PROPOSITION II: FUNCTIONS IMPLY SHEAVES OVER $\mathbb{F}_q(\!(t)\!)$, IWAHORI LEVEL

Over $F = \mathbb{F}_q(\!(t)\!)$ the trace of Frobenius turns the generators $\mathrm{IC}_w * _I \mathrm{IC}_0$ of the Iwahori-generated category into Hecke translates of the basic function. We show that Iwahori generation of functions forces every simple object of $\mathcal{C}_I$ to be a shifted summand of a generator, hence $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ over $\overline{\mathbb{F}}_q$. The converse is not given by the argument; a weighted triangularity condition implies both levels. Part (b) transfers ℓ-adic generation at one good closed point to $\mathbb{C}$ and to $\mathcal{D}$-modules, under level-uniformity hypotheses (Section 8). Part (a) is the Iwahori analogue of Proposition II of the spherical run [sat, trace.tex].

1. Setting

Groups. p is an odd prime, q a power of p, $k = \overline{\mathbb{F}}_q$, $\ell \neq p$. $G \supseteq B \supseteq T$ is a split connected reductive group over $\mathbb{F}_q$ and X a smooth affine G-variety over $\mathbb{F}_q$. $LG \supseteq K = L^+G \supseteq I$ are the loop group, the arc group and the Iwahori subgroup (the preimage of B), $\widetilde{W} = X_*(T) \rtimes W = W_{\mathrm{aff}} \rtimes \Omega$ with length ℓ and Bruhat order $\leq$, and $\mathrm{Fl} = LG/I$ with Schubert cells IwI/I and closures $\mathrm{Fl}_{\leq w}$.

Levels and pieces. A level presentation is $LX = \varinjlim_N Y_N$, $Y_N = \varprojlim_M Y_{N,M}$, $\pi_{N,M} \colon Y_N \to Y_{N,M}$, with I acting on $Y_{N,M}$ through $H_{N,M} = I/I_{N,M}$. In general I has infinitely many orbits on Y_N (for $\mathbb{A}^1$: the $t^k\mathcal{O}^\times$, $k \geq -N$, and $\{0\}$), while $H_{N,M}$ has finitely many on $Y_{N,M}$ (I1). A *piece* of level (N, M) is the set of points of Y_N whose image lies in one such orbit:

$$P = \pi_{N,M}^{-1}(o) = \{y \in Y_N : \pi_{N,M}(y) \in o\}, \qquad \mathcal{P}_{N,M} = \left\{\pi_{N,M}^{-1}(o) : o \in H_{N,M}\backslash Y_{N,M}\right\},$$

ordered by $P' \leq P \iff P' \subseteq \overline{P} = \pi_{N,M}^{-1}(\bar{o})$. A piece is an I-stable subset cut out by finitely many coordinates, a union of I-orbits but not always one: for $\mathbb{A}^1$ at level $(0, M)$, $t^M\mathcal{O}$ is one piece containing infinitely many orbits.

Lemma 1.1 (Pieces partition Y_N)**.** *The pieces of level (N, M) are nonempty, I-stable and pairwise disjoint, $Y_N = \bigsqcup_{P \in \mathcal{P}_{N,M}} P$, and $o \mapsto \pi_{N,M}^{-1}(o)$ is a bijection $H_{N,M}\backslash Y_{N,M} \to \mathcal{P}_{N,M}$; so $|\mathcal{P}_{N,M}| = \#(H_{N,M}\backslash Y_{N,M})$.*

Proof. $\pi_{N,M}$ is surjective, a limit of the surjective affine-space fibrations $Y_{N,M'} \to Y_{N,M}$, so each preimage is nonempty and $\pi_{N,M}(\pi_{N,M}^{-1}(o)) = o$, which gives injectivity. $\pi_{N,M}$ is I-equivariant and I acts through $H_{N,M}$, so $\pi_{N,M}(gy) = \bar{g}\,\pi_{N,M}(y)$ stays in the orbit of $\pi_{N,M}(y)$. The orbits partition $Y_{N,M}$ and preimages preserve disjoint unions. □

For $M' \geq M$ every piece of level (N, M') lies in exactly one piece of level (N, M); for $N' \geq N$ and $M \gg 0$ the pieces of level (N, M) are those of level (N', M) contained in Y_N.

