PROPOSITION VII: E_6/F_4, C_K NEEDS TWO GENERATORS

Proposition VIII shows that the basic function does not generate the unramified Hecke module of E_6/F_4. Here no single function does. Part (a) is algebra about Sakellaridis's restriction map and holds for every q. Part (b) holds over $\mathbb{F}_q(\!(t)\!)$ and over every p-adic field, p odd: by Proposition VI, Sakellaridis's Theorem 8.0.2 holds for an explicit integral model over both (audited run sakellaridis-char-p [SCP]). Part (c) says where each hypothesis of that theorem is proved.

1. Setting

Root data. As in Proposition VIII, §1: G is the split simply connected group of type E_6 (Bourbaki labels), θ the involution of type E IV with $G^\theta = F_4$, black nodes $2,3,4,5$ (a D_4), ρ_L the half-sum of its 12 positive roots, $\varphi = 1-\theta$ on $\Lambda_T = X_*(T)$, and $(a,b) := a\omega_1^\vee + b\omega_6^\vee$. Then

$$e_1 = \varphi(\alpha_1^\vee) = (2,-1), \qquad e_6 = \varphi(\alpha_6^\vee) = (-1,2), \qquad \Lambda_S = \mathbb{Z}e_1 \oplus \mathbb{Z}e_6,$$

$$\Lambda_S^+ = \{(a,b) : a,b \geq 0,\ a \equiv b \bmod 3\}.$$

The diagram automorphism σ ($1 \leftrightarrow 6$, $3 \leftrightarrow 5$) acts on Λ_S by $(a,b) \mapsto (b,a)$.

The ring. A point $q^\lambda \in A^* = \mathrm{Hom}(\Lambda_T, \mathbb{C}^\times)$ is $\mu \mapsto q^{\langle\lambda,\mu\rangle}$. On $A_X^* = \mathrm{Hom}(\Lambda_S, \mathbb{C}^\times)$ put $u = y(e_1)$, $v = y(e_6)$, so $y^{(a,b)} = u^{(2a+b)/3}v^{(a+2b)/3}$; $W_X = S_3$ acts by $s_1(a,b) = (-a, a+b)$, $s_6(a,b) = (a+b, -b)$, and $\sigma(u,v) = (v,u)$. Put

$$S := \mathbb{C}[\Lambda_S]^{W_X}, \qquad M_\kappa := \sum_{\kappa' \in W_X\kappa} y^{\kappa'}, \qquad m_\mu := \sum_{\nu \in W\mu} x^\nu,$$

$$r\colon \mathbb{C}[A^*]^W \to S, \quad r(f)(y) = f(q^{\rho_L}y), \qquad R := \mathrm{im}\, r, \qquad c_8 := q^{-3}+q^{-2}+q^{-1}+2+q+q^2+q^3,$$

$$P := M_{(3,0)} + M_{(0,3)}, \qquad D := M_{(3,0)} - M_{(0,3)}, \qquad y_1 := (q^{-5}, q), \quad \sigma y_1 = (q, q^{-5}) \ \text{ in } (u,v),$$

$$\xi := q^{\rho-\omega_4} \in A^*/W, \qquad \mathcal{K} := \{g \in S :\ g(y_1) = g(\sigma y_1)\}.$$

$\mathfrak{m}_\xi \subset \mathbb{C}[A^*]^W$ is the maximal ideal of ξ; $\mathbb{C}[A^*]^W$ acts on S through r. S is the target $\mathbb{C}[\delta(X)^{1/2}A_X^*]^{W_X}$ of Sakellaridis's restriction map [Sak, (8.2)], translated by $\delta(X)^{1/2} = q^{\rho_L}$ (Proposition VIII, Step 1).

The model. F is a non-archimedean local field with residue field $\mathbb{F}_q$, q a power of an odd prime p: either $\mathbb{F}_q(\!(t)\!)$ or a p-adic field of any ramification. $\mathcal{O}$ is its ring of integers, ϖ a uniformizer ($\varpi = t$ for $\mathbb{F}_q(\!(t)\!)$), and $t^\mu := \mu(\varpi)$ for $\mu \in \Lambda_T$. The model $\mathcal{M}_\theta$: $G_\mathbb{Z}$ is the split simply connected Chevalley group of type E_6 with pinning $(T, B, \{x_{\alpha_i}\})$, C its Chevalley involution ($C(t) = t^{-1}$ on T, $C(x_{\pm\alpha_i}(c)) = x_{\mp\alpha_i}(-c)$), σ_L the Tits lift of the longest element w_L of $W(D_4)$ on the black nodes, and

