PROPOSITION VIII: E_6/F_4, THE ONE DIVERGENCE

At $X = E_6/F_4$ the sheaf statement $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ holds while spherical generation (SG) fails. The sheaf half is proved over $\overline{\mathbb{F}}_q$. The function half now holds over $\mathbb{F}_q(\!(t)\!)$, the same field family, and over every p-adic field, p odd: by Proposition VI, Sakellaridis's Theorem 8.0.2 holds for an explicit integral model over both (audited run sakellaridis-char-p [SCP]). This note states both halves precisely and proves them. The sheaf half uses Propositions I and II; the function half uses Proposition VI and Proposition VII, which shows that C_K needs exactly two generators.

1. Setting

Root data. G is the split simply connected group of type E_6 (over $\mathbb{Z}[1/2]$), with Bourbaki labels: the chain 1–3–4–5–6 and node 2 attached to node 4. θ is the involution of type E IV, $G^\theta = F_4$ (connected, G simply connected), $X = G/G^\theta$, $x_0 = eG^\theta$. The Satake diagram has black nodes $2, 3, 4, 5$ (a D_4) and white nodes $1, 6$. S is a maximal θ-split torus, $T \supseteq S$ a θ-stable maximal torus, $L = Z_G(S)$ the Levi of the black nodes, ρ_L the half-sum of its 12 positive roots, and on $\Lambda_T = X_*(T) = \bigoplus_j \mathbb{Z}\alpha_j^\vee$

$$\theta = -w_L, \qquad \varphi = 1 - \theta, \qquad Z_{G^\theta}(S) = \mathrm{Spin}_8 \ \text{(connected)}.$$

Write $(a, b) := a\omega_1^\vee + b\omega_6^\vee$. Then φ kills $\alpha_2^\vee, \dots, \alpha_5^\vee$ and

$$e_1 := \varphi(\alpha_1^\vee) = (2, -1), \quad e_6 := \varphi(\alpha_6^\vee) = (-1, 2), \qquad \Lambda_S = \mathbb{Z}e_1 \oplus \mathbb{Z}e_6 = \{(a, b) : a \equiv b \bmod 3\},$$

$$\Lambda_S^+ = \{(a, b) : a, b \geq 0,\ a \equiv b \bmod 3\}, \qquad \Lambda_S^+ \setminus \varphi(\Lambda_T^+) = \{(3k, 0),\ (0, 3k) : k \text{ odd}\}.$$

The last equality is the lemma of §5 of the spherical workspace's E_6/F_4 page: a dominant $\lambda = \sum_i c_i\omega_i^\vee$ has $\varphi(\lambda) = (2c_1 + c_2 + 2c_3 + 2c_4 + c_5,\ c_2 + c_3 + 2c_4 + 2c_5 + 2c_6)$, so $\varphi(\lambda) = (a, 0)$ forces $c_2 = \cdots = c_6 = 0$ and $a = 2c_1$ even. The diagram automorphism σ exchanges nodes $1 \leftrightarrow 6$, $3 \leftrightarrow 5$.

Sheaf level. $k = \overline{\mathbb{F}}_q$, $p \neq 2$, $\ell \neq p$; coefficients $\overline{\mathbb{Q}}_\ell$. $LX = X(k(\!(t)\!))$, $L^+G = G(k[\![t]\!])$, $LX = \bigsqcup_{\mu \in \Lambda_S^+} \mathcal{O}_\mu$, $\mathcal{O}_\mu = L^+G \cdot x_\mu$, $\tau(x_\mu) = t^\mu$ for the Cartan embedding $\tau(gG^\theta) = g\theta(g)^{-1}$, $t^\lambda x_0 = x_{\varphi(\lambda)}$ [CY, Prop. 17]. $\nu \leq \mu$ is the dominance order, $\overline{\mathcal{O}_\mu} = \bigsqcup_{\nu \leq \mu} \mathcal{O}_\nu$ [CY, Thm. 18], and $\delta(\mu)$ is the codimension of $\mathcal{O}_0$ in $\overline{\mathcal{O}_\mu}$. IC_ν is normalized by $\mathrm{IC}_\nu|_{\mathcal{O}_\nu} = \overline{\mathbb{Q}}_\ell[\delta(\nu)]$. V_λ is the Satake sheaf of $\mathrm{Gr}^{\leq\lambda}$,

$$m\colon \mathrm{Gr}^{\leq\lambda} \mathbin{\widetilde{\times}} L^+X \to LX,\ [g, y] \mapsto gy, \qquad V_\lambda * \mathrm{IC}_0 = m_!(V_\lambda \mathbin{\widetilde{\boxtimes}} \mathrm{IC}_0) \simeq \bigoplus_{\nu, n} \mathrm{IC}_\nu[-n] \otimes E_{\nu,n}(\lambda),$$

the decomposition being input D2. $\mathcal{C}_K^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}_K}\{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\}$.