The shift e_M. $\pi_{N,M}$ has infinite-dimensional fibres, so Y_N has no relative dimension over $Y_{N,M}$; perversity is normalized level by level by $\Phi_{N,M} = \pi_{N,M}^*[e_M]$, $e_M \in \mathbb{Z}$. For the transition $f \colon Y_{N,M'} \to Y_{N,M}$ $(M \leq M')$ of fibre rank $d(M, M')$, $f^*[d(M, M')]$ is t-exact, and $\Phi_{N,M'}(f^*K[d(M, M')]) = \Phi_{N,M}(K)$ if and only if

$$e_M = e_{M'} + d(M, M').$$

Fix $e_{M_0} = 0$ at one level; this determines all e_M. For $P = \pi_{N,M}^{-1}(o)$ put $\delta(P) = \dim o + e_M$.

Example: $X = \mathbb{A}^1$, $G = \mathrm{GL}_1$. $I = \mathcal{O}^\times$, $Y_N = t^{-N}\mathcal{O}$, $Y_{N,M} = t^{-N}\mathcal{O}/t^M\mathcal{O} \cong \mathbb{A}^{N+M}$, $I_{N,M} = 1+t^{N+M}\mathcal{O}$ $(N+M \geq 1)$, $I/I_{N,M} = (\mathcal{O}/t^{N+M})^\times \cong \mathbb{G}_m \times \mathbb{A}^{N+M-1}$, $d(M, M') = M'-M$, $Y_N \subseteq Y_{N+1}$ the closed subset $a_{-N-1} = 0$. The orbits are $o_j = \{\mathrm{val} = j\} \cong \mathbb{G}_m \times \mathbb{A}^{M-j-1}$ $(-N \leq j < M)$ and $o_M = \{0\}$; the pieces are $t^j\mathcal{O}^\times$ and $t^M\mathcal{O}$, a chain, so $|\mathcal{P}_{N,M}| = N + M + 1$. With $e_0 = 0$: $e_M = -M$ and $\delta(t^j\mathcal{O}^\times) = -j$. For $\mathbb{A}^n$ under GL_n: $I_{N,M} = 1 + t^{N+M}M_n(\mathcal{O})$, $d(M, M') = n(M' - M)$, $e_M = -nM$, $|\mathcal{P}_{N,M}| = (N + M)n + 1$.

Sheaves over k. $\mathcal{C}_I^{(N,M)}$ is the image of $\pi_{N,M}^*$ (extended by zero) on $D_c^b(I/I_{N,M}\backslash Y_{N,M})$, with $\overline{\mathbb{Q}}_\ell$-coefficients, and $\mathcal{C}_I = \bigcup_{N,M} \mathcal{C}_I^{(N,M)}$. $\mathcal{H}_{c,I}$ is the Iwahori–Hecke category, IC_w the intersection complex of $\mathrm{Fl}_{\leq w}$ with $\mathrm{IC}_w|_{IwI/I} = \overline{\mathbb{Q}}_\ell[\ell(w)]$, IC_0 the constant sheaf on L^+X (unshifted), $\mathcal{G} *_I M = m_!(\mathcal{G} \widetilde{\boxtimes} M)$, and

$$G_w = \mathrm{IC}_w *_I \mathrm{IC}_0, \qquad \mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}_{\mathcal{C}_I}\{\mathcal{G} *_I \mathrm{IC}_0 : \mathcal{G} \in \mathcal{H}_{c,I}\} = \mathrm{Thick}\{G_w : w \in \widetilde{W}\}.$$

$\mathcal{S}_I$ is the set of isomorphism classes of simple objects of the glued perverse heart $\mathcal{P}_I$ of $\mathcal{C}_I$: the $\mathrm{IC}(P)$, P a piece of some level, with $\mathrm{IC}(P) = \mathrm{IC}(P^\circ)$ for the open piece $P^\circ \subseteq P$ of a finer level (Lemma 3.1). $\mathrm{Thick}(\mathcal{A})$ is the smallest full triangulated subcategory containing $\mathcal{A}$ and closed under isomorphisms, shifts, cones and direct summands. Finally $\mathcal{M}_I \subseteq \mathcal{S}_I$ the set of S such that $S[s]$ is a direct summand of some G_w in $\mathcal{C}_I$.

