# PROPOSITION V: FUNCTION-LEVEL GENERATION WITHOUT (A1)–(A5)

## 1. Setting

$F$ is a non-archimedean local field of either characteristic, with ring of integers $\mathcal{O}$, uniformizer $\varpi$, valuation val and finite residue field $k_F$; $q = |k_F|$ except in §4, where $q$ is a block size. $G$ is a split connected reductive group scheme over $\mathcal{O}$ with split maximal torus $T$ and Borel $B \supseteq T$, $t^\lambda = \lambda(\varpi)$ for $\lambda \in \Lambda_T^+$, and $X$ is a smooth affine $\mathcal{O}$-scheme with a $G$-action and a point $x_0 \in X(\mathcal{O})$. With Haar measure $\operatorname{vol} K = 1$,

$$K = G(\mathcal{O}), \quad \mathcal{H}_K = C_c(K\backslash G(F)/K), \quad (f *_K \varphi)(x) = \sum_{g \in G(F)/K} f(g)\,\varphi(g^{-1}x), \quad T_\lambda = \mathbf{1}_{Kt^\lambda K},$$

$$C_K = C_c^\infty(X(F))^K, \qquad \varphi_0 = \mathbf{1}_{X(\mathcal{O})} \in C_K, \qquad \text{(SG)} \quad \mathcal{H}_K *_K \varphi_0 = C_K.$$

**Proposition V.** *(a) Spheres. Let $n \geq 2$, $N(x) = \sum_{i=1}^n x_i x_{2n+1-i}$ on $L = \mathcal{O}^{2n}$, $Q = \{N = 1\}$, and let $G = \mathrm{Spin}_{2n}$ act on $X = Q$ through $\mathrm{SO}(L, N)$, $x_0 = e_1 + e_{2n}$. Then the $K$-orbits on $Q(F)$ are the shells $O_m = \{x \in Q(F) : \min_i \operatorname{val} x_i = -m\}$, $m \geq 0$, each compact open, with $O_0 = Q(\mathcal{O})$, and* (SG) *holds.*

*(b) Pair model of $\mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_{n-p})$. Let $1 \leq p \leq n - p$, let $X(R)$ be the idempotents $\pi \in M_n(R)$ of rank $n - p$, with $\mathrm{GL}_n$ acting by conjugation and base point $\pi_0$ of §3. Then $K\backslash X(F) \cong \{a \in \mathbb{Z}^p : a_1 \geq \cdots \geq a_p \geq 0\} = \Lambda_S^+$ and* (SG) *holds. If $2 \in \mathcal{O}^\times$, $\varepsilon \mapsto (1 + \varepsilon)/2$ identifies the involution model ($J$-model) with $X$, integral points and basic functions included.*

*(c) Split symmetric spaces over $\mathbb{F}_q(\!(t)\!)$. Let $q$ be odd and $X = G/G^\theta$ over $\mathbb{F}_q$ with $T \supseteq S$ split and $G^\theta$ connected. Assume D0, D3 and the two conditions of Proposition III:*

$$\text{(C1)} \quad Z_{G^\theta}(S) \text{ connected}, \qquad \text{(C2)} \quad \varphi(\Lambda_T^+) = \Lambda_S^+.$$

*Then* (SG) *holds over $\mathbb{F}_{q^r}(\!(t)\!)$ for every $r \geq 1$.*

*(d) Consistency. For the spheres, Sakellaridis's restriction map is onto: $r(m_{\varepsilon_1+\varepsilon_2}) = c_n(q)(u+u^{-1}) + \text{const}$ with $c_n(q) = \sum_{j=2}^n (q^{n-j} + q^{j-n}) > 0$; under (A1)–(A5), $\operatorname{char} F = 0$, Theorem E gives the answer of (a).*

*Parts (a) and (b) use none of (A1)–(A5), no Cartan decomposition of $X(F)$, and no characteristic-zero input.*

The symplectic families are Proposition IV.

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*Date*: 17 October 2026. Source: verdict §1 and §3 (Theorem 4, non-symplectic parts; Theorem 6, sphere part) of the run *Satake generation at two levels*, steps A5-1–A5-4, A1-3, A4-2, A4-8, A6-1, A6-3, audited in A7-20–A7-21 and C-4.