Function level. F is $\mathbb{F}_q(\!(t)\!)$ or a p-adic field, p odd, with ring of integers $\mathcal{O}_F$ and residue field $\mathbb{F}_q$, $K = G(\mathcal{O}_F)$, $\mathcal{H}_K = C_c(K\backslash G(F)/K) \simeq \mathbb{C}[A^*]^W$, $A^* = \mathrm{Hom}(\Lambda_T, \mathbb{C}^\times)$, $C_K = C_c^\infty(X(F))^K$, $\varphi_0 = \mathbf{1}_{X(\mathcal{O}_F)}$, and

$$\text{(SG)} \qquad \mathcal{H}_K \cdot \varphi_0 = C_K.$$

$A_X^* = \mathrm{Hom}(\Lambda_S, \mathbb{C}^\times) \hookrightarrow A^*$, $y \mapsto y \circ \varphi$, with coordinates $u = y(e_1)$, $v = y(e_6)$; $\delta(X)^{1/2} = q^{\rho_L}$; $W_X = S_3$; and

$$r\colon \mathbb{C}[A^*]^W \to \mathbb{C}[\Lambda_S]^{W_X}, \quad r(f)(y) = f(q^{\rho_L} y), \qquad M_\kappa = \textstyle\sum_{\kappa' \in W_X \kappa} y^{\kappa'}.$$

Inputs.

(A):
Sakellaridis's standing hypotheses [Sak, §1.7, §7, §8] for an $\mathcal{O}_F$-model of X at F, stated on Proposition VI: F of characteristic 0 and G split with simply connected derived group (A1) ((A1$_p$): the same with $F = \mathbb{F}_q(\!(t)\!)$); an $\mathcal{O}_F$-model whose open B-orbit is smooth, surjective and B-homogeneous over $\mathcal{O}_F$, with eigenmeasures (A2); rank-one models over $\mathcal{O}_F$ (A3a), the Cartan decomposition (A3b) and Iwahori orbits (A3c) (Axioms 2.2.1, 2.4.1, 2.4.2); no maximal-rank pair (Y, α) of type N, L_γ classical, Statements 6.3.1 and 7.1.5 (A4); X homogeneous affine with $\mathring{X}(F)$ one $B(F)$-orbit (A5). *Status for E_6/F_4: proved* for the model $\mathcal{M}_\theta = (G_{\mathbb{Z}}/G_{\mathbb{Z}}^{\theta_0}) \otimes \mathcal{O}_F$ of Proposition VII, θ_0 an involution of type E IV defined over $\mathbb{Z}$, p odd. Over every p-adic F all of them hold, (A3a) included (Theorem 2.8; audited in Summary 6.9); over $\mathbb{F}_q(\!(t)\!)$, (A1$_p$), (A2), (A3a), (A3b), (A3c), (A5) and (A4) hold, except the analytic part of Statement 6.3.1 (Summary 3.12; audited in Summary 5.13), and Theorem 8.0.2 holds by transfer from the totally ramified fields $\mathbb{Q}_q(p^{1/e})$ (Theorem 4.8; audited in Audit 5.6). The cited inputs are listed in Proposition VII, Step 11.

D0:
Chen–Yi's orbit geometry over $\overline{\mathbb{F}}_q$, $p \neq 2$: Prop. 17, Thm. 18, the support bound $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{\mathcal{O}_{\varphi(\lambda)}}$, the stabilizer lemma. Caveat: Chen–Yi prove Prop. 17(i), the orbit parametrization, by citing Gaitsgory–Nadler, Thm. 8.2.9, and Nadler (2004), Thm. 4.2, which are over $\mathbb{C}$ (found independently by two agents of the run and confirmed by its audit: Audit 3.1, Audit 4.3, Audit 5.7). For E_6/F_4 the function-level parametrization $K\backslash X(\mathbb{F}_q(\!(t)\!)) \simeq \Lambda_S^+$ is now proved directly (Theorem 3.8, Proposition 4.5).

D1:
Proper base change and Grothendieck–Lefschetz on convolution fibres.

D2:
The decomposition of $V_\lambda * \mathrm{IC}_0$ into shifted IC_ν with constant coefficients [Sun, Thm. 1.2], [CY, Prop. 70].

D3:
Lang's theorem for L^+G and the stabilizers (used only in part (c)).