$$\theta = \theta_0 := \mathrm{Int}(\sigma_L) \circ C.$$

θ_0 is an involution of $G_\mathbb{Z}$ with $\theta_0|_T = -w_L$, trivial on $\mathrm{Spin}_8 = [L,L]$, of type E IV with fixed group of type F_4 on every fibre of characteristic $\neq 2$, and the standard parabolic $P \supseteq B$

of $\{2,3,4,5\}$ is minimal θ_0-split with θ_0-stable Levi L (audit node S4-11). Put $H_{\mathcal{O}} := G^{\theta}_{\mathcal{O}}$, $X = \mathcal{M}_\theta := (G_{\mathbb{Z}}/G^{\theta_0}_{\mathbb{Z}}) \otimes \mathcal{O}$, $x_0 = eH$, $K = G(\mathcal{O})$, J the Iwahori subgroup, and

$$\mathcal{H}_K = C_c(K\backslash G(F)/K) \simeq \mathbb{C}[A^*]^W, \qquad C_K = C_c^\infty(X(F))^K, \qquad \varphi_0 = \mathbf{1}_{X(\mathcal{O})}.$$

For $\kappa \in \Lambda_S^+$ let $x_\kappa := t^\mu x_0$ for any $\mu \in \Lambda_T$ with $\varphi(\mu) = \kappa$, so that $\tau(x_\kappa) = t^\kappa$ for $\tau(gx_0) = g\theta(g)^{-1}$, and $O_\kappa := Kx_\kappa$.

Inputs. Sakellaridis's hypotheses are stated on Proposition VI: (A1), F of characteristic 0 and G split over $\mathcal{O}$ with connected fibres and simply connected derived group ((A1$_p$): the same with $F = \mathbb{F}_q(\!(t)\!)$); (A2), an integral model, quasi-affine, whose open Borel orbit is smooth and surjective over $\mathcal{O}$ with B-homogeneous special fibre, with eigenmeasures; (A3a), rank-one models over $\mathcal{O}$ (Axiom 2.2.1); (A3b), the Cartan decomposition (Axiom 2.4.1); (A3c), Iwahori orbits (Axiom 2.4.2) [Sak, pp. 9–11]; (A4), no maximal-rank pair (Y,α) of type N, classical Levis L_γ, Statements 6.3.1 and 7.1.5; (A5), X homogeneous and affine with $\mathring{X}(F)$ one $B(F)$-orbit. Two inputs of the proof, both proved:

(Sak):

(Theorem 8.0.2 for $\mathcal{M}_\theta$ at F.) There is an isomorphism of $\mathcal{H}_K$-modules $\Phi\colon C_K \xrightarrow{\sim} S$ with $\Phi(\varphi_0) = 1$, $\Phi(h * \phi) = r(\hat{h})\,\Phi(\phi)$ ($\hat{h}$ the Satake transform) and $\Phi(\mathbf{1}_{O_\kappa}) = P_\kappa$ for $\kappa \in \Lambda_S^+$, where P_κ is the polynomial (8.1) of [Sak, p. 64]: its lowest weight for the order $\succ_2$ defined by the cone of colour valuations is κ, with nonzero coefficient, and it depends only on q. Φ is the only isomorphism with $\Phi(\varphi_0) = 1$. *Status: proved* over $\mathbb{F}_q(\!(t)\!)$ and every p-adic F, p odd (Proposition VI; Steps 7, 11, 12).

(A3a):

(Rank-one models over $\mathcal{O}$, for $\mathcal{M}_\theta$.) For each Borel orbit Y_F of maximal rank, with $Y \subseteq X$ its $\mathcal{O}$-scheme (the complement of the closure of the complement of Y_F in its closure), and each simple root α, $YP_\alpha/R(P_\alpha)$ is one of the four standard PGL_2-spaces over $\mathcal{O}$, and YP_α is the closure of its generic fibre minus the closure of the generic boundary. *Status: proved* over every discrete valuation ring with 2 invertible, mixed or equal characteristic, $p = 3$ included (run sakellaridis-char-p, Theorem 2.8; audited in Summary 6.9; Step 12).