## 2. The triangular criterion

**Lemma 2.1** (Lemma A(2)). *For $\lambda \in \Lambda_T^+$ and $x \in X(F)$, $(T_\lambda *_K \varphi_0)(x) = \#\{gK \subseteq Kt^\lambda K : g^{-1}x \in X(\mathcal{O})\} \in \mathbb{Z}_{\geq 0}$, $\operatorname{supp}(T_\lambda *_K \varphi_0) = Kt^\lambda X(\mathcal{O})$, and $(T_\lambda *_K \varphi_0)(t^\lambda x_0) \geq 1$.*

*Proof.* $Kt^\lambda K/K$ is finite and $(T_\lambda *_K \varphi_0)(x) = \sum_{gK \subseteq Kt^\lambda K} \mathbf{1}_{X(\mathcal{O})}(g^{-1}x)$. A term is nonzero iff $x \in gX(\mathcal{O})$, so the support is $Kt^\lambda K \cdot X(\mathcal{O}) = Kt^\lambda X(\mathcal{O})$. At $x = t^\lambda x_0$ the coset $t^\lambda K$ contributes 1. □

**Lemma 2.2** (Proposition B). *Suppose $X(F) = \bigsqcup_{\mu \in I} O_\mu$ is a partition into compact open $K$-orbits and $\leq$ is a partial order on $I$ with finite lower sets. If for each $\mu$ there is $f_\mu \in \mathcal{H}_K$ with $\operatorname{supp}(f_\mu *_K \varphi_0) \subseteq \bigcup_{\nu \leq \mu} O_\nu$ and $f_\mu *_K \varphi_0 \neq 0$ on $O_\mu$, then* (SG) *holds.*

*Proof.* A $K$-invariant $\varphi \in C_c^\infty(X(F))$ is constant on each orbit, and its compact support meets finitely many of the open orbits; so $\{\mathbf{1}_{O_\mu}\}$ spans $C_K$. Write $f_\mu *_K \varphi_0 = c_\mu \mathbf{1}_{O_\mu} + \sum_{\nu<\mu} c_{\mu\nu} \mathbf{1}_{O_\nu}$ with $c_\mu \neq 0$. By induction on $\#\{\nu \leq \mu\}$, every $\mathbf{1}_{O_\nu}$, $\nu < \mu$, lies in $\mathcal{H}_K *_K \varphi_0$, hence so does $\mathbf{1}_{O_\mu}$. □

**Corollary 2.3** (Corollary C). *In Lemma 2.2, suppose each $\mu$ has $\lambda_\mu \in \Lambda_T^+$ with $t^{\lambda_\mu} x_0 \in O_\mu$ and $Kt^{\lambda_\mu} X(\mathcal{O}) \subseteq \bigcup_{\nu \leq \mu} O_\nu$. Then* (SG) *holds, with $f_\mu = T_{\lambda_\mu}$.*

*Proof.* Lemma 2.1 and Lemma 2.2. □

*Remark* 2.4. On the classical page, Lemma A(1) obtains the compact open orbits from Borel–Serre finiteness, which needs $\operatorname{char} F = 0$. Here the partition into compact open $K$-orbits is a hypothesis, verified directly in §3–§5.

## 3. Spheres

$B(x,y) = N(x+y) - N(x) - N(y)$ satisfies $B(e_i, e_j) = \delta_{i+j,2n+1}$, so it is perfect on $L$ in every characteristic. $\mathrm{SO}(L)$ is the kernel of the Dickson invariant $\mathrm{O}(L,N) \to \mathbb{Z}/2$ [1, Thm. C.2.11, Rem. C.2.12], and $K = \mathrm{Spin}_{2n}(\mathcal{O})$ acts on $Q$ through its image in $\mathrm{SO}(L)(\mathcal{O})$. Put $x_m = \varpi^m e_1 + \varpi^{-m} e_{2n} \in O_m$, $y_m = \varpi^{2m} e_1 + e_{2n}$, $P_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_{2n}$ and $M_0 = \bigoplus_{i=2}^{2n-1} \mathcal{O}e_i \cong H^{n-1}$, where $H$ is the hyperbolic plane.

