^1(F, \mathrm{Spin}_8) = 1$ (Kneser). Since $H^1(F, F_4) = 1$, $X(F) = G(F)x_0$, so every K-orbit meets $T(F)x_0$. The elements $\dot{n}_i = \sigma_{\alpha_i}\theta(\sigma_{\alpha_i}) \in H(\mathcal{O}) \cap N_G(T)$, $i = 1, 6$, act on labels by the simple reflections of W_X and move every point into $A_X^+ x_0$; with Step 15, (A3b) holds. In characteristic p, A3-4 proves the single class directly and Bruhat–Tits III gives $H^1 = 1$. Cited, not re-proved: Delorme–Sécherre with Helminck–Wang inside it; Vust and Richardson; Kneser; Bruhat–Tits for $K = \mathrm{Stab}(x)$.

Step 17 ((A3a) over every $\mathcal{O}$ with 2 invertible). For $\mathcal{M}_\theta$ and every DVR $\mathcal{O}$ with $2 \in \mathcal{O}^\times$, $p = 3$ included (A2-8; audit A6-9 with repairs A6-3, A6-5, A6-6; the 45×6 table recomputed in A6-1): the B_F-orbits are the 45 orbits $B\dot{w}x_0$, $x = ww_L w^{-1} \in \mathcal{C}_K$, all of maximal rank; Sakellaridis's scheme of each is the smooth orbit scheme $Y_x \simeq B/B_x$, $B_x = (B \cap \theta_x B)^{\theta_x}$; $Y_x P_\alpha/R(P_\alpha) \simeq \mathrm{PGL}_2 \backslash \mathrm{PGL}_2$ for the 90 pairs with $\alpha \in \Phi_x$ and $\simeq B_2\backslash \mathrm{PGL}_2 = \mathbb{P}^1_{\mathcal{O}}$ for the other 180, P_α-equivariantly; and the complement clause holds for all 270 pairs. The proof in five lines. (1) Every θ-stable maximal torus is H-conjugate to T: with $a = \dim T'_+$, r, c the numbers of real and complex roots and i_c, i_n those of imaginary roots with $\theta = \pm 1$, $52 = a + \frac{r+c}{2} + i_c$ and $26 = (6-a) + \frac{r+c}{2} + i_n$, so $i_c - i_n = 32 - 2a$, which forces $a = 4$ (A2-3). (2) Two Borel subgroups share a maximal torus [Con, Thm. 1.4.12], so every orbit

meets $N_G(T)x_0$; the Bruhat cell of τ separates the 45 orbits, and $B_x = T^{\theta_x} \ltimes U^{\theta_x}_{\Phi^+\cap\theta_x\Phi^+}$ is smooth over $\mathbb{Z}[1/2]$ [Con, Thm. 5.1.13, Prop. 5.1.16] (A2-3, A2-4). (3) Through τ, each orbit is a smooth locally closed $Y_x \simeq B/B_x$, which is Sakellaridis's Y with complements read inside closures (A2-4, A6-7). (4) For $\alpha \in \Phi_x$, $P_\alpha Y_x = Y_x$ and the quotient is a point; otherwise $P_\alpha Y_x = Y_x \sqcup Y_{s_\alpha x s_\alpha}$ with quotient PGL_2/B_2 (A2-5, A2-6). (5) By induction along type-U raisings from the closed orbit $x = w_0$, closures over $\mathcal{O}$ have the expected special fibres, which gives the complement clause (A2-7, A6-6). The proof uses the sign convention of S4-11 for C.
Step 18 ($\mathcal{M}_\theta$ satisfies (T1)–(T5)). (T1): H_Λ smooth with connected F_4 fibres (S4-4), $(T^\theta)^\circ$ a split maximal torus (A4-4). (T2): by the density argument of A4-5, or $H^1(E, F_4) = 1$. (T3): τ is a closed immersion over Λ (A4-3). (T4): Delorme–Sécherre at p-adic places (Step 16). (T5): Step 11, since (A1)–(A5) hold over every $\mathcal{O}'_e$, with (A3a) by Step 17; the label of Kx_κ is $\kappa \in \Lambda_S^+ = \Lambda_X^+$ for every e (A4-9, A5-7).

7. Part (d): the criterion for E_6/F_4

Step 19 (E_6/F_4 in both characteristics). Over every p-adic F, (A1)–(A5) hold for $\mathcal{M}_\theta$ (Steps 13–17, the ramified case by Step 11), so (a) applies; over $\mathbb{F}_q(\!(t)\!)$, (T1)–(T5) hold (Step 18), so (b) applies with the images P_κ of the p-adic fields. Hence $C_c^\infty(X)^K \simeq \mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}$ with $\varphi_0 \mapsto 1$ and (SG) holds iff r is onto. Unconditionally (S4-7), $\operatorname{im} r = \{g \in \mathbb{C}[\Lambda_S]^{W_X} : g(y_1) = g(\sigma y_1)\}$, a subring of codimension one, for every q; so (SG) fails, C_K is not cyclic, and $C_K = \mathcal{H}_K\varphi_0 \oplus \mathbb{C}\,\mathbf{1}_{Kx_{3\omega_1^\vee}}$ needs exactly two generators, with $C_K/\mathcal{H}_K\varphi_0$ the character of Satake parameter $q^{\rho-\omega_4}$ (S4-8; A4-8 over $\mathbb{F}_q(\!(t)\!)$). The second generator uses the image of an orbit indicator (Steps 5 and 12). Residue characteristic 2 is not covered. Proposition VII proves the algebra; Proposition VIII compares it with the sheaf level.
Other families. $\mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$ and $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$: r is onto, so (SG) holds under (A1)–(A5) (Proposition F of the classical page). The spheres $\mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}$: Proposition V proves (SG) directly. SL_2/T: (A5) fails. Over $\mathbb{F}_q(\!(t)\!)$, part (b) covers symmetric X with (T1)–(T5).

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[Run] The run *sakellaridis-char-p* of these pages: returns A1–A6, audits A5 (two rounds) and A6.