PROPOSITION IX: SHEAF GENERATION DOES NOT IMPLY FUNCTION GENERATION

Proposition IIc proves, under (R), (C) and D0–D3, that spherical generation over $\mathbb{F}_q(\!(t)\!)$ (any odd q) forces sheaf generation over $\overline{\mathbb{F}}_q$. Proposition IX, of separate interest, says that the converse fails under the same hypotheses; one counterexample proves it. Section 2 proves part (a), E_6/F_4, from Propositions VI–VIII, and Section 3 proves part (b), the unitary twists of the group case, an elementary counterexample outside (R). Section 4 proves the function half of a split, non-symmetric example of Sakellaridis, which is not part of the statement. Sections 5–12 are a self-contained account of the categorical trace of Gaitsgory–Kazhdan–Rozenblyum–Varshavsky [GKRV] and Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky [AGKRRV3]. They explain why the trace does not turn the categorical statement into the classical one, and when it does. Part (b) and Section 4 come from a /solve-maths run (three searchers, two auditors), where the proposition has the working name IId; step numbers such as "Theorem 2.6" refer to that run. The run ended before Proposition VI covered $\mathbb{F}_q(\!(t)\!)$ and left the case of (R) open; part (a) closes it.

1. Setting and statement

q is odd, $k=\overline{\mathbb{F}}_q$, $F=\mathbb{F}_q(\!(t)\!)$, $\mathcal{O}=\mathbb{F}_q[\![t]\!]$, and $F_n=\mathbb{F}_{q^n}(\!(t)\!)$. $X=G/H$ is a spherical variety over $\mathbb{F}_q$ with base point x_0, $K=G(\mathcal{O})$ and $K_n=G(\mathbb{F}_{q^n}[\![t]\!])$. On the function side,

$$\mathcal{H}_{K_n}=C_c(K_n\backslash G(F_n)/K_n),\qquad C_{K_n}=C_c^{\infty}(X(F_n))^{K_n},\qquad \varphi_0=\mathbf{1}_{X(\mathbb{F}_{q^n}[\![t]\!])},$$

$$(\mathrm{SG})_n:\quad \mathcal{H}_{K_n}*\varphi_0=C_{K_n},$$

with $(\mathrm{SG})=(\mathrm{SG})_1$. On the sheaf side, $\mathcal{C}_{K,k}$ is the category of bounded L^+G-equivariant constructible $\overline{\mathbb{Q}}_\ell$-complexes on LX supported on finitely many orbit closures [CY], Sph_k the spherical Hecke category on Gr with Satake sheaves V_λ, IC_0 the constant sheaf on L^+X, and

$$\mathcal{C}_{K,k}^{\mathrm{Sat}}=\mathrm{Thick}_{\mathcal{C}_{K,k}}\{V_\lambda*\mathrm{IC}_0:\lambda\in\Lambda_T^+\},$$

the closure under shifts, cones and retracts. A Weil object is (M,Φ) with $\Phi\colon \mathrm{Fr}^*M\xrightarrow{\sim}M$, and $\mathrm{tr}(M)(x)=\sum_i(-1)^i\,\mathrm{tr}(\Phi_{\bar{x}},\mathcal{H}^i(M)_{\bar{x}})$ for $x\in LX(\mathbb{F}_q)=X(F)$.

The hypotheses of Proposition II are: (R) G,θ are defined over $\mathbb{F}_q$, the tori $T\supseteq S$ are split, and there is a θ-stable Borel pair (B_0,T_0) over $\mathbb{F}_q$ with T_0 split; (C) G^θ and $Z_{G^\theta}(S)$ are connected; D0–D3, the orbit geometry over k, proper base change with the trace formula on the convolution fibres, the decomposition theorem for $V_\lambda*\mathrm{IC}_0$, and Lang's theorem for L^+G and the stabilizers.

Proposition IX. *Under (R), (C) and D0–D3, sheaf generation over $\overline{\mathbb{F}}_q$ does not imply spherical generation over F:*

$$\mathcal{C}_{K,k}^{\mathrm{Sat}}=\mathcal{C}_{K,k}\quad\not\Longrightarrow\quad\mathcal{H}_K*\varphi_0=C_K.$$

(a) *Let G be the split simply connected group of type E_6, $\theta=\theta_0$ the involution of type E IV of Proposition VII, with $G^\theta=F_4$, and $X=G/G^\theta$ with its model $\mathcal{M}_\theta$. Then X satisfies*

(R) and (C), $\mathcal{C}^{\mathrm{Sat}}_{K,k} = \mathcal{C}_{K,k}$*, and*

$$C_K = \mathcal{H}_K * \varphi_0 \oplus \mathbb{C}\,\mathbf{1}_{O_{3\omega_1^\vee}},$$

so $(\mathrm{SG})_n$ *fails for every* n*.*

(b) *Let* $H \neq 1$ *be split connected reductive over* $\mathbb{F}_q$ *with a Chevalley involution* ϑ*,* $G = \mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} H$*,* σ *its Galois involution,* $\theta = \vartheta \circ \sigma$ *and* $X = G/G^\theta$*. Then* $\mathcal{C}^{\mathrm{Sat}}_{K,k} = \mathcal{C}_{K,k}$*,* (SG) *fails over* F*, and* $(\mathrm{SG})_n$ *holds exactly for even* n*.* X *satisfies (C) and D0–D3, and every clause of (R) except the splitness of* T *and* T_0*.*

Remark 1.1 (not part of the statement). For Sakellaridis's split, non-symmetric $X = \mathrm{SL}_2^3\backslash(\mathrm{Sp}_4 \times \mathrm{Sp}_4)$ [Sak, Ex. 8.0.1], $(\mathrm{SG})_n$ fails for every q and n (Section 4), and $\mathcal{C}^{\mathrm{Sat}}_{K,k} = \mathcal{C}_{K,k}$ given Chen–Yi's ℓ-adic framework for this X. The categorical trace of Frobenius turns $F \mapsto F * \mathrm{IC}_0$ into $f \mapsto f * \varphi_0$, but generation does not make that map onto: the image is controlled by Frobenius-twisted endomorphisms, not by objects (Theorem 9.1). In (b) Satake objects lose their Weil structure; in (a) every Satake object is Weil and every row of the multiplicity matrix is nonzero, yet two spectral parameters glue (Proposition VIII(a)); in Section 4 two classes fuse.