2. Statement

Proposition VII. *Let $X = E_6/F_4$, with the notation above.*

(a) ***Algebra, unconditional.*** *For every real $q > 1$:*

(1) *$S^\sigma = \mathbb{C}[M_{(1,1)}, P]$, a polynomial ring, and $S = S^\sigma \oplus S^\sigma D$.*

(2) *$R = \mathcal{K} = S^\sigma \oplus \mathfrak{n}D$, where $\mathfrak{n} = \big(M_{(1,1)} - M_{(1,1)}(y_1),\ P - P(y_1)\big)$ is the maximal ideal of S^σ at y_1. Hence $S = R \oplus \mathbb{C}\,M_{(3,0)}$, and $\mathbb{C}[A^*]^W$ acts on $S/R \simeq \mathbb{C}$ by $f \mapsto f(\xi)$.*

(3) *$S^\times = \mathbb{C}^\times$. Hence S is not a cyclic $\mathbb{C}[A^*]^W$-module, $S = R \cdot 1 + R \cdot M_{(3,0)}$ needs exactly two generators, and $\dim_{\mathbb{C}} S/\mathfrak{m}_\xi S = 2$.*

(b) ***Hecke module.*** *Let p be odd, let F be $\mathbb{F}_q(\!(t)\!)$ or a p-adic field, and let $X = \mathcal{M}_\theta$. Then:*

(1) *C_K is not cyclic: $\mathcal{H}_K\phi \neq C_K$ for every $\phi \in C_K$; and C_K needs exactly two generators.*

(2) *$C_K/\mathcal{H}_K\varphi_0 \simeq \mathbb{C}$, and $\mathcal{H}_K$ acts on it by the character with Satake parameter $q^{\rho-\omega_4}$.*

(3) *$\dim \mathrm{Hom}_{\mathcal{H}_K}(C_K, \chi_\xi) = 2$ for the character χ_ξ of ξ.*

(4) $C_K = \mathcal{H}_K\varphi_0 \oplus \mathbb{C}\,\mathbf{1}_{O_{3\omega_1^\vee}} = \mathcal{H}_K\varphi_0 \oplus \mathbb{C}\,\mathbf{1}_{O_{3\omega_6^\vee}}$; *so* φ_0 *and* $\mathbf{1}_{O_{3\omega_1^\vee}}$ *generate* C_K.

(c) ***Hypotheses.*** *For* $\mathcal{M}_\theta$, *p odd, every hypothesis of* [Sak, Thm. 8.0.2] *holds over every p-adic F: Sakellaridis's hypotheses* (A1), (A2), (A3a), (A3b), (A3c), (A4), (A5) *(field and group, integral model, rank-one models over* $\mathcal{O}$, *Cartan decomposition, Iwahori orbits, spherical roots and colours, one open orbit). Over* $\mathbb{F}_q(\!(t)\!)$, (A1$_p$), (A2), (A3a), (A3b), (A3c), (A5) *and* (A4) *hold, except the analytic part of Statement 6.3.1, which the transfer does not need; and Theorem 8.0.2 holds by transfer from the fields* $\mathbb{Q}_q(p^{1/e})$. *Hence* (Sak) *holds over both fields, and (b) follows from (a).*

Part (a) uses no hypothesis; part (b) uses Proposition VI and the cited inputs listed in Step 11. At the sheaf level the uncovered labels $3k\omega_1^\vee$ are summands of Satake objects (Proposition VIII).

3. Proof of (a): the image of r

Step 1 (the σ-invariants). σ acts on Λ_S as $-w_0$ of the restricted A_2, so on S it acts as $-1 \in \mathrm{GL}(\Lambda_S)$ and $S^\sigma = \mathbb{C}[\Lambda_S]^{W_X\times\{\pm1\}}$. $W_X \times \{\pm1\}$ is the Weyl group of G_2 on Λ_S, with short roots $W_X(1,1) = \{\pm(1,1), \pm(2,-1), \pm(-1,2)\}$ and long roots $\pm(3,0), \pm(0,3), \pm(3,-3)$. The short roots span Λ_S, so Λ_S is the root lattice of G_2, which is its weight lattice; the fundamental weights are $(1,1)$ and $(3,0)$, with $W(G_2)$-orbit sums $M_{(1,1)}$ and P. By [Bou, VI §3.4, Thm. 1], $S^\sigma = \mathbb{C}[M_{(1,1)}, P]$ with $M_{(1,1)}, P$ algebraically independent. S is generated by $M_{(1,1)}, M_{(3,0)}, M_{(0,3)}$ (Proposition VIII, Step 2), and

$$M_{(0,3)} = P - M_{(3,0)}, \qquad M_{(3,0)}M_{(0,3)} = M_{(3,3)} + 3 \in S^\sigma, \qquad M_{(3,0)}^2 = P\,M_{(3,0)} - M_{(3,0)}M_{(0,3)},$$

so $S = S^\sigma + S^\sigma M_{(3,0)} = S^\sigma + S^\sigma D$. As $\sigma D = -D$, splitting into σ-eigenspaces gives $S = S^\sigma \oplus S^\sigma D$. This proves (a)(1).