D5:
The comparison $\overline{\mathbb{F}}_q \leftrightarrow \mathbb{C}$ of "IC_μ is a summand of $V_\lambda * \mathrm{IC}_0$" (spreading out, [BBD, §6.1]; [BF, Prop. 5]).

2. Statement

Proposition VIII. *Let $X = E_6/F_4$ as above.*

(a) ***Functions fail.*** *Let p be odd, let F be $\mathbb{F}_q(\!(t)\!)$ or a p-adic field, and let $X = \mathcal{M}_\theta$; or let F be p-adic and X any $\mathcal{O}_F$-model satisfying (A). Put $y_1 = (q^{-5}, q)$ and $y_2 = (q^{-5}, q^4)$ in (u, v)-coordinates. Then $q^{\rho_L} y_1$ and $q^{\rho_L} y_2$ are $W(E_6)$-conjugate, y_1 and y_2 are not*

W_X-conjugate, and

$$M_{(3,0)}(y_1) = q^{-9} + q^3 + q^6 \neq q^{-6} + q^{-3} + q^9 = M_{(3,0)}(y_2).$$

Hence $\operatorname{im} r \subseteq \{g : g(y_1) = g(y_2)\} \subsetneq \mathbb{C}[\Lambda_S]^{S_3}$, *$r$ is not onto, and (SG) fails, for every q; more precisely, C_K needs exactly two generators (Proposition VII). For $\mathcal{M}_\theta$ no hypothesis remains open: (A) is proved, and Theorem 8.0.2 holds over both fields (Proposition VI).*

(b) ***Sheaves hold.*** *Over $\overline{\mathbb{F}}_q$, $p \neq 2$, under* D0–D2*: for odd k put*

$$\mu_k = (3k, 0), \qquad \mu'_k = \mu_k + e_6 = (3k-1, 2), \qquad \lambda_k = \tfrac{3k-1}{2}\omega_1^\vee + \omega_6^\vee.$$

Then $\varphi(\lambda_k) = \mu'_k$, $\{\nu : \mu_k \leq \nu \leq \mu'_k\} = \{\mu_k, \mu'_k\}$, $\operatorname{codim} = 8$, both fibres $F_k = m^{-1}(x_{\mu_k})$ and $F'_k = m^{-1}(x_{\mu'_k})$ lie in Gr^{λ_k}, $F'_k = \{t^{\lambda_k}\}$, and $\dim F_k \geq 7$. Consequently IC_{μ_k} is a direct summand of $V_{\lambda_k} * \mathrm{IC}_0$; with the mirror labels and the covered labels, every IC_μ is a summand of a Satake object and $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$. Through D5 *the same holds in the Betti category over $\mathbb{C}$.*

(c) ***Why the levels split.*** *Over $\mathbb{F}_q(\!(t)\!)$, q odd, under* D0–D3*, $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ says that every row of the multiplicity matrix is nonzero, while (SG) says that its columns span. For E_6/F_4 every row is nonzero, by (b), and the columns do not span, by (a) over $\mathbb{F}_q(\!(t)\!)$: the first does not imply the second. Both halves now concern $\mathbb{F}_q(\!(t)\!)$ and its geometric counterpart over $\overline{\mathbb{F}}_q$.*

3. Proof of (a): functions

Step 1 (Theorem E). Let $X = \mathcal{M}_\theta$ over $\mathbb{F}_q(\!(t)\!)$ or a p-adic F, p odd, or let X be any model satisfying (A) at a p-adic F. By [Sak, Thm. 8.0.2], and over $\mathbb{F}_q(\!(t)\!)$ by Proposition VI (Theorem 4.8), there is an isomorphism of $\mathcal{H}_K$-modules

$$C_K \simeq \mathbb{C}[\delta(X)^{1/2} A_X^*]^{W_X}, \qquad \varphi_0 \mapsto 1,$$

with $\mathcal{H}_K \simeq \mathbb{C}[A^*]^W$ acting through r. So $\mathcal{H}_K\varphi_0$ corresponds to $\operatorname{im} r$, and (SG) holds if and only if r is onto ([Sak, Cor. 8.0.4(1)] for $\Leftarrow$). For symmetric X, $L(X) = L$, $A_X \simeq S$, $\Lambda_X = \varphi(\Lambda_T) = \Lambda_S$ [Sak, §2.1], and $\delta(X)$, the modular character of $L \cap B$, gives $\delta(X)^{1/2} = q^{\pm\rho_L}$; the two signs give intertwined maps, so surjectivity does not depend on the sign. Each class of W_X has a representative w with $w\rho_L = \rho_L$, so $\operatorname{im} r \subseteq \mathbb{C}[\Lambda_S]^{W_X}$.