2. Part (a): E_6/F_4

Lemma 2.1. *$X = E_6/F_4$ with $\theta = \theta_0$ satisfies (R) and (C): $T \supseteq S$ are split, T lies in a θ-stable Borel subgroup defined over $\mathbb{F}_q$, and $G^\theta = F_4$ and $Z_{G^\theta}(S) = \mathrm{Spin}_8$ are connected.*

Proof. $\theta_0 = \mathrm{Int}(\sigma_L) \circ C$ is defined over $\mathbb{Z}$ and acts on $X_*(T)$ by $-w_L$, where w_L is the longest element of $W(D_4)$ on the black nodes $2, 3, 4, 5$ (Proposition VII). Put $V_L = \bigoplus_{i=2}^5 \mathbb{R}\alpha_i^\vee$. Since $w_L = -1$ on V_L and $w_L = 1$ on $V_L^\perp$,

$$X_*(T)^\theta \otimes \mathbb{R} = V_L, \qquad X_*(S) \otimes \mathbb{R} = V_L^\perp,$$

and $S \subseteq T$ is split because T is. No root of E_6 vanishes on V_L: such a root would be orthogonal to the D_4 on nodes $2, 3, 4, 5$, and of the 72 roots none is (a direct check; the 48 roots outside D_4 restrict to S as the six roots of A_2, each with multiplicity 8). Choose $v \in V_L$ off the finitely many root hyperplanes and put $\Phi_v^+ = \{\alpha : \langle \alpha, v\rangle > 0\}$. Since $\theta v = v$, $\theta\Phi_v^+ = \Phi_v^+$, and since θ_0 maps U_α to $U_{\theta\alpha}$, the Borel subgroup $B_0 = T \cdot \prod_{\alpha \in \Phi_v^+} U_\alpha$ is θ-stable. It is defined over $\mathbb{F}_q$ because T is split, and $(B_0, T_0) = (B_0, T)$ is the Borel pair of (R). $G^\theta = F_4$ is connected because G is simply connected [Ste, Thm. 8.1], and $Z_{G^\theta}(S) = \mathrm{Spin}_8$ (Proposition VIII). □

Proof of part (a). Functions. For $\mathcal{M}_\theta$ over F, p odd, Proposition VII(b) gives $C_K = \mathcal{H}_K * \varphi_0 \oplus \mathbb{C}\,\mathbf{1}_{O_{3\omega_1^\vee}}$. Its input is Sakellaridis's Theorem 8.0.2 over F, which Proposition VI proves by transfer from the fields $\mathbb{Q}_q(p^{1/e})$; the algebra of Proposition VII(a) holds for every $q > 1$. The same argument over F_n, with q^n in place of q, shows that $(\mathrm{SG})_n$ fails for every n.

Sheaves. Under D0–D2, Proposition VIII(b) shows that for odd k, $\mathrm{IC}_{3k\omega_1^\vee}$ is a direct summand of $V_{\lambda_k} * \mathrm{IC}_0$, and likewise for the mirror labels $3k\omega_6^\vee$. Every other label has a dominant lift. So every IC_μ is a summand of a Satake object, and Proposition I gives $\mathcal{C}^{\mathrm{Sat}}_{K,k} = \mathcal{C}_{K,k}$.

Hypotheses. Lemma 2.1. □

3. Part (b): the unitary twist

Lemma 3.1 (run, Construction 2.4 and Lemma 2.5)**.** *Over k, $G_k = H \times H$ with σ the swap and $\mathrm{Fr}(g_1, g_2) = (\mathrm{Fr}g_2, \mathrm{Fr}g_1)$. The map $(g_1, g_2) \mapsto g_1\vartheta(g_2)^{-1}$ identifies X_k with H acted on by $(g_1, g_2) \cdot x = g_1 x \vartheta(g_2)^{-1}$. The orbits on LX are $\mathcal{O}_c = L^+H\, t^c\, L^+H$, $c \in \Lambda_H^+$, with connected stabilizers, and $V_{(c,0)} * \mathrm{IC}_0 \simeq \mathrm{IC}_c$. Hence $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$. Over F, with $E = \mathbb{F}_{q^2}(\!(t)\!)$,*

$$X(F) = \{x \in H(E) : \vartheta(\sigma x) = x^{-1}\}, \qquad g \cdot x = g\, x\, \vartheta(\sigma g)^{-1}, \qquad K = H(\mathcal{O}_E),$$

and the K-orbits are $O_c = K \cdot t^c$, $c \in \Lambda_H^+$. For $H = \mathrm{GL}_n$ these are the nondegenerate hermitian matrices with $g \cdot x = gxg^*$.

Proof. $\mathbb{F}_{q^2} \otimes k = k \times k$ gives $G_k = H \times H$; Frobenius of the Weil restriction is the swap composed with Fr, and ϑ commutes with σ, so $\theta(g_1, g_2) = (\vartheta g_2, \vartheta g_1)$. Then $\tau(g) = g\theta(g)^{-1} = (x, \vartheta(x)^{-1})$ with $x = g_1\vartheta(g_2)^{-1}$, which gives the model of X_k. Since $\vartheta(L^+H) = L^+H$, the L^+G-orbits are the double cosets $L^+Ht^cL^+H$ (Cartan decomposition), and $\mathrm{Stab}(t^c) \simeq L^+H \cap t^cL^+Ht^{-c}$ is connected. Over $\mathrm{Gr}_{\le (c,0)}$ the convolution space is $\overline{L^+Ht^cL^+H}$, mapped isomorphically onto $\overline{\mathcal{O}_c}$; since IC_0 is constant, $V_{(c,0)} * \mathrm{IC}_0$ is the smooth pullback of $\mathrm{IC}(\mathrm{Gr}_{\le c})$, that is IC_c. Every IC_c is thus a Satake object, and Proposition I gives $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$. Over F, $G(F) = H(E)$ embeds by $e \mapsto (e, \sigma e)$, which gives the equations; $\vartheta(\sigma t^c) = t^{-c}$, so $t^c \in X(F)$ is a rational point of $\mathcal{O}_c$, and connected stabilizers with Lang's theorem make $\mathcal{O}_c(\mathbb{F}_q)$ one K-orbit. □