Step 2 (the even part lies in R). For dominant $\mu \in \Lambda_T$, $r(m_\mu) = \sum_\kappa \big(\sum_{\nu\in W\mu,\ \varphi(\nu)=\kappa} q^{\langle\rho_L,\nu\rangle}\big) M_\kappa$. Summing exactly over the orbits of $\omega_2^\vee$ (72 elements) and $\omega_1^\vee + \omega_6^\vee$ (270 elements):

$$\begin{aligned} r(m_{\omega_2^\vee}) &= c_8 M_{(1,1)} + c_0, & c_0 &= q^{\pm5} + q^{\pm4} + 3q^{\pm3} + 3q^{\pm2} + 4q^{\pm1}, \\ r(m_{\omega_1^\vee+\omega_6^\vee}) &= M_{(2,2)} + c_8 P + d_1 M_{(1,1)} + d_0, & d_0 &= 3\,(q^{\pm6} + q^{\pm4} + q^{\pm2} + 2), \end{aligned}$$

with $d_1 = q^{\pm6} + 2q^{\pm5} + 3q^{\pm4} + 2q^{\pm3} + 3q^{\pm2} + 4q^{\pm1} + 2$, where $q^{\pm k}$ stands for $q^k + q^{-k}$. With $M_{(1,1)}^2 = M_{(2,2)} + 2P + 2M_{(1,1)} + 6$,

$$r(m_{\omega_1^\vee+\omega_6^\vee}) - M_{(1,1)}^2 = (c_8 - 2)\,P + (d_1 - 2)\,M_{(1,1)} + (d_0 - 6), \qquad c_8 - 2 = q^{\pm3} + q^{\pm2} + q^{\pm1} > 0.$$

So $M_{(1,1)} = (r(m_{\omega_2^\vee}) - c_0)/c_8 \in R$ and then $P \in R$, as R is a ring. Hence $S^\sigma \subseteq R$.

Step 3 (gluing bounds the odd part). σ fixes ρ_L and normalizes W and W_X, so $r \circ \sigma^* = \sigma^* \circ r$ and R is σ-stable. With Step 2, $R = S^\sigma \oplus I\,D$ for the ideal $I := \{a \in S^\sigma : aD \in R\}$ of S^σ. By Proposition VIII, Step 3, $q^{\rho_L}y_1$ and $q^{\rho_L}\sigma y_1$ are $W(E_6)$-conjugate to $q^{\rho-\omega_4}$, so $R \subseteq \mathcal{K}$. For $a \in I$, using $a(\sigma y_1) = a(y_1)$ and $D(\sigma y_1) = -D(y_1)$,

$$(aD)(y_1) - (aD)(\sigma y_1) = 2\,a(y_1)\,D(y_1), \qquad D(y_1) = -\big(q^3 - q^{-3}\big)\big(q^{3/2} - q^{-3/2}\big)^2\big(q^3 + 1 + q^{-3}\big) \neq 0,$$

since $M_{(3,0)}(y_1) = q^{-9} + q^3 + q^6$ and $M_{(0,3)}(y_1) = q^{-6} + q^{-3} + q^9$. So $a(y_1) = 0$, and $I \subseteq \mathfrak{n}$.

Step 4 (two exact identities). Put $d := c_8\, M_{(1,1)}(y_1)$, where $M_{(1,1)}(y_1) = q^{\pm 5} + q^{\pm 4} + q^{\pm 1}$, and $c := q^{\pm 9} + q^{\pm 6} + q^{\pm 3} + 2 = P(y_1) + 2$. Exact expansion over the orbits of $\omega_1^\vee + \omega_3^\vee$ (432 elements) and $3\omega_1^\vee$ (27 elements) gives

$$r(m_{\omega_1^\vee+\omega_3^\vee}) = c_8 M_{(4,1)} + d\, M_{(0,3)} + (\text{terms in } M_{(2,2)}, M_{(1,1)}), \qquad r(m_{3\omega_1^\vee}) = M_{(6,0)} + c\, M_{(0,3)},$$

the coefficient d being an identity of Laurent polynomials in q. With $\sigma^* m_\mu = m_{\sigma\mu}$, $M_{(1,1)} D = M_{(4,1)} - M_{(1,4)}$, $M_{(3,0)}^2 = M_{(6,0)} + 2M_{(0,3)}$ and $M_{(0,3)}^2 = M_{(0,6)} + 2M_{(3,0)}$:

$$r(m_{\omega_1^\vee+\omega_3^\vee}) - r(m_{\omega_5^\vee+\omega_6^\vee}) = c_8 \left(M_{(1,1)} - M_{(1,1)}(y_1)\right) D, \qquad r(m_{3\omega_1^\vee}) - r(m_{3\omega_6^\vee}) = \left(P - P(y_1)\right) D.$$