Step 2 (the restriction map for E_6/F_4). The 12 positive roots of the black D_4 give $2\rho_L = 6\alpha_2 + 6\alpha_3 + 10\alpha_4 + 6\alpha_5$ and $\langle \rho_L, \alpha_j^\vee \rangle_{j=1..6} = (-3, 1, 1, 1, 1, -3)$. So $x = q^{\rho_L} y$ satisfies

$$x^{\alpha_1^\vee} = q^{-3}u, \qquad x^{\alpha_6^\vee} = q^{-3}v, \qquad x^{\alpha_j^\vee} = q \quad (j = 2, \ldots, 5).$$

The elements $w_1 = w_0(D_5^{\{1..5\}})w_0(D_4)$ and $w_6 = w_0(D_5^{\{2..6\}})w_0(D_4)$ fix ρ_L, commute with φ and act by $s_1(a,b) = (-a, a+b)$, $s_6(a,b) = (a+b, -b)$; on A_X^*, $s_1(u,v) = (u^{-1}, uv)$, $s_6(u,v) = (uv, v^{-1})$. So $W_X = S_3$, the restricted root system is A_2 with all multiplicities 8, and the target is $\mathbb{C}[\Lambda_S]^{S_3}$, generated as a ring by $M_{(1,1)}, M_{(3,0)}, M_{(0,3)}$, since $(1,1), (3,0), (0,3)$ generate the monoid Λ_S^+ and $M_\kappa M_{\kappa'} = M_{\kappa+\kappa'} +$ lower terms. Finally $r(m_{\omega_2^\vee}) = c_8(q)M_{(1,1)} + \text{const}$ with $c_8 = q^{-3} + q^{-2} + q^{-1} + 2 + q + q^2 + q^3$, and σ commutes with r; so r is onto if and only if $M_{(3,0)} \in \operatorname{im} r$.

Step 3 (a glued pair of points). Write q^λ for the point x with $x^\mu = q^{\langle\lambda,\mu\rangle}$. By Step 2,

$$q^{\rho_L} y_1 = q^{\lambda_1^*}, \quad \lambda_1^* = \rho - 9\omega_1 - 3\omega_6 = (-8,1,1,1,1,-2),$$
$$q^{\rho_L} y_2 = q^{\lambda_2^*}, \quad \lambda_2^* = \rho - 9\omega_1 = (-8,1,1,1,1,1)$$

in fundamental-weight coordinates. Reflecting at negative coordinates, the words (applied left to right)

$$\lambda_1^*:\ s_1s_3s_4s_2s_5s_4s_3s_1s_6s_5s_4s_2s_3s_1s_4s_3s_5s_6, \qquad \lambda_2^*:\ s_1s_3s_4s_2s_5s_4s_3s_1s_6s_5s_4s_2s_3$$

carry both weights to the same dominant weight $(1,1,1,0,1,1) = \rho - \omega_4$. So the two points are W-conjugate, and every $f \in \mathbb{C}[A^*]^W$ takes equal values there: $g(y_1) = g(y_2)$ for $g \in \operatorname{im} r$. In $(\log_q u, \log_q v)$,

$$W_X y_1 = \{(-5,1),(-4,-1),(-1,5),(1,4),(4,-5),(5,-4)\},$$
$$W_X y_2 = \{(-5,4),(-4,5),(-1,-4),(1,-5),(4,1),(5,-1)\},$$

which are disjoint. With $W_X(3,0) = \{(3,0),(-3,3),(0,-3)\}$ and $y^{(a,b)} = u^{(2a+b)/3} v^{(a+2b)/3}$, $M_{(3,0)} = u^2v + u^{-1}v + u^{-1}v^{-2}$ takes the two values of (a), which differ for $q > 1$. Hence $M_{(3,0)} \notin \operatorname{im} r$ and r is not onto; by Step 1, (SG) fails. Only the W-conjugacy of the integral weights λ_1^*, λ_2^* is used, so this holds for every q. Moreover $\sigma y_1 \in W_X y_2$: the point y_1 is glued to its σ-image. Proposition VII strengthens this: $\operatorname{im} r$ is exactly $\{g : g(y_1) = g(\sigma y_1)\}$, of codimension one.