Theorem 3.2 (run, Theorem 2.6, audited twice)**.** *Let $T_a = \mathbf{1}_{H(\mathcal{O}_E)t^aH(\mathcal{O}_E)}$, $a \in \Lambda_H^+$, the basis of $\mathcal{H}_K$. Then:*

1. $\mathrm{supp}(T_a * \varphi_0) \subseteq \bigcup_{c \le 2a} O_c$ *and* $(T_a * \varphi_0)(t^{2a}) \ge 1$*;*
2. $\mathbf{1}_{O_c} \in \mathcal{H}_K * \varphi_0 \Rightarrow c \in 2\Lambda_H^+$*, so* (SG) *fails over F;*
3. $(\mathrm{SG})_n$ *fails for odd n and holds for even n.*

Proof. Hecke basis. σ swaps the two factors of $\Lambda_T = \Lambda_H \oplus \Lambda_H$, so the rational dominant coweights are (a, a), $a \in \Lambda_H^+$, and $Kt^{(a,a)}K = H(\mathcal{O}_E)t^aH(\mathcal{O}_E)$.

Support and top value. $\mathrm{supp}(T_a * \varphi_0) = Kt^aX(\mathcal{O})$. For $k' \in K$ and $y \in X(\mathcal{O}) \subseteq H(\mathcal{O}_E)$, using $\sigma(t^a) = t^a$ and $\vartheta(t^a) = t^{-a}$,

$$k't^a \cdot y = k'\, t^a\, y\, t^a\, \vartheta(\sigma k')^{-1} \in Kt^aKt^aK \subseteq \bigcup_{\nu \le 2a} Kt^\nu K,$$

the last inclusion because convolution maps $\mathrm{Gr}_{\le a} \mathbin{\widetilde{\times}} \mathrm{Gr}_{\le a}$ into $\mathrm{Gr}_{\le 2a}$. Since $t^a \cdot 1 = t^{2a}$ and all contributions are ≥ 0, $(T_a * \varphi_0)(t^{2a}) \ge 1$.

Leading terms. The functions $T_a * \varphi_0$ have distinct leading orbits $2a$ in a partial order with finite lower sets; a linear combination of them has a leading term at some $2a$. So an orbit indicator $\mathbf{1}_{O_c}$ in their span forces $c = 2a$. If every dominant coweight were in $2\Lambda_H$, then for $\lambda \in \Lambda_H$ and $N \gg 0$ both $\lambda + 2N\xi$ and $2N\xi$ ($\xi = 2\rho^\vee$, or any ξ for a torus) would be dominant, so $\Lambda_H = 2\Lambda_H$ and $H = 1$. Hence some dominant $c \notin 2\Lambda_H$ exists and (SG) fails.

Other n. For odd n, $\mathbb{F}_{q^2} \otimes \mathbb{F}_{q^n}$ is a field and the argument repeats. For even n, $G_{F_n} \simeq H \times H$, $X(F_n) \simeq H(F_n)$ with $g_1 x \vartheta(g_2)^{-1}$, and $T_{(c,0)} * \varphi_0 = \mathbf{1}_{K_H t^c K_H}$, so $(\mathrm{SG})_n$ holds. □

Remark 3.3. For $H = \mathbb{G}_m$, $X(F) = F^\times$ under $z \cdot x = N(z)x$, and $\mathcal{H}_K * \varphi_0$ is the span of the indicators of even valuations. For $H = \mathrm{SL}_2$, $X(F)$ is a single $G(F)$-orbit, every row of the matrix $m^{(1)}$ is nonzero, and still $\mathbf{1}_{K \cdot \mathrm{diag}(t,t^{-1})} \notin \mathcal{H}_K * \varphi_0$. The split, Galois and unitary forms of $(H \times H)/H$ have the same $\mathcal{C}_{K,k}$, while (SG) holds, holds and fails respectively.

4. A split, non-symmetric example

Let V_1, V_2 be symplectic of dimension 4 and $G = \mathrm{Sp}(V_1) \times \mathrm{Sp}(V_2)$. Then X is the variety of triples (W_1, W_2, φ) with $W_i \subset V_i$ nondegenerate planes and $\varphi\colon W_1 \to W_2$ an isometry, with stabilizer $H = \mathrm{Sp}(W_1^{\perp}) \times \mathrm{Sp}(W_2^{\perp}) \times \mathrm{SL}_2^{\Delta} \simeq \mathrm{SL}_2^3$. Its dual group is the image of $\iota\colon \mathrm{SL}_2 \times \mathrm{SL}_2^{\mathrm{diag}} \times \mathrm{SL}_2 \to \mathrm{SO}_5 \times \mathrm{SO}_5$, $(g_1, g_2, g_3) \mapsto ([g_1, g_2], [g_2, g_3])$; for $a \neq \pm b^{\pm 1}$ the points (a, b, a) and (b, a, b) of $\check{G}_X$ are $\check{G}$-conjugate but not $\check{G}_X$-conjugate (run, Lemma 3.3).

Lemma 4.1 (run, Lemma 3.1, audited)**.** *Let F be a non-archimedean local field and $X(F) = G(F)x_0$ one orbit. If $\mathcal{H}_K * \varphi_0 = C_K$, then $\dim \mathrm{Hom}_{H(F)}(\pi, \mathbb{C}) \leq 1$ for every irreducible unramified π.*

Proof. For $\pi^K = \mathbb{C}v_0$, $\ell \mapsto \Psi_\ell$, $\Psi_\ell(gx_0) = \ell(\pi(g)^{-1}v_0)$, embeds $\mathrm{Hom}_{H(F)}(\pi, \mathbb{C})$ into the K-invariant χ_π-eigenfunctions, and $\Psi \mapsto \int_X (-)\Psi\, dx$ embeds those into $\mathrm{Hom}_{\mathcal{H}_K}(C_K, \chi_\pi \circ \vee)$ (in the integral form of the Hecke action). If $C_K = \mathcal{H}_K * \varphi_0$ is cyclic, an $\mathcal{H}_K$-linear functional to a character is determined by its value at φ_0, so that space has dimension ≤ 1. □