Both left sides lie in R, and $c_8 \neq 0$, so $M_{(1,1)} - M_{(1,1)}(y_1)$ and $P - P(y_1)$ lie in I. They generate $\mathfrak{n}$, because S^σ is the polynomial ring in $M_{(1,1)}, P$. With Step 3, $I = \mathfrak{n}$.

Step 5 (the image of r). By Steps 2–4, $R = S^\sigma \oplus \mathfrak{n} D$. An element $g = a + bD$ $(a, b \in S^\sigma)$ satisfies $g(y_1) - g(\sigma y_1) = 2\, b(y_1) D(y_1)$, so $g \in \mathcal{K}$ iff $b(y_1) = 0$: $\mathcal{K} = S^\sigma \oplus \mathfrak{n} D = R$. Since $S^\sigma/\mathfrak{n} = \mathbb{C}$, $S/R \simeq S^\sigma D/\mathfrak{n} D \simeq \mathbb{C}$, and $M_{(3,0)} = \frac{1}{2}(P + D) \equiv \frac{1}{2} D$ modulo R, so $S = R \oplus \mathbb{C}\, M_{(3,0)}$. The functional $\ell(g) := g(y_1) - g(\sigma y_1)$ has kernel R, and $\ell(r(f)g) = f(q^{\rho_L} y_1)\, \ell(g) = f(\xi)\, \ell(g)$. This proves (a)(2).

Step 6 (units and generators). $S \subseteq \mathbb{C}[\Lambda_S] = \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, whose units are the $c\, y^\kappa$, $c \in \mathbb{C}^\times$. If $c\, y^\kappa$ is W_X-invariant then $s_1\kappa = s_6\kappa = \kappa$, i.e. $\kappa = 0$; so $S^\times = \mathbb{C}^\times$. If $S = R\, s$ for one $s \in S$, then $1 = r(f)s$ for some f, so $s \in S^\times = \mathbb{C}^\times$ and $S = R$, contradicting Step 5. By Step 5, $S = R \cdot 1 + R \cdot M_{(3,0)}$, so exactly two generators are needed. Fibre: y_1 and σy_1 are not W_X-conjugate (Proposition VIII, Step 3), so their maximal ideals $\mathfrak{m}_{y_1} \neq \mathfrak{m}_{\sigma y_1}$ of S both contain $\mathfrak{m}_\xi S$; by the Chinese remainder theorem [Stacks, Tag 00DT], $S/\mathfrak{m}_\xi S \twoheadrightarrow S/\mathfrak{m}_{y_1} \times S/\mathfrak{m}_{\sigma y_1} = \mathbb{C}^2$, so $\dim S/\mathfrak{m}_\xi S \geq 2$. Conversely $S/\mathfrak{m}_\xi S$ is spanned by $1, M_{(3,0)}$ over $R/r(\mathfrak{m}_\xi)$, a quotient of $\mathbb{C}[A^*]^W/\mathfrak{m}_\xi = \mathbb{C}$. This proves (a)(3).

4. Proof of (b): the Hecke module

Step 7 (Theorem 8.0.2 holds; transport). (Sak) holds for $\mathcal{M}_\theta$ over $\mathbb{F}_q(\!(t)\!)$ and every p-adic F, p odd. Over a p-adic F, Steps 11–12 verify every hypothesis of [Sak, Thm. 8.0.2], so the theorem applies, and its proof (p. 64) gives $\Phi(\mathbf{1}_{O_\kappa}) = P_\kappa$; for ramified F the paper's normalizations are checked and the results it takes from [Sak08] are stated there for every local field of characteristic 0 (Audit 4.7, Audit 5.5, Audit 7.3). Over $\mathbb{F}_q(\!(t)\!)$, Proposition VI transfers the theorem from the totally ramified fields $\mathbb{Q}_q(p^{1/e})$, $e \gg 0$, where $\mathbb{Q}_q$ is the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$: their rings of integers agree with $\mathbb{F}_q[\![t]\!]$ modulo a high power of the uniformizer, the Cartan decomposition $K\backslash X(F) \simeq \Lambda_S^+$ holds (Theorem 3.8, Proposition 4.5), every structure constant $(T_\lambda * \mathbf{1}_{O_\mu})(x_\nu)$ agrees with its value over $\mathbb{Q}_q(p^{1/e})$ for e large (Proposition 4.6), and so $\mathbf{1}_{O_\kappa} \mapsto P_\kappa$ is $\mathcal{H}_K$-linear (Theorem 4.8; audited in Audit 5.1, Audit 5.3, Objection 5.4, Audit 5.6). Uniqueness: two such isomorphisms differ by an R-linear automorphism of S fixing 1, which fixes the nonzero ideal $\{g : g(y_1) = g(\sigma y_1) = 0\} \subseteq R$ (Step 5) pointwise, hence all of the domain S. So Φ carries $\mathcal{H}_K\varphi_0$ onto R and $\mathcal{H}_K\phi$ onto $R\,\Phi(\phi)$. The translation $\delta(X)^{1/2} A_X^* \to A_X^*$, the sign of the shift and left versus right actions change r by automorphisms of source and target and change none of the statements (Proposition VIII, Step 1); ξ is fixed by $-w_0$, since σ fixes node 4, so inversion does not move it.