Step 4 (Sakellaridis's hypotheses hold for the model). For the model $\mathcal{M}_\theta$ of Proposition VII, p odd. The field and group (A1), (A1$_p$) hold: E_6 is split and simply connected; F is p-adic, or $F = \mathbb{F}_q(\!(t)\!)$. The integral model (A2) holds: θ_0 is defined over $\mathbb{Z}$, $G^\theta_{\mathcal{O}_F}$ is smooth with connected fibres of type F_4, X is quasi-affine, and the open Borel orbit is $U_P \times A_X$, smooth and surjective over $\mathcal{O}_F$ with one $B_{\mathbb{F}_q}$-orbit as special fibre. Rank-one models (A3a) hold over every discrete valuation ring with 2 invertible: the 45 Borel orbits $B\dot{w}x_0$ are all of maximal rank, and $YP_\alpha/R(P_\alpha)$ is $\mathrm{PGL}_2\backslash\mathrm{PGL}_2$ (90 pairs) or $B_2\backslash\mathrm{PGL}_2 = \mathbb{P}^1_{\mathcal{O}_F}$ (180 pairs) (Theorem 2.8; audited in Summary 6.9). The Cartan decomposition (A3b), $G(F) = K\,T(F)\,G^\theta(F)$, holds by Delorme–Sécherre with $H^1(F, \mathrm{Spin}_8) = 1$, and the Iwahori orbits (A3c) at every label; over $\mathbb{F}_q(\!(t)\!)$ both are reproved in characteristic p (Theorem 3.8, Audit 3.9). The spherical roots and colours (A4) hold: the spherical roots $\gamma_1 = 2\omega_1 - \omega_6$ and $\gamma_6 = -\omega_1 + 2\omega_6$ have type G, no maximal-rank pair has type N, each L_γ is of type D_5, and Statements 6.3.1 and 7.1.5 hold; over $\mathbb{F}_q(\!(t)\!)$ all but the analytic part of Statement 6.3.1 is reproved (Summary 3.12), and the transfer does not need that part. The single open orbit (A5) holds: for $g = ul \in B \cap G^\theta$ with $u \in U_P$, $l \in B_L$, $\theta(u) = u\,l\theta(l)^{-1} \in U_{P^-} \cap P = 1$, so $B \cap G^\theta = B_L \cap L^\theta$, a Borel subgroup of $L^\theta = \mathrm{Spin}_8$, connected; hence $H^1(F, B \cap G^\theta) = 1$ and $\mathring{X}(F)$ is one $B(F)$-orbit. Over $\mathbb{F}_q(\!(t)\!)$, Theorem 8.0.2 holds by transfer from $\mathbb{Q}_q(p^{1/e})$, $e \gg 0$, where $\mathbb{Q}_q$ is the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ (Lemma 4.2, Proposition 4.5, Proposition 4.6, Theorem 4.8; audited in Audit 5.1, Audit 5.3, Objection 5.4, Audit 5.6). The cited inputs not re-proved are listed in Proposition VII, Step 11.

Remark (the family $H_3(K_d)$). The restricted root system A_2 with multiplicity d occurs for $d = 1, 2, 4, 8$; r is onto for $d = 4$ ($\mathrm{SL}_6/\mathrm{Sp}_6$) and not onto for $d = 8$ (E_6/F_4).

4. Proof of (b): sheaves

Step 5 (the standing inputs). Over $\overline{\mathbb{F}}_q$, H1–H3 of Proposition I hold for E_6/F_4 by D0: closures by [CY, Thm. 18], finite-type models by [CY, Thm. 36], connected stabilizers by [CY, Ex. 88], i.e. $Z_{G^\theta}(S) = \mathrm{Spin}_8$ connected. D2 gives the decomposition with constant coefficients; D1 computes stalks by fibres.

Step 6 (labels). $\Lambda_S \cap \sum \mathbb{Z}_{\geq 0}\alpha_j^\vee = \mathbb{Z}_{\geq 0}e_1 \oplus \mathbb{Z}_{\geq 0}e_6$: $e_1 = 2\alpha_1^\vee + \alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee$, and $p = \frac{1}{2}(1-\theta)$ fixes Λ_S and sends $\sum c_j\alpha_j^\vee$ to $\frac{c_1}{2}e_1 + \frac{c_6}{2}e_6$. So $\nu \leq \mu$ iff $\mu - \nu \in \mathbb{Z}_{\geq 0}e_1 + \mathbb{Z}_{\geq 0}e_6$, and e_6 spans an edge: the interval $[\mu_k, \mu_k']$ is $\{\mu_k, \mu_k'\}$. By [CY, Prop. 64] (LX connected as G is simply connected; its proof is valid for $p \neq 2$), $\operatorname{codim}_{\overline{\mathcal{O}_\mu}} \mathcal{O}_\nu = \langle \rho, \mu - \nu \rangle$, so $\delta(a,b) = 8(a+b)$ and the codimension is $\langle \rho, e_6 \rangle = 8$. The label λ_k is integral (its class $c_k + 2 = \frac{3(k+1)}{2}$ vanishes in $\mathbb{Z}/3$), dominant, $\varphi(\lambda_k) = (2c_k, 2) = \mu_k'$, and $\dim \mathrm{Gr}^{\lambda_k} = \langle 2\rho, \lambda_k \rangle = 16(c_k+1) = 8(3k+1) = \delta(\mu_k')$, with $c_k = \frac{3k-1}{2}$.