**Lemma 3.1** (Hensel). *If $f \in \mathcal{O}[a]$ and $\bar{a}_0 \in k_F$ is a simple root of $\bar{f}$, then $f$ has a root $a \equiv a_0$.*

*Proof.* $\mathcal{O}$ is complete, hence henselian [2, Tag 04GM], and a henselian local ring has this property for every (not necessarily monic) $f$ [2, Tag 04GG(2)]. □

**Lemma 3.2** (Hyperbolic lifting). *Let $M$ be free of rank $2s$ over $\mathcal{O}$ with a quadratic form whose polar form is perfect. If $\bar{M} \cong H^s$ over $k_F$, then $M \cong H^s$ over $\mathcal{O}$.*

*Proof.* Induction on $s$. Lift a hyperbolic pair $\bar{e}, \bar{f}$ of $\bar{M}$ to $e, f \in M$. The polynomial $g(a) = N(e + af) = N(f)a^2 + B(e,f)a + N(e)$ has $\bar{g}(0) = 0$ and $\bar{g}'(0) = 1$, so Lemma 3.1 gives $a \in \varpi\mathcal{O}$ with $e' = e + af$ isotropic. Put $f' = f/B(e', f)$ and $f'' = f' - N(f')e'$: then $N(f'') = 0$, $B(e', f'') = 1$, and $e', f''$ reduce to $\bar{e}, \bar{f}$. So $M = (\mathcal{O}e' \oplus \mathcal{O}f'') \oplus M'$ with $\bar{M}' = \langle \bar{e}, \bar{f} \rangle^\perp \cong H^{s-1}$. □

**Lemma 3.3** (shells). *For every $F$ and $n \geq 2$: $Q(F) = \bigsqcup_{m \geq 0} O_m$, each $O_m$ is compact open, $O_0 = Q(\mathcal{O})$, and $O_m = Kx_m$.*

*Proof. Shells.* If $x \in \varpi L$ then $N(x) \in \varpi^2\mathcal{O}$, so $m := -\min_i \operatorname{val} x_i \geq 0$ on $Q(F)$. $O_m = Q(F) \cap (\varpi^{-m}L \smallsetminus \varpi^{1-m}L)$ is open, closed and bounded in the closed set $Q(F)$, hence compact open, and $O_0 = Q(F) \cap L = Q(\mathcal{O})$. $\mathrm{SO}(L)(\mathcal{O})$ preserves every $\varpi^k L$, hence every $O_m$.

*Step A: a hyperbolic plane through $y$.* Let $x \in O_m$ and $y = \varpi^m x$, primitive with $N(y) = \varpi^{2m}$. As $B$ is perfect and $y$ primitive, $B(y,\cdot)\colon L \to \mathcal{O}$ is onto; choose $z_1$ with $B(y,z_1) = 1$, so $N(z_1 + ay) = \varpi^{2m}a^2 + a + N(z_1)$. If $m \geq 1$, the reduction $a + \bar{N}(z_1)$ has a simple root, and Lemma 3.1 gives $a$ with $z_2 = z_1 + ay$ isotropic. If $m = 0$, Witt's extension theorem over $k_F$ (valid in every characteristic for a nondegenerate polar form [1, proof of Prop. C.3.1, citing Chevalley I.4.1]) gives $\eta \in \mathrm{O}(\bar{L})$ with $\eta(\bar{e}_1 + \bar{e}_{2n}) = \bar{y}$; then $\bar{s} = \eta\bar{e}_2$, $\bar{t} = \eta\bar{e}_{2n-1}$ is a hyperbolic pair in $\bar{y}^\perp$. Lift it to $s, t \in y^\perp$ (a direct summand of $L$ reducing onto $\bar{y}^\perp$). For $c, d \in \mathcal{O}$,

$$N(z_1 + cs + dt) \equiv (\bar{c} + \bar{B}(z_1,t))(\bar{d} + \bar{B}(z_1,s)) + \bar{N}(z_1) - \bar{B}(z_1,s)\bar{B}(z_1,t) \pmod{\varpi}$$

takes every residue, so we may assume $N(z_1) \in \varpi\mathcal{O}$; then $a^2 + a + N(z_1)$ has the simple root 0 modulo $\varpi$, and Lemma 3.1 gives $a \in \varpi\mathcal{O}$ with $z_2 = z_1 + ay$ isotropic. In both cases $u = B(y,z_2) = 1 + 2a\varpi^{2m} \in \mathcal{O}^\times$; put $z = u^{-1}z_2$, so $N(z) = 0$ and $B(y,z) = 1$.