Theorem 4.2 (run, Theorem 3.5, audited)**.** *For every non-archimedean F, in particular $F = \mathbb{F}_{q^n}(\!(t)\!)$ for every q and n, and for χ outside a proper Zariski-closed set of unramified characters of the torus of* Sp_4*, with $\pi = I(\chi)$,*

$$\dim \mathrm{Hom}_{H(F)}(\pi \boxtimes \pi, \mathbb{C}) \geq 2, \qquad \textit{hence} \qquad \mathcal{H}_K * \varphi_0 \subsetneq C_K.$$

Proof. Two quotients (run, Lemma 3.4). For $V = P \oplus P^{\perp}$, the restriction of $I(\chi)$ to $\mathrm{Sp}(P) \times \mathrm{Sp}(P^{\perp})$ has a Mackey filtration along the four orbits on the flag variety: open, codimension one, and two closed. The open and codimension-one orbits have stabilizers whose tori act on the flag through (t, t^{-1}) and (t, t). Ext-vanishing between distinct characters of a split torus shows that, for generic χ, there are nonzero maps $f_j\colon I(\chi) \to \mathbf{1} \boxtimes \sigma_{c_j}$, $j = 1, 2$, with $c_1 = \chi_1\chi_2^{-1}|\cdot|^{s_1}$, $c_2 = \chi_1\chi_2|\cdot|^{s_2}$, and $\sigma_{c_1} \not\simeq \sigma_{c_2}$ irreducible principal series of $\mathrm{SL}_2(F)$.

Two functionals. Apply this to $V_i = W_i^{0\perp} \oplus W_i^0$ on both factors, and let β_j be an invariant pairing on $\sigma_{c_j} \simeq \sigma_{c_j}^{\vee}$. Then $\ell_j(v \otimes w) = \beta_j(f_j(v), f_j'(w))$ is $H(F)$-invariant. $(f_1, f_2)\colon \pi \to \sigma_{c_1} \oplus \sigma_{c_2}$ is onto, since the factors are irreducible and non-isomorphic, so ℓ_1, ℓ_2 are pulled back from functionals on different summands and are independent.

Conclusion. $\pi \boxtimes \pi$ is irreducible and unramified, and $X(F)$ is one orbit, so Lemma 4.1 shows that C_K is not cyclic. Nothing depends on the characteristic of F. □

Remark 4.3 (the sheaf half). Over k, every convolution map $\mathrm{Gr}_{\leq\lambda} \,\widetilde{\times}\, L^+X \to LX$ is semismall. The number of top-dimensional components of its fibre over $\mathcal{O}_\nu$ is the branching number $c_{\nu\lambda} = [\mathrm{Res}^{\mathrm{SO}_5 \times \mathrm{SO}_5}_{\iota\check{G}_X} V_\lambda : V_\nu]$, and every orbit label occurs (run, Propositions 3.6 and 3.8, checked by brute-force counts at $q = 3, 5, 7$). Given Chen–Yi's framework for this X, namely Sakellaridis–Wang's degeneration in odd characteristic, the dimension theory $\delta = 3d_1 + 3d_2 + e$, and Chen–Yi's Appendix A with Prop. 70, the decomposition theorem gives $V_\lambda * \mathrm{IC}_0 \simeq \bigoplus_\nu \mathrm{IC}_\nu^{c_{\nu\lambda}}$ and Proposition I gives $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$.

5. Traces in a symmetric monoidal category

Definition 5.1 ([GKRV, §3.1.1])**.** In a symmetric monoidal category $(\mathcal{O}, \otimes, \mathbf{1})$, an object o is *dualizable* if there are $o^{\vee}$, unit$\colon \mathbf{1} \to o \otimes o^{\vee}$ and counit$\colon o \otimes o^{\vee} \to \mathbf{1}$ satisfying the zigzag

identities. The trace of $F \in \mathrm{End}(o)$ is

$$\mathrm{Tr}(F, o) = \big(\mathbf{1} \xrightarrow{\text{unit}} o \otimes o^\vee \xrightarrow{F\otimes \mathrm{id}} o \otimes o^\vee \xrightarrow{\text{counit}} \mathbf{1}\big) \in \mathrm{End}_{\mathcal{O}}(\mathbf{1}).$$

For finite-dimensional vector spaces this is the usual trace, and for perfect complexes the alternating trace on cohomology. It is multiplicative, $\mathrm{Tr}(F_1 \otimes F_2) = \mathrm{Tr}(F_1)\,\mathrm{Tr}(F_2)$, and cyclic, $\mathrm{Tr}(f \circ g, o_2) = \mathrm{Tr}(g \circ f, o_1)$ [GKRV, §3.1.2, §3.1.4]. At this level a trace is a number, and nothing compares the traces of two different objects.

6. The trace of an endofunctor

Definition 6.1 ([GKRV, §3.4.1], [AGKRRV3, §0.2.2], [GR, Ch. 1])**.** DGCat is the symmetric monoidal 2-category of presentable stable $\overline{\mathbb{Q}}_\ell$-linear categories, continuous functors and natural transformations, with Lurie's tensor product and unit Vect. Since $\mathrm{End}(\mathrm{Vect}) = \mathrm{Vect}$, a continuous endofunctor F of a dualizable $\mathcal{C}$ has a trace $\mathrm{Tr}(F, \mathcal{C}) \in \mathrm{Vect}$, a chain complex.

Proposition 6.2 ([GKRV, §3.5.1]). A compactly generated $\mathcal{C}$ is dualizable, with $\mathcal{C}^\vee = \mathrm{Ind}((\mathcal{C}^c)^{\mathrm{op}})$. Let g be a compact generator, $E = \mathrm{End}(g)$, and $Q = \mathrm{Hom}(g, Fg)$, an E-bimodule by $a \cdot q \cdot b = F(a) \circ q \circ b$. Then

$$\mathrm{Tr}(F, \mathcal{C}) = E \otimes^{\mathbf{L}}_{E\otimes E^{\mathrm{op}}} Q = \mathrm{HH}_\bullet(E; Q),$$

and if E and Q are concentrated in degree 0, $H_0\,\mathrm{Tr}(F, \mathcal{C}) = Q/\langle F(a) \circ q - q \circ a\rangle$. The left side depends only on $(\mathcal{C}, F)$, so the right side does not depend on g (Morita invariance). With several generators, use the category on them and $Q(i, j) = \mathrm{Hom}(g_i, Fg_j)$.