Step 8 (not cyclic; the cokernel). If $\mathcal{H}_K\phi = C_K$, then $S = R\,\Phi(\phi)$, which (a)(3) excludes; and $C_K = \mathcal{H}_K\varphi_0 + \mathcal{H}_K\Phi^{-1}(M_{(3,0)})$ by (a)(3). This is (b)(1). By (a)(2), $C_K/\mathcal{H}_K\varphi_0 \simeq S/R \simeq \mathbb{C}$ with h acting by $\hat{h}(\xi)$: this is (b)(2).

Step 9 (the second generator). By (Sak), $\Phi(\mathbf{1}_{O_{3\omega_1^\vee}}) = P_{(3,0)}$: over a p-adic F from the proof of [Sak, Thm. 8.0.2], over $\mathbb{F}_q(\!(t)\!)$ from the transfer, with the same polynomial (Theorem 4.8, step 5). $P_{(3,0)}$, defined by (8.1), has lowest weight $(3,0)$ for $\succ_2$, with nonzero coefficient [Sak, p. 64]. The colour valuations of E_6/F_4 are e_1, e_6 (Step 11), and the dominant κ' with $(3,0) - \kappa' \in \mathbb{N}e_1 + \mathbb{N}e_6$ are $(3,0)$, $(1,1) = (3,0) - e_1$ and $(0,0) = (3,0) - 2e_1 - e_6$. Explicitly, by (7.4) with $\Theta = \check{\Phi}_X \times \{(+,4)\}$, up to a nonzero scalar and with $\kappa \in \{(3,0),(0,3)\}$ fixed by the left/right convention,

$$P_{(3,0)} \;\propto\; (1+q^{-4})\, M_\kappa + (1-q^{-8})\, M_{(1,1)} + (1-q^{-4}-q^{-8}+q^{-12}),$$

which is $1+t$ times Macdonald's Hall–Littlewood polynomial of A_2 at $\lambda = 3\varpi_1$, $t = q^{-4}$. Its σ-odd part is $\pm\frac{1}{2}(1+q^{-4})D \neq 0$, so $P_{(3,0)} \notin R$ and, as $\dim S/R = 1$, $S = R \oplus \mathbb{C}\, P_{(3,0)}$. Transporting by Φ gives (b)(4); the label $3\omega_6^\vee$ is the same with σ.

Step 10 (two eigenfunctionals). For $y \in \{y_1, \sigma y_1\}$, $\phi \mapsto \Phi(\phi)(y)$ is an $\mathcal{H}_K$-eigenfunctional with character $h \mapsto \hat{h}(q^{\rho_L} y) = \hat{h}(\xi)$. The two differ, on $\Phi^{-1}(M_{(3,0)})$. Every eigenfunctional with character ξ vanishes on $\mathfrak{m}_\xi C_K$, and $\dim S/\mathfrak{m}_\xi S = 2$, so $\dim \mathrm{Hom}_{\mathcal{H}_K}(C_K, \chi_\xi) = 2$: this is (b)(3). The criterion is unconditional: a cyclic module $\mathcal{H}_K/\mathfrak{a}$ has $\dim \mathrm{Hom}(\mathcal{H}_K/\mathfrak{a}, \chi) \leq 1$ for every character χ, so (b)(3) re-proves (b)(1). This is the multiplicity at glued points that Sakellaridis describes after [Sak, Cor. 8.0.4].