Step 7 (both fibres in the open cell). $F_x \simeq \{gL^+G \in \mathrm{Gr}^{\leq \lambda_k} : g^{-1}x \in L^+X\}$ is closed in $\mathrm{Gr}^{\leq \lambda_k}$. If such g lies in $\mathrm{Gr}^{\lambda''}$ with $x \in \{x_{\mu_k}, x_{\mu_k'}\}$, then $x \in L^+Gt^{\lambda''}L^+X$, so by the support bound $\mu_k \leq \varphi(\lambda'') \leq \varphi(\lambda_k) = \mu_k'$. $\varphi(\lambda'') = \mu_k$ is impossible (μ_k is uncovered). If $\varphi(\lambda'') = \mu_k'$ and $\lambda'' < \lambda_k$, then $\kappa = \lambda_k - \lambda'' = \sum_{j=2}^5 c_j\alpha_j^\vee \neq 0$, $c_j \geq 0$, and dominance of λ'' with $\langle \alpha_j, \lambda_k \rangle = 0$ gives $\langle \alpha_j, \kappa \rangle \leq 0$ for $j = 2..5$, contradicting $(\kappa, \kappa) = \sum_j c_j(\alpha_j^\vee, \kappa) > 0$. So $F_k, F_k' \subseteq \mathrm{Gr}^{\lambda_k}$, where $V_{\lambda_k} = \overline{\mathbb{Q}}_\ell[\delta(\mu_k')]$; by proper base change, $(V_{\lambda_k} * \mathrm{IC}_0)_x \simeq R\Gamma(F_x, \overline{\mathbb{Q}}_\ell)[\delta(\mu_k')]$. No stalk of $\mathrm{IC}(\mathrm{Gr}^{\leq \lambda_k})$ enters.

Step 8 (the covering fibre is a point). This replaces a dimension count in a congruence quotient, which is wrong as written. $T' = (T^\theta)^\circ$ fixes every x_μ, since $\tau(sx_\mu) = st^\mu s^{-1} = t^\mu$; so F_k' is T'-stable. No root of E_6 vanishes on T': such a root is $a\omega_1 + b\omega_6$ with $2(a^2+ab+b^2) = 3$. So a generic cocharacter η of T' has $Z_G(\eta) = T$, and its fixed points on Gr are the t^ν [Zhu, proof of Lemma 5.3.7]. A fixed point t^ν of F_k' has $\nu \in W\lambda_k$ and $\varphi(\nu) = \mu_k'$; then $\nu - \lambda_k \in \ker\varphi$, which is orthogonal to λ_k, and $|\nu| = |\lambda_k|$ force $\nu = \lambda_k$. In a T-equivariant projective embedding with linear action (lattice model of $\mathrm{Gr}^{\leq \lambda_k}$, Plücker coordinates), a non-fixed point has two distinct limits under η, both fixed, in its own component. Hence $F_k' = \{t^{\lambda_k}\}$, and $E_{\mu_k', n}(\lambda_k) = \overline{\mathbb{Q}}_\ell$ for $n = 0$, 0 otherwise.

Step 9 (the special fibre is large). $Z_k = Z_{G^\theta}(\mu_k)^\circ$ fixes x_{μ_k} ($\tau(hx_{\mu_k}) = ht^{\mu_k}h^{-1} = t^{\mu_k}$), hence acts on F_k. $Z_G(\mu_k)$ is the Levi D_5 on nodes 2..6, and $\mathrm{Lie}\, Z_k = \mathfrak{t}^\theta \oplus \bigoplus_{\alpha \in \Phi_L} \mathfrak{g}_\alpha \oplus \bigoplus_{\{\alpha, \theta\alpha\}} k(X_\alpha + \theta X_\alpha)$ has dimension $4 + 24 + 8 = 36$: type B_4, $Z_k \simeq \mathrm{Spin}_9$. The points $t^\nu L^+G$ of F_k are the $\nu \in W\lambda_k$ with $\varphi(\nu) = \mu_k$: exactly eight, with $\langle \rho_L, \nu \rangle \in \{-3, -2, -1, 0, 0, 1, 2, 3\}$. At each, $\dim(\mathrm{Lie}\, Z_k \cap \mathrm{Lie}\, P(-\nu)) = 4 + 18 + 7 = 29$, where $P(-\nu)$ is the stabilizer of $t^\nu L^+G$ in G. So $\dim Z_k \cdot t^\nu L^+G \geq 36 - 29 = 7$ and $\dim F_k \geq 7$.