For $\mathcal{C} = \mathrm{Ind}\,\mathcal{C}_{K,k}$ and $F = \mathrm{Fr}_* \simeq (\mathrm{Fr}^*)^{-1}$ (Frobenius is a universal homeomorphism), $\mathrm{Tr}(\mathrm{Fr}_*, \mathcal{C}_{K,k})$ is the Hochschild homology of $\mathcal{C}_{K,k}$ with coefficients in $(M, N) \mapsto \mathrm{Hom}(\mathrm{Fr}^* M, N)$; we write $\mathrm{Tr}(-, \mathcal{C}_{K,k})$ for the trace on $\mathrm{Ind}\,\mathcal{C}_{K,k}$.

7. Classes and functoriality

Definition 7.1 ([GKRV, §3.4.3], [AGKRRV3, Rem. 0.6.5])**.** For compact $c \in \mathcal{C}$ and $\alpha\colon c \to F(c)$, the functor $T_c\colon \mathrm{Vect} \to \mathcal{C}$, $V \mapsto V \otimes c$, has a continuous right adjoint, and α is a 2-morphism $T_c \Rightarrow F \circ T_c$. Its trace is an element $\mathrm{cl}(c, \alpha) \in H_0\,\mathrm{Tr}(F, \mathcal{C})$. For $F = \mathrm{Fr}_*$, α corresponds to $\alpha^\flat\colon \mathrm{Fr}^* c \to c$; for a Weil object (M, Φ) we write $\mathrm{cl}(M, \Phi)$.

Definition 7.2 ([GKRV, §3.2.1, §3.4.1])**.** A 1-morphism $(\Psi, \gamma)\colon (\mathcal{C}, F_{\mathcal{C}}) \to (\mathcal{D}, F_{\mathcal{D}})$ is a continuous Ψ with continuous right adjoint (for compactly generated categories: Ψ preserves compact objects) and a natural transformation $\gamma\colon \Psi \circ F_{\mathcal{C}} \Rightarrow F_{\mathcal{D}} \circ \Psi$. Pasting γ and the adjunction into the unit–trace–counit composite gives $\mathrm{Tr}(\Psi, \gamma)\colon \mathrm{Tr}(F_{\mathcal{C}}, \mathcal{C}) \to \mathrm{Tr}(F_{\mathcal{D}}, \mathcal{D})$.

Theorem 7.3 ([GKRV, §3.2.3, §3.3.1, §3.4.2–3.4.3])**.** (a) $\mathrm{Tr}(\Psi' \circ \Psi, \gamma' \circ \gamma) = \mathrm{Tr}(\Psi', \gamma') \circ \mathrm{Tr}(\Psi, \gamma)$, *and* Tr *is a symmetric monoidal functor on pairs.*

(b) $\mathrm{Tr}(\Psi, \gamma)(\mathrm{cl}(c, \alpha)) = \mathrm{cl}(\Psi c, \gamma_c \circ \Psi(\alpha))$.

(c) $\mathrm{cl}(c \oplus c', \alpha \oplus \alpha') = \mathrm{cl}(c, \alpha) + \mathrm{cl}(c', \alpha')$, *and* $\mathrm{cl}(c, \beta \circ u) = \mathrm{cl}(c', F(u) \circ \beta)$ *for* $u\colon c \to c'$, $\beta\colon c' \to F(c)$.

(d) *For* $\mathcal{C} = \mathrm{Vect}$, $F = \mathrm{Id}$ *and* V *perfect,* $\mathrm{cl}(V, \alpha) = \sum_i (-1)^i \operatorname{tr} H^i(\alpha)$.

Proof. (b) follows from (a) since $(\Psi, \gamma) \circ (T_c, \alpha) = (T_{\Psi c}, \gamma_c \circ \Psi(\alpha))$. (c) In the model of Proposition 6.2 with a generator containing $c \oplus c'$, $q \mapsto [q]$ is linear and the cyclicity identity is the relation defining H_0. (d) $\mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}) = \overline{\mathbb{Q}}_\ell$ and $\mathrm{Tr}(T_V, \alpha)$ is the trace of the dualizable V. □

Proposition 7.4 ([GKRV, §3.3.2–3.3.3, §3.4.4, Prop. 3.4.6]). If a monoidal $\mathcal{R}$ acts on $\mathcal{M}$, with multiplication and action preserving compact objects, $F_{\mathcal{R}}$ right-lax monoidal and $F_{\mathcal{M}}$ right-lax compatible, then $\mathrm{Tr}(F_{\mathcal{R}}, \mathcal{R})$ is an algebra, $\mathrm{Tr}(F_{\mathcal{M}}, \mathcal{M})$ a module, and $\mathrm{cl}(r, \alpha)$ acts by $\mathrm{Tr}(r * -, \alpha_{r,\mathcal{M}})$.

Corollary 7.5. *With* $\mathcal{R} = \mathrm{Ind}\,\mathrm{Sph}_k$, $\mathcal{M} = \mathrm{Ind}\,\mathcal{C}_{K,k}$, $F = \mathrm{Fr}_*$ *and* Φ_0 *the canonical Weil structure of* IC_0, $\mathrm{Tr}(\mathrm{act})(x) = x \cdot \mathrm{cl}(\mathrm{IC}_0, \Phi_0)$. *So* $\mathrm{Tr}(\mathrm{act})$ *is onto iff* $\mathrm{Tr}(\mathrm{Fr}_*, \mathcal{C}_{K,k})$ *is generated by* $\mathrm{cl}(\mathrm{IC}_0, \Phi_0)$ *as a* $\mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Sph}_k)$*-module.*

Proof. act is $\mathrm{Sph}_k \simeq \mathrm{Sph}_k \otimes \mathrm{Vect} \xrightarrow{\mathrm{Id}\otimes T_{\mathrm{IC}_0}} \mathrm{Sph}_k \otimes \mathcal{C}_{K,k} \xrightarrow{*} \mathcal{C}_{K,k}$. Apply Theorem 7.3(a) and Proposition 7.4. □

So "IC_0 generates $\mathcal{C}_{K,k}$ as a Sph_k-module category" ($\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$) becomes, after the trace, "$\mathrm{cl}(\mathrm{IC}_0)$ generates $\mathrm{Tr}(\mathrm{Fr}_*, \mathcal{C}_{K,k})$ as a module" ((SG), by Section 7). The first is built from objects under shifts, cones and retracts; the second from classes under linear combinations.