*Step B: the plane is standard.* $P = \mathcal{O}y \oplus \mathcal{O}z$ has Gram matrix $\left(\begin{smallmatrix} 2\varpi^{2m} & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of unit determinant, so $L = P \oplus P^\perp$, and $\iota\colon y \mapsto y_m$, $z \mapsto e_1$ is an isometry $P \to P_0$.

*Step C: the complement is hyperbolic.* Witt's theorem over $k_F$ extends $\bar{\iota}$ to an isometry of $\bar{L}$, which maps $\overline{P^\perp} = \bar{P}^\perp$ onto $\bar{M}_0 \cong H^{n-1}$. By Lemma 3.2, $P^\perp \cong M_0$ over $\mathcal{O}$, and $g = \iota \oplus (P^\perp \cong M_0)$ is an isometry of $L$ with $g(y) = y_m$.

*Step D: orientation.* Let $f, f'$ be a hyperbolic pair of $P^\perp$ and $w = f + f'$, so $N(w) = 1$. The reflection $r_w(v) = v - B(v,w)w$ fixes $P$ and equals $-1$ times the swap $f \leftrightarrow f'$ on $\langle f, f'\rangle$; the swap has Dickson invariant 1 [1, Rem. C.3.3] and $-1$ on a hyperbolic plane lies in SO. So $g$ or $g \circ r_w$ lies in $\mathrm{SO}(L)(\mathcal{O})$ and maps $x = \varpi^{-m}y$ to $\varpi^{-m}y_m = x_m$: $\mathrm{SO}(L)(\mathcal{O})$ is transitive on $O_m$.

*Step E: $K$ has the same orbits.* From $1 \to \mu_2 \to \mathrm{Spin} \to \mathrm{SO} \to 1$ (fppf) and Kummer theory with $\operatorname{Pic}\mathcal{O} = 0$, the image of $K$ in $\mathrm{SO}(L)(\mathcal{O})$ is the kernel of the spinor norm $\mathrm{sp}\colon \mathrm{SO}(L)(\mathcal{O}) \to H^1_{\mathrm{fppf}}(\mathcal{O}, \mu_2) = \mathcal{O}^\times/\mathcal{O}^{\times 2}$, and $\mathrm{sp}(r_v) = [N(v)]$ for $N(v) \in \mathcal{O}^\times$ [1, §C.5, p. 349]. For $u \in \mathcal{O}^\times$ put $v_1 = e_2 + e_{2n-1}$, $v_u = e_2 + ue_{2n-1}$: then $r_{v_1}r_{v_u}$ acts as $\mathrm{diag}(u, u^{-1})$ on $\langle e_2, e_{2n-1}\rangle$, fixes the other $e_i$ (so fixes $x_m$), and has spinor norm $[u]$. Hence $\mathrm{SO}(L)(\mathcal{O}) = \mathrm{im}(K) \cdot \mathrm{Stab}(x_m)$ and $O_m = Kx_m$. □

**Theorem 3.4.** (SG) *holds for* $\mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}$ *in the model $Q$, for every $n \geq 2$ and every $F$.*

*Proof.* Order the labels $m \geq 0$ as integers. $\lambda_m = (m, m, 0, \ldots, 0)$ lies in $\Lambda_T^+ = \{x \in \mathbb{Z}^n : \sum x_i \text{ even},\ x_1 \geq \cdots \geq x_{n-1} \geq |x_n|\}$, its image in $\mathrm{SO}(L)$ is $\mathrm{diag}(\varpi^m, \varpi^m, 1, \ldots, 1, \varpi^{-m}, \varpi^{-m})$, and $t^{\lambda_m}x_0 = x_m \in O_m$. Since $t^{\lambda_m}L \subseteq \varpi^{-m}L$ and $K$ preserves $\varpi^{-m}L$,

$$Kt^{\lambda_m}X(\mathcal{O}) \subseteq Q(F) \cap \varpi^{-m}L = \textstyle\bigsqcup_{m' \leq m} O_{m'}.$$