Step 10 (detection lemma). Let $\mathcal{P} \simeq \bigoplus_{\nu,n} \mathrm{IC}_\nu[-n] \otimes E_{\nu,n}$ (finite, constant coefficients) and $\mu < \mu'$ with (1) $E_{\nu,n} = 0$ unless $\nu \leq \mu'$; (2) $[\mu, \mu'] = \{\mu, \mu'\}$; (3) $E_{\mu',n} = 0$ for $n \neq 0$. Then $H^j(\mathcal{P}_{x_\mu}) \simeq E_{\mu, j+\delta(\mu)}$ for $j \geq -\delta(\mu)$. Indeed only $\nu \in \{\mu, \mu'\}$ have stalks at x_μ [CY, Thm. 18]; $\mathrm{IC}_\mu[-n]$ contributes $E_{\mu,n}$ in degree $n - \delta(\mu)$; and $H^j(\mathrm{IC}_{\mu'})_{x_\mu} = 0$ for $j \geq -\delta(\mu)$ by the degree bound $\deg P_{\mu;\mu'} \leq \frac{1}{2}(\delta(\mu') - \delta(\mu) - 1)$ [CY, Cor. 14(ii)].

Step 11 (the uncovered label is a summand). Apply Step 10 to $\mathcal{P} = V_{\lambda_k} * \mathrm{IC}_0$, $\mu = \mu_k$, $\mu' = \mu_k'$: (1) is the support bound, (2) is Step 6, (3) is Step 8. By Step 7, for $n \geq 0$,

$$E_{\mu_k, n}(\lambda_k) = H^{n - \delta(\mu_k) + \delta(\mu_k')}(F_k) = H^{n+8}(F_k).$$

F_k is projective of dimension $d \geq 7$, so $H^{2d}(F_k) \neq 0$ and $E_{\mu_k,2d-8} \neq 0$, $2d-8 \geq 6$; also $E_{\mu_k,0} = H^8(F_k) \neq 0$. So IC_{μ_k} is a direct summand of $V_{\lambda_k} * \mathrm{IC}_0$, a retract, and lies in $\mathcal{C}_K^{\mathrm{Sat}}$.

Step 12 (mirror labels, covered labels, generation). The labels $(0, 3k)$: rerun Steps 6–11 with nodes 1 and 6 exchanged, $\lambda = \omega_1^\vee + c_k\omega_6^\vee$, Levi D_5 on nodes 1..5; the numbers are identical (eight fixed points over $(0, 3k)$, one over $(2, 3k-1)$, orbit bound 7). Transport by σ would first require normalizing σ to preserve G^θ. Covered labels $\mu = \varphi(\lambda)$: the restriction of $V_\lambda * \mathrm{IC}_0$ to $\mathcal{O}_\mu$ is a nonzero sum of shifted constant sheaves and its support lies in $\overline{\mathcal{O}_\mu}$ (Proposition III, step 6 of the spherical workspace), so some $E_{\mu,n}(\lambda) \neq 0$. Thus every IC_μ is a summand, and Proposition I (summand criterion) gives $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$. Each summand statement involves finite-type data defined over $\mathbb{Z}[1/2]$; by D5 it holds over $\mathbb{C}$ for the corresponding label, and Proposition I there gives $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ in the Betti category.

Step 13 (heuristic refinement, not used). This step is a conjecture; nothing else uses it. If $F_k \cong Q^7$, the quadric of isotropic lines of the 9-dimensional representation of Spin_9 ($\chi(Q^7) = 8$ matches the eight fixed points), then $E_{\mu_k,\bullet} = \overline{\mathbb{Q}}_\ell^7$ in degrees $-6, -4, \ldots, 6$, $P_{\mu_k;\mu_k'} = 1$, and the graded trace is $q^{-3} + \cdots + q^3 = c_8(q) - 1$, matching the coefficient of $M_{(3,0)}$ in $r(m_{\omega_1^\vee+\omega_6^\vee})$.