8. The Hecke square

Definition 8.1 ([AGKRRV1, §22.2], [AGKRRV3, (0.9), Rem. 0.6.5])**.** For a quasi-compact algebraic stack Y over $\mathbb{F}_q$ there is a local-term map $\mathrm{LT}_Y \colon \mathrm{Tr}((\mathrm{Frob}_Y)_*, \mathrm{Shv}(Y)) \to \mathrm{Funct}(Y(\mathbb{F}_q))$ with

$$\mathrm{LT}_Y(\mathrm{cl}(M, \alpha))(x) = \textstyle\sum_i (-1)^i \,\mathrm{tr}(\alpha_{\bar{x}}^{\flat}, \mathcal{H}^i(M)_{\bar{x}}).$$

On Gr the arc group acts on each $\mathrm{Gr}_{\leq\lambda}$ through a finite-type quotient; on LX the run defines the map by the same formula on classes (Theorem 1.3, step 1). This gives

$$\mathrm{LT}_{\mathrm{Sph}} \colon \mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Sph}_k) \to \mathcal{H}_K, \qquad \mathrm{LT}_{\mathcal{C}_K} \colon \mathrm{Tr}(\mathrm{Fr}_*, \mathcal{C}_{K,k}) \to C_K.$$

Proposition 8.2. In the square

$$\begin{array}{ccc} \mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Sph}_k) & \xrightarrow{\mathrm{Tr}(\mathrm{act})} & \mathrm{Tr}(\mathrm{Fr}_*, \mathcal{C}_{K,k}) \\ \downarrow{\scriptstyle \mathrm{LT}_{\mathrm{Sph}}} & & \downarrow{\scriptstyle \mathrm{LT}_{\mathcal{C}_K}} \\ \mathcal{H}_K & \xrightarrow[f \mapsto f * \varphi_0]{} & C_K \end{array}$$

(a) the square commutes on classes of Weil objects: $\mathrm{tr}(F * \mathrm{IC}_0) = \mathrm{tr}(F) * \varphi_0$;

(b) if every Frobenius-stable orbit has connected stabilizer, both local-term maps are onto, and with $W = \mathrm{span}\{\mathrm{cl}(V_\lambda * \mathrm{IC}_0, \Phi) : \sigma\lambda = \lambda\}$, $\mathrm{LT}_{\mathcal{C}_K}(W) = \mathcal{H}_K * \varphi_0$, so (SG) $\iff$ $\mathrm{LT}_{\mathcal{C}_K}(W) = C_K$;

(c) under (R), (C) and D0–D3, both traces are concentrated in degree 0 and spanned by classes, both local-term maps are isomorphisms, and (SG) $\iff$ $\mathrm{Tr}(\mathrm{act})$ is onto.

Proof. (a) is the Grothendieck–Lefschetz trace formula with proper base change on the fibres of the convolution map (Proposition IIa). (b) T_λ ($\sigma\lambda = \lambda$) and $\mathbf{1}_{\mathcal{O}_\nu(\mathbb{F}_q)}$ are the trace functions of the constant sheaves on Frobenius-stable cells and orbits, extended by zero; Lang's theorem and connected stabilizers make each set of rational points one K-orbit, and an orbit with a rational point is Frobenius-stable. V_λ is Weil iff $\sigma\lambda = \lambda$ (run, Lemma 2.8), and the $\mathrm{tr}(V_\lambda * \mathrm{IC}_0)$ with $\sigma\lambda = \lambda$ span $\mathcal{H}_K * \varphi_0$. (c) is the run's Theorem 1.3, audited with three repairs: formality from free even cohomology with scalar Frobenius, a written proof of additivity of twisted Hochschild homology along the recollement, and LT defined by functoriality. Given (c), $\mathrm{Tr}(\mathrm{act})$ is determined by (a), and it is onto iff its image under the isomorphism $\mathrm{LT}_{\mathcal{C}_K}$ is C_K. □

9. Generation versus surjectivity

Theorem 9.1. *Let $(\Psi,\gamma)\colon(\mathcal{C},F_{\mathcal{C}})\to(\mathcal{D},F_{\mathcal{D}})$ be a 1-morphism and $\{c_i\}$ compact objects with $\mathcal{C}^c=\mathrm{Thick}\{c_i\}$ and $\mathcal{D}^c=\mathrm{Thick}\{\Psi c_i\}$. Suppose $\mathrm{Hom}(c_i,c_j)$, $\mathrm{Hom}(c_i,F_{\mathcal{C}}c_j)$, $\mathrm{Hom}(\Psi c_i,\Psi c_j)$ and $\mathrm{Hom}(\Psi c_i,F_{\mathcal{D}}\Psi c_j)$ are concentrated in degree 0. Then:*

(i) $H_0\,\mathrm{Tr}(F_{\mathcal{D}},\mathcal{D})=\bigoplus_i\mathrm{Hom}(\Psi c_i,F_{\mathcal{D}}\Psi c_i)/\langle q\circ f-F_{\mathcal{D}}(f)\circ q\rangle$, *with $f\colon\Psi c_i\to\Psi c_j$ and $q\colon\Psi c_j\to F_{\mathcal{D}}\Psi c_i$, and every element is a sum of classes;*

(ii) $\mathrm{im}\,H_0\,\mathrm{Tr}(\Psi,\gamma)=\mathrm{span}\{\mathrm{cl}(\Psi c_i,\gamma\circ\Psi(\alpha)):\alpha\in\mathrm{Hom}(c_i,F_{\mathcal{C}}c_i)\}$;

(iii) $\mathrm{Tr}(\Psi,\gamma)$ *is onto in degree 0 iff* $\bigoplus_i\mathrm{Hom}(\Psi c_i,F_{\mathcal{D}}\Psi c_i)=\bigoplus_i\gamma\circ\Psi(\mathrm{Hom}(c_i,F_{\mathcal{C}}c_i))+\langle q\circ f-F_{\mathcal{D}}(f)\circ q\rangle$.