Lemma 3.3 and Corollary 2.3 give (SG). □

*Remark* 3.5. For residue characteristic $\neq 2$ this is A1-3 (via the classification of unimodular lattices). The audit of A5-2 noted two slips, corrected above: $B(y,z_2) = 1 + 2a\varpi^{2m}$ must be normalized, and for $m = 0$ the residue of $N(z_1)$ must be chosen so that the reduced quadratic has a simple root. A further imprecision is avoided here: for an arbitrary pair $\bar{s}, \bar{t}$ with $\bar{B}(\bar{s},\bar{t}) = 1$ the quadratic part $\bar{N}(\bar{s})c^2 + cd + \bar{N}(\bar{t})d^2$ can be degenerate when $q$ is odd; a hyperbolic pair, supplied by Witt's theorem, makes the residue map visibly onto. The lift of $P^\perp$ uses Lemma 3.2 in place of the smoothness of the isometry scheme.

## 4. The pair model of $\mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$

In this section $q = n - p \geq p \geq 1$ is a block size. Over $F$, $\pi \in X(F)$ is the pair $(U, V) = (\operatorname{im}\pi, \ker\pi)$ with $F^n = U \oplus V$, $\dim V = p$; $X(\mathcal{O})$ consists of the pairs with $L_0 = (L_0 \cap U) \oplus (L_0 \cap V)$, $L_0 = \mathcal{O}^n$. The base point $\pi_0$ has $U_0 = \langle e_i + e_{n+1-i}\ (i \leq p),\ e_j\ (p < j \leq q)\rangle$, $V_0 = \langle e_{n+1-i}\ (i \leq p)\rangle$, which split $L_0$. A finite $\mathcal{O}$-module $E \cong \bigoplus_i \mathcal{O}/\varpi^{b_i}$ has *type* $b_1 \geq b_2 \geq \cdots$ [3, Tag 0ASV], and

$$(*) \qquad \dim_{k_F} \varpi^{j-1}E/\varpi^j E = \#\{i : b_i \geq j\}.$$

For a lattice $L$ let $\ell(U, V; L)$ be the type of $\pi_V(L)/(L \cap V)$, $\pi_V$ the projection onto $V$ along $U$; it has at most $p$ parts, $\ell(gU, gV; gL) = \ell(U, V; L)$, and $\ell := \ell(\cdot; L_0)$ is $K$-invariant. $O_a := \{\ell = a\}$.

**Lemma 4.1** (normal form)**.** *Let $(U, V) \in X(F)$, $a = \ell(U, V)$, $h = \#\{i : a_i > 0\}$. There are bases $u_1, \ldots, u_q$ of $U$ and $v_1, \ldots, v_p$ of $V$ with*

$$L_0 = \bigoplus_{i \leq h} \left(\varpi^{a_i}\mathcal{O}u_i \oplus \mathcal{O}(u_i + v_i)\right) \oplus \bigoplus_{h < i \leq q} \mathcal{O}u_i \oplus \bigoplus_{h < j \leq p} \mathcal{O}v_j.$$

*Hence $\ell\colon K\backslash X(F) \to \{a_1 \geq \cdots \geq a_p \geq 0\}$ is injective, and every $K$-orbit is compact open.*