5. Proof of (c): the hypotheses

Step 11 (Sakellaridis's hypotheses hold for the model). For $\mathcal{M}_\theta$, p odd, every hypothesis of [Sak, Thm. 8.0.2] other than (A3a) holds over every p-adic F (audit nodes S4-3, S4-4, S4-5, S4-10 to S4-13 of the first check), and over $\mathbb{F}_q(\!(t)\!)$ except the analytic part of Statement 6.3.1 (run sakellaridis-char-p, Summary 3.12; audited in Summary 5.13). The invariants agree in both characteristics: 45 maximal-rank Borel orbits, spherical roots γ_1, γ_6 of type G, colours with valuations e_1, e_6, $W_X \simeq S_3$.

- (A1), (A1$_p$) (field and group): F has characteristic 0, or $F = \mathbb{F}_q(\!(t)\!)$; $G_{\mathcal{O}}$ is split with connected fibres and simply connected derived group, for every p.
- (A2) (integral model): θ_0 is defined over $\mathbb{Z}$ (audit node S4-11); $H_{\mathcal{O}}$ is smooth with connected reductive fibres of type F_4 (first check: [Ed, Prop. 3.4], [St, Thm. 8.1]; over any complete discrete valuation ring with 2 invertible: [Con, Lemma 2.2.4] and a direct proof of connectedness, Proposition 3.2); X is smooth, separated and quasi-affine [Con, Thm. 2.3.1]; $B \cap H = B_L \cap L^\theta$ with $L^\theta = \mathrm{Spin}_8$; $B/(B\cap H) \to X$ is an open immersion with image $U \simeq U_P \times A_X$, and $\mathring{X} = U$ by [EGA, IV, 21.12.7], so $\mathring{X}$ is smooth, surjective, with one $B_{\mathbb{F}_q}$-orbit as special fibre. Invariant measures exist because $\langle 2\rho - 2\rho_L, \alpha_j^\vee \rangle = 0$ for $j = 2, \ldots, 5$. Also $X(\mathcal{O}) = Kx_0$, since $H^1(\mathcal{O}, H_{\mathcal{O}}) = H^1(\mathbb{F}_q, H_{\mathbb{F}_q}) = 1$ (Hensel, Lang), so $\varphi_0 = \mathbf{1}_{O_0}$.
- (A3b) (Cartan decomposition): by Delorme–Sécherre [DS, Thm. 1.1],

  $$G(F) = \textstyle\bigcup_{j\in J} H(F)\, y_j\, T(F)\, K$$

  over the $H(F)$-classes of maximal (θ, F)-split tori, and $|J| = 1$ because $H^1(F, \mathrm{Spin}_8) = 1$ [BT3, Thm. 4.7], the tori forming one class by a direct argument (Lemma 3.3, Proposition 3.4); so $X(F) = K \cdot T(F)x_0$. The elements $\dot{n}_i = \sigma_{\alpha_i}\theta(\sigma_{\alpha_i}) \in H(\mathcal{O}) \cap N_G(T)$ $(i = 1, 6)$ act on labels by the simple reflections of W_X, so every K-orbit meets $A_X^+ x_0$ (audit

nodes S4-12, S4-13). Distinctness: $Kx_\kappa = Kx_{\kappa'}$ iff $\kappa = \kappa'$, since $\tau(kx) = k\tau(x)\theta(k)^{-1}$, $\theta(K) = K$ and $G(F) = \bigsqcup_{\nu\in\Lambda_T^+} Kt^\nu K$. Over $\mathbb{F}_q(\!(t)\!)$ Delorme–Sécherre's theorem covers the field and its proof was checked (Audit 3.5); so $K\backslash X(F) \simeq \Lambda_S^+$ (Theorem 3.8, and by counting from $\mathbb{Q}_q(p^{1/e})$, Proposition 4.5).