Proof. Step 1. $\mathcal{D}^c=\mathrm{Thick}\{\Psi c_i\}$ makes $\mathcal{D}$ the module category of the full subcategory on the Ψc_i, an ordinary linear category by the degree hypothesis, and Proposition 6.2 computes the trace as its Hochschild homology with coefficients in $Q(i,j)=\mathrm{Hom}(\Psi c_i,F_{\mathcal{D}}\Psi c_j)$. In degree 0 the cyclic bar complex gives the quotient in (i), and $q\in Q(i,i)$ maps to $\mathrm{cl}(\Psi c_i,q)$. *Step 2.* The same for $\mathcal{C}$: $H_0\,\mathrm{Tr}(F_{\mathcal{C}},\mathcal{C})$ is spanned by the $\mathrm{cl}(c_i,\alpha)$. *Step 3.* Theorem 7.3(b) sends $\mathrm{cl}(c_i,\alpha)$ to $\mathrm{cl}(\Psi c_i,\gamma_{c_i}\circ\Psi(\alpha))$, which proves (ii); (iii) follows. □

The generation hypothesis serves only to compute $\mathrm{Tr}(F_{\mathcal{D}},\mathcal{D})$ from the Ψc_i. Surjectivity is decided by (iii), a fullness condition on Frobenius-twisted endomorphisms modulo commutators, which generation does not imply. It fails in two ways:

(M1) *Missing Weil structures.* A generator with $\mathrm{Hom}(c_i,F_{\mathcal{C}}c_i)=0$ contributes no class, while its image may be $F_{\mathcal{D}}$-fixed and carry twisted endomorphisms. Generation without retracts does not help (Toy B).

(M2) *Retracts.* If $\Psi c_i\simeq N\oplus N'$ and the projector onto N is not in $\Psi(\mathrm{End}\,c_i)$ modulo commutators, $\mathrm{cl}(N)$ is not reached; the trace contains only combinations such as $\mathrm{cl}(N)+\mathrm{cl}(N')$.

Without the degree-0 hypothesis, H_0 of a trace need not be spanned by classes [AGKRRV3, Rem. 0.6.11], and (ii) gives only "$\supseteq$".

10. Three toy models

The categories below are semisimple and the generators lie in the heart, so Theorem 9.1 applies.

Example 10.1 (Toy A, a retract)**.** $\Psi\colon\mathrm{Vect}\to\mathrm{Vect}\times\mathrm{Vect}$, $V\mapsto(V,V)$, $F=\mathrm{Id}$. $(\overline{\mathbb{Q}}_\ell,0)$ is a retract of $\Psi(\overline{\mathbb{Q}}_\ell)$, so Ψ generates. $\mathrm{Tr}(\Psi)\colon\overline{\mathbb{Q}}_\ell\to\overline{\mathbb{Q}}_\ell^2$ is $1\mapsto(1,1)$: $\mathrm{End}(\Psi\overline{\mathbb{Q}}_\ell)=\overline{\mathbb{Q}}_\ell^2$ is larger than $\Psi(\mathrm{End}\,\overline{\mathbb{Q}}_\ell)=\overline{\mathbb{Q}}_\ell(1,1)$, and the idempotent $(1,0)$ is missed (M2). Here $\mathrm{Vect}^2=(\overline{\mathbb{Q}}_\ell\times\overline{\mathbb{Q}}_\ell)$-mod and Ψ is the free-module functor. In the same way, $\mathcal{C}^{\mathrm{Sat}}_{K,k}=\mathcal{C}_{K,k}$ makes $\mathrm{Ind}\,\mathcal{C}_{K,k}$ the category of A-modules in $\mathrm{Ind}\,\mathrm{Sph}_k$, $A=\mathrm{act}^R(\mathrm{IC}_0)$, and act the free-module functor (run, Theorem 1.3(d)). For $\mathrm{Perf}(\overline{\mathbb{Q}}_\ell[x]/x^2)$ the nilpotent class $\mathrm{cl}(\overline{\mathbb{Q}}_\ell[x]/x^2,x)$ is missed, but if Frobenius acts on x by $q\neq1$ the relation $x\sim qx$ kills it: Frobenius weights remove nilpotent classes, not idempotent ones.

Example 10.2 (Toy B, no Weil structure)**.** $\Psi\colon(\mathrm{Vect}^2,\mathrm{swap})\to(\mathrm{Vect},\mathrm{Id})$, $(V_1,V_2)\mapsto V_1\oplus V_2$, γ the canonical isomorphism. $\Psi(\overline{\mathbb{Q}}_\ell,0)=\overline{\mathbb{Q}}_\ell$, so Ψ generates without retracts. For $c=(\overline{\mathbb{Q}}_\ell,\overline{\mathbb{Q}}_\ell)$ and $f=(1,0)$, the relation $q\circ f-\mathrm{swap}(f)\circ q=(q_1,-q_2)$ spans $\mathrm{Hom}(c,\mathrm{swap}\,c)$, so

$\mathrm{Tr}(\mathrm{swap}, \mathrm{Vect}^2) = 0$ and $\mathrm{Tr}(\Psi, \gamma)\colon 0 \to \overline{\mathbb{Q}}_\ell$. A pair $\alpha = (\phi_1, \phi_2)$ goes to $\left(\begin{smallmatrix} 0 & \phi_2 \\ \phi_1 & 0 \end{smallmatrix}\right)$, of trace 0. The Weil object $(\overline{\mathbb{Q}}_\ell, +1)$ is a summand of $(\overline{\mathbb{Q}}_\ell^2, \mathrm{swap}) = (\overline{\mathbb{Q}}_\ell, +1) \oplus (\overline{\mathbb{Q}}_\ell, -1)$, but its class is not reached (M1). For $\mathrm{swap}^2 = \mathrm{Id}$ the map $(a, b) \mapsto a + b$ is onto.