*Proof.* Put $A = \pi_U L_0 \supseteq A' = L_0 \cap U$, $B = \pi_V L_0 \supseteq B' = L_0 \cap V$. The map $\psi\colon A/A' \to B/B'$, $\pi_U l \mapsto \pi_V l$, is a well-defined isomorphism (if $\pi_U l \in A' \subseteq L_0$ then $\pi_V l = l - \pi_U l \in B'$), and $L_0 = \{\alpha + \beta \in A \oplus B : \psi(\bar\alpha) = \bar\beta\}$. A stacked basis (Smith form over the discrete valuation ring $\mathcal{O}$: take an entry of minimal valuation, clear its row and column, induct) gives a basis $u_i$ of $A$ with $A' = \bigoplus_{i \leq h} \varpi^{a_i}\mathcal{O}u_i \oplus \bigoplus_{i > h} \mathcal{O}u_i$, the exponents being the type of $A/A' \cong B/B'$, that is $a$. The classes $\psi(\bar u_i)$, $i \leq h$, reduce to a basis of $B/(B' + \varpi B)$; lift them to $v_i \in B$, extend by Nakayama to a basis of $B$, and subtract from each added vector a combination of the $v_i$ to put it in $B'$. Then $\varpi^{a_i} v_i \in B'$ and comparing lengths gives $B' = \bigoplus_{i \leq h} \varpi^{a_i}\mathcal{O}v_i \oplus \bigoplus_{j > h} \mathcal{O}v_j$; the condition $\psi(\bar\alpha) = \bar\beta$ reads $\alpha_i \equiv \beta_i \bmod \varpi^{a_i}$, which is the display. Two points with the same label then differ by the $g \in \mathrm{GL}_n(F)$ matching the bases, which preserves $L_0$, so $g \in K$. Orbits are compact as images of $K$; open because for $\pi$ near $\pi_1$, $g(\pi) = \pi\pi_1 + (1 - \pi)(1 - \pi_1) \in K$ and $g(\pi)\pi_1 g(\pi)^{-1} = \pi$. □

**Lemma 4.2** (lift)**.** *For $a_1 \geq \cdots \geq a_p \geq 0$, $\lambda_a = (a_1, \ldots, a_p, 0, \ldots, 0) \in \Lambda_T^+$ and $\ell(t^{\lambda_a}\pi_0) = a$. So $\ell$ is a bijection onto $\Lambda_S^+$, and $X(\mathcal{O}) = O_0 = K\pi_0$.*

*Proof.* $t^{\lambda_a}U_0 = \langle \varpi^{a_i} e_i + e_{n+1-i}, e_j\rangle$, $t^{\lambda_a}V_0 = V_0$, and the projection onto $V_0$ sends $e_i \mapsto -\varpi^{-a_i} e_{n+1-i}$, kills the $e_j$ and fixes $V_0$; so $\pi_{V_0}(L_0)/(L_0 \cap V_0) \cong \bigoplus \mathcal{O}/\varpi^{a_i}$. A pair splits $L_0$ iff $\pi_V(L_0) = L_0 \cap V$ iff its label is 0. □

**Lemma 4.3** (componentwise support bound)**.** *$Kt^{\lambda_a}X(\mathcal{O}) \subseteq \bigsqcup_{b \leq a} O_b$, where $b \leq a$ means $b_i \leq a_i$ for all $i$.*

*Proof.* By Lemma 4.2, $Kt^{\lambda_a}X(\mathcal{O}) = Kt^{\lambda_a}K\pi_0$. For $g = k_1 t^{\lambda_a} k_2$, $\ell(g\pi_0) = \ell(\pi_0; L')$ with $L' = g^{-1}L_0 = k_2^{-1}t^{-\lambda_a}L_0 \supseteq L_0$ and $L'/L_0 \cong \bigoplus_{i \leq p} \mathcal{O}/\varpi^{a_i}$. Since $\pi_0$ splits $L_0$, $\pi_{V_0}(L_0) = L_0 \cap V_0 \subseteq L' \cap V_0$, so $\pi_{V_0}(L')/(L' \cap V_0)$ is a quotient of $L'/L_0$. By $(*)$, a quotient has $\#\{i : b_i \geq j\}$ no larger for every $j$, so $b \leq a$ componentwise. □

**Theorem 4.4.** (SG) *holds for the pair model, for every $1 \leq p \leq q$ and every $F$.*

*Proof.* The componentwise order has finite lower sets; Lemmas 4.1–4.3 and Corollary 2.3 with $f_a = T_{\lambda_a}$. □