5. Proof of (c): why the levels split

Step 14 (rows and columns). Over $\mathbb{F}_q(\!(t)\!)$, q odd, under D0–D3, put $m_{\nu\lambda} = \sum_i (-1)^i \operatorname{tr}(\mathrm{Fr} \mid E_{\nu,i}(\lambda))$, rows indexed by orbit labels ν, columns by Satake labels λ. By Proposition II (the trace dictionary), $K\backslash X(F) = \Lambda_S^+$, the $\operatorname{tr}\mathrm{IC}_\nu$ are unitriangular in the orbit indicators, $\operatorname{tr}(V_\lambda * \mathrm{IC}_0) = \operatorname{tr}(V_\lambda) * \varphi_0$, and the $\operatorname{tr} V_\lambda$ span $\mathcal{H}_K \otimes \overline{\mathbb{Q}}_\ell$; so the column span is $\mathcal{H}_K\varphi_0 \otimes \overline{\mathbb{Q}}_\ell$, and (SG) over $\mathbb{F}_q(\!(t)\!)$ holds iff the columns span. By Proposition I, $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ iff every row of $(E_{\nu,\bullet}(\lambda))$ is nonzero. At $\nu = \mu_k$ every column with $E_{\mu_k,\bullet}(\lambda) \neq 0$ has $\varphi(\lambda) > \mu_k$, so no column supported in $\{\nu \leq \mu_k\}$ is nonzero at μ_k: a nonzero row detects IC_{μ_k} but produces no triangular generator, and whether a combination of columns isolates μ_k depends on the values at the given q. For E_6/F_4 the columns do not span: (SG) fails over $\mathbb{F}_q(\!(t)\!)$ by (a), and the indicator of the orbit of $\mu_1 = 3\omega_1^\vee$ is not in the column span (Proposition VII(b)(4)). Step 3 kills the span by a Weyl coincidence of shifted points that move with q; in [Sak, Ex. 8.0.1], where $\delta(X)$ is trivial, they do not.

Step 15 (one field). For $\mathcal{M}_\theta$, p odd, every arrow of the chain is available:

$$\mathcal{C}_{K,\mathbb{C}} \overset{\text{D5}}{\longleftrightarrow} \mathcal{C}_{K,\overline{\mathbb{F}}_q} \xrightarrow{\text{tr}} C_K\big(\mathbb{F}_q(\!(t)\!)\big) \overset{\text{close fields}}{\longleftrightarrow} C_K\big(\mathbb{Q}_q(p^{1/e})\big) \overset{\text{Thm. 8.0.2}}{\longleftrightarrow} \operatorname{im} r.$$

For $e \gg 0$ the rings of integers of $\mathbb{F}_q(\!(t)\!)$ and $\mathbb{Q}_q(p^{1/e})$ agree modulo a high power of the uniformizer, and membership in Cartan and Iwasawa cells is decided modulo a power independent of the field (Lemma 4.2). So the Cartan decomposition and every structure constant of the Hecke action on C_K agree over the two fields (Proposition 4.5, Proposition 4.6), and Theorem 8.0.2 over $\mathbb{Q}_q(p^{1/e})$ gives the $\mathcal{H}_K$-isomorphism over $\mathbb{F}_q(\!(t)\!)$ with $\varphi_0 \mapsto 1$ (Theorem 4.8; audited in Audit 5.1, Audit 5.3, Objection 5.4, Audit 5.6). This is the comparison of C_K between p-adic and Laurent-series fields that was missing. The computation of r in Steps 2–3 does not depend on the field, so (SG) fails over $\mathbb{F}_q(\!(t)\!)$ and C_K needs exactly two generators there (Proposition VII); in particular $\dim \mathrm{Hom}_{\mathcal{H}_K}(C_K, \mathbb{C}_\chi) = 2$ at the character of $q^{\rho-\omega_4}$, which a cyclic module cannot have [Sak, Cor. 8.0.4(2)]. A Hecke count on the $E_6(\mathcal{O}_F)$-orbits of $\{N = 1\} \subset H_3(\mathbb{O})$ would be a second route; it is not needed.

Step 16 (conclusion). By (a) and (b), the E_6/F_4 row is a divergence over one field: $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ over $\overline{\mathbb{F}}_q$, $p \neq 2$, under D0–D2 (Betti via D5), while (SG) fails for $\mathcal{M}_\theta$ over $\mathbb{F}_q(\!(t)\!)$ and over every p-adic field, p odd, where C_K needs exactly two generators (Proposition VII). The function half has no open hypothesis, granted the cited inputs of Proposition VII, Step 11. By Proposition II no case of the reverse divergence occurs over $\mathbb{F}_q(\!(t)\!)$. □

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