- (A3c) (Iwahori orbits): $Jx_\kappa \subseteq B(\mathcal{O})x_\kappa$ for every $\kappa \in \Lambda_S^+$, from $J = U_P(\mathcal{O})(J\cap L)U_P^-(\mathfrak{p})$ at labels with a dominant lift, and at the uncovered labels $3k\omega_1^\vee, 3k\omega_6^\vee$ (k odd) from the wall parabolic with Levi D_5 and a central half-translation z, $z^2 = \zeta(\varpi)$, $\zeta = 3\omega_1^\vee$; over $\mathbb{F}_q(\!(t)\!)$ this needs $F(\sqrt{t})/F$ separable, true as p is odd (Audit 3.9).
- (A4) (spherical roots and colours): $\Delta_X = \{\gamma_1 = 2\omega_1 - \omega_6,\ \gamma_6 = -\omega_1 + 2\omega_6\}$ with $\gamma_1 = \alpha_1 + w_L\alpha_1$, $(\alpha_1, w_L\alpha_1) = 0$, both of type G; the $|W|/|W_X||W_{P(X)}| = 51840/(6\cdot 192) = 45$ maximal-rank Borel orbits meet only rank-one types G and U, since no root lies in $\mathbb{Q}\omega_1 \oplus \mathbb{Q}\omega_6$, so none has type N; L_{γ_i} of type D_5; two colours with valuations $e_1 = \check\gamma_1$, $e_6 = \check\gamma_6$; $\Theta = \check\Phi_X \times \{(+,4)\}$; Statement 7.1.5 holds, and Statement 6.3.1 holds by [Sak, Thm. 6.3.2], with $B_{w_\gamma}(\tilde\chi) = -e^{\tilde\gamma}(1-q^{-4}e^{-\tilde\gamma})/(1-q^{-4}e^{\tilde\gamma})$ (cross-check: [Sak, §6.19], $n = 5$). Over $\mathbb{F}_q(\!(t)\!)$ the types, colours and valuation cone are reproved without characteristic-0 inputs, and of Statement 6.3.1 the combinatorial part is checked (Proposition 3.6, Proposition 3.7, Proposition 3.10).
- (A5) (one open orbit): X_F is affine as F_4 is reductive, and $\mathring{X}(F)$ is one $B(F)$-orbit since $B\cap H$ is connected, solvable and split, for every odd p (Lemma 3.3).

Cited, not re-proved: Delorme–Sécherre [DS, Thm. 1.1]; Bruhat–Tits [BT, BT3]; Hensel and Lang; Conrad, Edixhoven and Steinberg [Con, Ed, St]; EGA IV [EGA]; SGA 3, Exp. XXIII [SGA3]. The first check's characteristic-0 citations (Kneser, Vust, Richardson, Helminck–Wang, Alper, Knop) are replaced by arguments valid for every odd p, in both characteristics (Audit 3.1, Summary 3.12), and its simplification "F unramified over $\mathbb{Q}_p$" is removed (Audit 4.7, Audit 5.5).

Step 12 (rank-one models over $\mathcal{O}$). (A3a) holds for $\mathcal{M}_\theta$ over every discrete valuation ring with 2 invertible, $p = 3$ included (Theorem 2.8; audit Summary 6.9). Counting θ-eigenspaces on $\mathfrak{g}$ ($\dim\mathfrak{g}^\theta = 52$, $\dim\mathfrak{g}^{-\theta} = 26$) shows that every θ-stable maximal torus is F_4-conjugate to T. Over an algebraically closed field of characteristic $\neq 2$ every Borel orbit meets $N_G(T)x_0$, so the Borel orbits are the 45 orbits $B\dot w x_0$, all of maximal rank. Their stabilizers B_x are smooth over $\mathbb{Z}[1/2]$, and $B/B_x \to X$ is an immersion with smooth image. For each of the $45\times 6 = 270$ pairs (Y,α),

$$YP_\alpha/R(P_\alpha) \simeq \begin{cases} \mathrm{PGL}_2\setminus\mathrm{PGL}_2 & \text{(90 pairs, type } G\text{)},\\ B_2\backslash\mathrm{PGL}_2 = \mathbb{P}^1_{\mathcal{O}} & \text{(180 pairs, type } U\text{)}, \end{cases}$$

and types T, N never occur; the table was recomputed independently (Audit 6.1). By induction on closures from the closed orbit, the flat closure of each generic orbit has the expected special fibre, so Sakellaridis's $\mathcal{O}$-scheme Y is the orbit scheme and the complement clause, read inside closures, holds for all 270 pairs (Lemma 2.7; audited in Audit 6.6, Audit 6.7).

Step 13 (conclusion). For E_6/F_4, model $\mathcal{M}_\theta$, over $\mathbb{F}_q(\!(t)\!)$ and over every p-adic field, p odd: (SG) fails, and C_K needs exactly two generators, φ_0 and $\mathbf{1}_{O_{3\omega_1^\vee}}$; the cokernel of $\mathcal{H}_K\varphi_0$ is the character $q^{\rho-\omega_4}$. No hypothesis remains open: the proof rests on Proposition VI and on the cited inputs of Step 11. The algebra of (a) holds for every q and strengthens Proposition VIII,

Step 3, which stands as stated. At the sheaf level, over $\overline{\mathbb{F}}_q$, every IC sheaf is a summand of a Satake object (Proposition VIII). □

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