*Remark* 4.5 (models). Let $J$ be the permutation matrix of $i \leftrightarrow n+1-i$ ($i \le p$). If $2 \in \mathcal{O}^\times$, $\varepsilon \mapsto (1+\varepsilon)/2$ is a $\mathrm{GL}_n$-equivariant bijection from the involutions $\varepsilon \in M_n(R)$ of type $(q,p)$ to $X(R)$ for every $\mathcal{O}$-algebra $R$, with inverse $\pi \mapsto 2\pi - 1$; it sends $J$ to the projection onto $U_0$ along $\langle e_i - e_{n+1-i}\rangle$, which lies in $X(\mathcal{O}) = K\pi_0$. So (SG) for the pages' $J$-model is Theorem 4.4. In residue characteristic 2 the models differ (C-4: over $\mathbb{Q}_2$, $n=2$, the point $t^{(1,0)}\pi_0$ has label 1 but Cartan label 0); the theorem is for the pair model. C-4 also proves $Kt^\lambda X(\mathcal{O}) \subseteq \bigcup_{b \le \mathrm{dom}\,\varphi(\lambda)} O_b$ for every dominant $\lambda$, by bounding the minors of $t^\lambda \pi t^{-\lambda}$; it is not needed here.

## 5. Split symmetric spaces over $\mathbb{F}_q(\!(t)\!)$

Here $q$ is odd, $F = \mathbb{F}_q(\!(t)\!)$, $K = L^+G(\mathbb{F}_q)$, $X(F) = LX(\mathbb{F}_q)$, $X(\mathcal{O}) = L^+X(\mathbb{F}_q)$; $\mathcal{O}_\mu = L^+G \cdot x_\mu$ are the geometric orbits over $\overline{\mathbb{F}}_q$, $\tau(x_\mu) = t^\mu$, and $\varphi(\lambda) = \lambda - \theta\lambda$.

- D0 Chen–Yi's orbit geometry over $\overline{\mathbb{F}}_q$, $p \neq 2$: $LX = \bigsqcup_{\mu \in \Lambda_S^+} \mathcal{O}_\mu$ (Prop. 17), the closure order (Thm. 18), the support bound $L^+G\,t^\lambda L^+X \subseteq \bigsqcup_{\nu \le \varphi(\lambda)} \mathcal{O}_\nu$, and $\pi_0 \,\mathrm{Stab}_{L^+G}(x_\mu) \cong \pi_0 Z_{G^\theta}(\mu)$ with $Z_{G^\theta}(\mu) = Z_{G^\theta}(\mu)^\circ Z_{G^\theta}(S)$.
- D3 Lang's theorem for $L^+G = \varprojlim G(\mathbb{F}_q[t]/t^k)$ and for $\mathrm{Stab}_{L^+G}(x_\mu)$.

**Lemma 5.1** (A4-2). *Under D0, D3 and (C1): $K\backslash X(F) \leftrightarrow \Lambda_S^+$, $O_\mu := \mathcal{O}_\mu(\mathbb{F}_q) = Kx_\mu$ is compact open.*

*Proof.* $X(F) = \bigsqcup_\mu \mathcal{O}_\mu(\mathbb{F}_q)$, each $\mathcal{O}_\mu$ being Frobenius-stable ($x_\mu$ is rational because $\tau$ is injective and $t^\mu \in S(F)$). For $y = gx_\mu$ rational, $\mathrm{Fr}(g)^{-1}g \in \mathrm{Stab}(x_\mu)$ defines a class in $H^1(\mathbb{F}_q, \mathrm{Stab}(x_\mu))$; by D3 ($H^1(\mathbb{F}_q, L^+G) = 1$) this is a bijection $K\backslash\mathcal{O}_\mu(\mathbb{F}_q) \cong H^1(\mathbb{F}_q, \mathrm{Stab}(x_\mu)) = H^1(\mathbb{F}_q, \pi_0 Z_{G^\theta}(\mu))$, trivial under (C1). The orbit map is smooth, so $K$-orbits are open; they are compact as images of $K$. □

**Theorem 5.2** (A4-8). *Under D0, D3, (C1), (C2),* (SG) *holds over $\mathbb{F}_{q^r}(\!(t)\!)$ for every $r \ge 1$.*

*Proof.* Order $\Lambda_S^+$ by dominance (finite lower sets, H1(a)). For $\mu$ choose $\lambda \in \Lambda_T^+$ with $\varphi(\lambda) = \mu$ (C2). Then $\tau(t^\lambda x_0) = t^{\varphi(\lambda)}$ gives $t^\lambda x_0 = x_\mu \in O_\mu$, and by D0

$$Kt^\lambda X(\mathcal{O}) \subseteq (L^+G\,t^\lambda L^+X)(\mathbb{F}_q) \subseteq \bigsqcup_{\nu \le \mu} \mathcal{O}_\nu(\mathbb{F}_q).$$