Example 10.3 (Toy C, fusion)**.** Res: $\mathrm{Rep}\,\mathrm{SL}_2 \to \mathrm{Rep}\,T$, $F = \mathrm{Id}$. The weight z^m is a summand of $\mathrm{Res}\, V_{|m|}$, so Res generates. For semisimple $\mathrm{Rep}\,H$, $\mathrm{Tr}(\mathrm{Id}, \mathrm{Rep}\,H) = R(H) \otimes \overline{\mathbb{Q}}_\ell$, and Tr(Res) is restriction of characters $\overline{\mathbb{Q}}_\ell[z + z^{-1}] \to \overline{\mathbb{Q}}_\ell[z^{\pm 1}]$, with image the W-invariants. The class $\mathrm{cl}(z)$, the weight projector of $\mathrm{Res}\, V_1 = z \oplus z^{-1}$, is missed (M2): t and t^{-1} fuse in SL_2. For finite groups $H \subseteq G$, restriction of class functions is onto iff no two H-classes fuse in G ($\mathbb{Z}/3 \subset S_3$ is the smallest failure).

11. The two geometric failures

Unitary twist: Toy B with Vect *replaced by* Sph_H. $\mathrm{Sph}_G = \mathrm{Sph}_H \otimes \mathrm{Sph}_H$ with $\mathrm{Fr}_* = \mathrm{swap} \circ (\mathrm{Fr}_* \otimes \mathrm{Fr}_*)$. By the string diagram for $\mathrm{tr}(\mathrm{swap} \circ (f_1 \otimes f_2)) = \mathrm{tr}(f_2 f_1)$, $\mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Sph}_G)$ is the Hecke algebra of $H(\mathbb{F}_{q^2}((t))) = G(F)$. Here $V_{(c,0)} * \mathrm{IC}_0 \simeq \mathrm{IC}_c$, but $\mathrm{Fr}^* V_\lambda \simeq V_{\sigma^{-1}\lambda}$, so $V_{(c,0)}$ is not Weil. Weil Satake objects come from fixed labels and from orbits $\{\lambda, \sigma\lambda\}$ with the swap structure, which contribute 0 to tr_n for odd n. The objects $V_{(c,0)} * \mathrm{IC}_0$ carry Weil structures in $\mathcal{C}_{K,k}$, but for $c \notin 2\Lambda_H$ none comes from Sph_G (run, Lemma 2.8). This is (M1), and Theorem 3.2 is its outcome.

Sakellaridis's X: Toy C with $\mathrm{SL}_2 \supset T$ *replaced by* $\mathrm{SO}_5^2 \supset \iota\check{G}_X$. Every label is Frobenius-fixed, so (M1) cannot occur. Given the decomposition theorem for X, $m_{\nu\lambda}^{(n)} = c_{\nu\lambda} q^{n(\langle 2\rho, \lambda\rangle - \delta(\nu))/2}$, so up to diagonal powers of q, Tr(act) is restriction $R(\check{G}) \otimes \overline{\mathbb{Q}}_\ell \to R(\iota\check{G}_X) \otimes \overline{\mathbb{Q}}_\ell$. The fusion $(a, b, a) \sim (b, a, b)$ makes it not onto, and $\mathrm{tr}\,\mathrm{IC}_{(0,0,2)} \notin \mathcal{H}_{K_n} * \varphi_0$ for every n (run, Corollary 3.9). This is (M2); Theorem 4.2 proves the function failure unconditionally.

12. When the argument works, and what AGKRRV prove

Proposition 12.1. If $(\Psi', \gamma')\colon (\mathcal{D}, F_\mathcal{D}) \to (\mathcal{C}, F_\mathcal{C})$ satisfies $(\Psi, \gamma) \circ (\Psi', \gamma') \simeq (\mathrm{Id}, \mathrm{id})$, then $\mathrm{Tr}(\Psi, \gamma) \circ \mathrm{Tr}(\Psi', \gamma') = \mathrm{id}$, so $\mathrm{Tr}(\Psi, \gamma)$ is split onto. This covers Frobenius-compatible localizations whose fully faithful right adjoint preserves compact objects, but not act, which is a free-module functor.

Proposition 12.2 (run, Proposition 1.8, audited). Under (R), (C) and D0–D3, if every orbit label μ has a dominant lift $\mu = \varphi(\lambda_\mu)$, the columns λ_μ of $m^{(n)}$ are triangular with nonzero diagonal, and $(\mathrm{SG})_n$ holds for every n. In the language of Theorem 9.1, each IC_μ is the top summand of $V_{\lambda_\mu} * \mathrm{IC}_0$ and is reached modulo lower labels without any projector.

Theorem 12.3 ([AGKRRV3, Main Thm. 0.2.6 = Cor. 4.1.4; Thm. 0.6.8 = Thm. 5.2.3; Cor. 0.6.9])**.** *For a smooth projective curve over* $\mathbb{F}_q$ *and* G *as in* [AGKRRV3, §0.2.1],

$$\mathrm{Tr}\left((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\right) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

Assuming [AGKRRV1, Conj. 14.1.8]*, this isomorphism is the trace of the inclusion into* $\mathrm{Shv}(\mathrm{Bun}_G)$ *followed by the local-term map. This is the Trace Conjecture* [AGKRRV1, Conj. 22.3.7]*; the proof uses the self-duality of* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *from* [AGKRRV2].

This theorem computes the trace of one category. Its local analogue here is Proposition 8.2(c), $\mathrm{Tr}(\mathrm{Fr}_*, \mathcal{C}_{K,k}) \simeq C_K$, which identifies the target. Given $\mathcal{C}_{K,k}^{\mathrm{Sat}} = \mathcal{C}_{K,k}$ it gives

$\mathrm{Tr}(\mathrm{Fr}_*, \mathcal{C}^{\mathrm{Sat}}_{K,k}) \simeq C_K$, which is a tautology. (SG) concerns the map Tr(act), the module action on $\mathrm{cl}(\mathrm{IC}_0)$ (Corollary 7.5), and neither theorem makes that module cyclic. AGKRRV draw the same line: for general Y it is "emphatically not true" that $\mathrm{Tr}((\mathrm{Frob}_Y)_*, \mathrm{Shv}(Y)) \simeq \mathrm{Funct}_c(Y(\mathbb{F}_q))$ [AGKRRV3, §0.4.2]. Without the singular-support condition there would be enough sheaves to recover all functions, but "the relations imposed by sheaves would not match the relations on functions" [AGKRRV3, Rem. 0.6.10]. Generation gives enough sheaves. (SG) asks that the relations among the classes in the image of one functor be the relations among functions.

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