Lemma 5.1 and Corollary 2.3. Replacing $q$ by $q^r$ changes nothing. □

Spheres and $\mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$ satisfy (C1) and (C2); for them Theorem 5.2 reproves (a), (b) over $\mathbb{F}_q(\!(t)\!)$, $q$ odd, but needs D0 and D3, which (a), (b) do not.

## 6. Consistency with Sakellaridis's restriction map

With $\delta(X)^{1/2} = q^{\rho_L}$ (A6-1; the sign of the shift does not affect surjectivity), $r(f)(y) = f(q^{\rho_L}y)$ on $A_X^* = \mathrm{Hom}(\Lambda_S, \mathbb{C}^\times)$. For the spheres, $\Lambda_S = \mathbb{Z}\cdot 2\varepsilon_1$, $u = y(2\varepsilon_1)$, $W_X = \mathbb{Z}/2$, $\Phi_L$ of type $D_{n-1}$, $\rho_L = (0, n-2, \dots, 1, 0)$, and $x^\mu \mapsto q^{\langle \rho_L, \mu\rangle} u^{\mu_1}$. The roots $\pm\varepsilon_1 \pm \varepsilon_j$ give

$$r(m_{\varepsilon_1+\varepsilon_2}) = c_n(q)(u + u^{-1}) + \mathrm{const}, \qquad c_n(q) = \sum_{j=2}^{n} (q^{n-j} + q^{j-n}) > 0,$$

so $\mathrm{im}\, r = \mathbb{C}[u+u^{-1}]$ is the whole target. Theorem E [4, Thm. 8.0.2, Cor. 8.0.4] then gives (SG) for $p$-adic $F$ under (A1)–(A5); (A1) and (A5) hold for the spheres, (A2)–(A4) are not verified. The answer agrees with Theorem 3.4, which needs none of them.

## 7. Audit record

- A5-1 (minor): the strategy pairs the symplectic row $\lambda^*(b)$ with reverse dominance, whose lower sets are infinite; the orders used here ($m' \le m$; componentwise) have finite lower sets. The symplectic repair is Proposition IV.
- A5-2, A7-20 (minor): the normalization of $B(y, z_2)$ and the residue choice for $m = 0$; corrected in Step A.
- A5-4, A7-21 (correct): the normal form is written out for the indices with $a_i = 0$ in Lemma 4.1.
- A4-2 (minor): "exactly when" needs the converse, which is not used; Sakellaridis's Axiom 2.4.1 is not what is proved: the bijectivity is the one he expects after Axiom 2.4.2, and it needs $Z_{G^\theta}(S)$, not $G^\theta$, connected.
- A6-1 (minor): the sign of $\delta(X)^{1/2}$ is a convention and left versus right modules change $r$ by an automorphism; neither affects surjectivity.
- C-4: support bound for every dominant $\lambda$ in the pair model; the route through the Cartan embedding fails in residue characteristic 2.

## References

[1] B. Conrad, *Reductive group schemes*, in: Autour des schémas en groupes I, Panoramas et Synthèses 42–43 (2014); author's file https://math.stanford.edu/~conrad/papers/luminysga3.pdf: Thm. C.1.5, Thm. C.2.11, Rem. C.2.12, Rem. C.3.3, proof of Prop. C.3.1, §C.5.
[2] The Stacks Project, Commutative Algebra, Tags 04GG, 04GM (Lemmas 10.153.3, 10.153.9).
[3] The Stacks Project, More on Algebra, Tag 0ASV (Lemma 15.126.9).
[4] Y. Sakellaridis, *Spherical functions on spherical varieties*, Amer. J. Math. 135 (2013), arXiv:0905.4244v3: Axioms 2.4.1–2.4.2 and the remark after them, Thm. 8.0.2, Cor. 8.0.4.
[5] T.-H. Chen, L. Yi, arXiv:2310.20006v4: Prop. 17, Thm. 18.