Functions over F. $F = \mathbb{F}_q(\!(t)\!)$, $\mathcal{O} = \mathbb{F}_q[\![t]\!]$, $I(\mathbb{F}_q) \subseteq G(F)$ the Iwahori subgroup, and

$$\mathcal{H}_I = C_c(I\backslash G(F)/I), \quad T_w = \mathbf{1}_{IwI}, \qquad C_I = C_c^\infty(X(F))^I, \quad \varphi_0 = \mathbf{1}_{X(\mathcal{O})},$$

$$(f * \varphi)(x) = \sum_{gI \in G(F)/I} f(g)\varphi(g^{-1}x),$$

so $\mathrm{vol}\, I = 1$. Iwahori generation of functions is

$$\text{(IG)} \qquad \mathcal{H}_I * \varphi_0 = C_I.$$

For a field E of characteristic 0, $\mathcal{H}_I * \varphi_0 \otimes E$ is the E-span of the $\mathbb{Z}$-valued functions $T_w * \varphi_0$ and $C_I \otimes E$ the E-valued functions in C_I. At one level, $C_I^{(N,M)}$ is the space of I-invariant functions supported in $Y_N(\mathbb{F}_q)$ and constant on the fibres of $\pi_{N,M}$, and $P(\mathbb{F}_q) = Y_N(\mathbb{F}_q) \cap \pi_{N,M}^{-1}(o(\mathbb{F}_q))$ for $P = \pi_{N,M}^{-1}(o)$.

Weil objects and traces. A Weil object is (M, θ) with $M \in \mathcal{C}_I$ and $\theta\colon \mathrm{Fr}^*M \xrightarrow{\sim} M$, Fr the q-Frobenius. For $x \in LX(\mathbb{F}_q) = X(F)$ and $\bar{x}$ over x,

$$\mathrm{tr}(M)(x) = \sum_i (-1)^i \,\mathrm{tr}\left(\theta_{\bar{x}}, \mathcal{H}^i(M)_{\bar{x}}\right) \in \overline{\mathbb{Q}}_\ell.$$

It is additive on Fr-compatible distinguished triangles, $\mathrm{tr}(M[1]) = -\mathrm{tr}(M)$, and multiplicative on tensor products [Lau, (1.1.1.1)–(1.1.1.2)]. tr_n is the trace of the n-th iterate of θ on $X(F_n)$, $F_n = \mathbb{F}_{q^n}(\!(t)\!)$.

Hypotheses. (Lev), I1, I2 and (DT) are those of Proposition I:

(Lev) each $Y_{N,M'} \to Y_{N,M}$ $(M' \geq M)$ is a Zariski-locally trivial affine-space fibration, equivariant for $I/I_{N,M'} \to I/I_{N,M}$; each $Y_N \subseteq Y_{N+1}$ is a closed embedding with $Y_N = \pi_{N+1,M}^{-1}(Y_{N,M})$ for $M \gg 0$; each $I/I_{N,M}$ is a connected linear algebraic group and $I_{N,M}$ is pro-unipotent; and $\pi_{N,M}^*$ is fully faithful with image $\mathcal{C}_I^{(N,M)}$;

I1 (i) for every $\mathcal{F} \in \mathcal{C}_I$ there are (N, M) and $\mathcal{F}_0 \in D^b_c(I/I_{N,M} \backslash Y_{N,M})$ with $\mathcal{F} \simeq \pi^*_{N,M}\mathcal{F}_0$ (extended by zero from Y_N); (ii) $|\mathcal{P}_{N,M}| < \infty$ for every (N, M). Equivalently $\mathcal{C}_I = \bigcup_{N,M} \mathcal{C}_I^{(N,M)}$ and $|\mathcal{P}_{N,M}| < \infty$; these are what Lemma 3.3 uses;

I2 every stabilizer of $I/I_{N,M}$ on $Y_{N,M}$ is connected;

(DT) every G_w is, in $\mathcal{C}_I$, a finite direct sum of shifted simple objects.

(Dict) (trace dictionary over $\mathbb{F}_q$) consists of:

(a) *Rationality.* $G \supseteq B \supseteq T$ are split over $\mathbb{F}_q$; X, the action, the level presentation and its transition maps are defined over $\mathbb{F}_q$, so that $LX(\mathbb{F}_q) = X(F)$; the ranks $d(M, M')$ of $Y_{N,M'} \to Y_{N,M}$ do not depend on N.

(b) *Topology.* $Y_N(\mathbb{F}_q)$ is compact open in $X(F)$, $\bigcup_N Y_N(\mathbb{F}_q) = X(F)$, and the fibres of $\pi_{N,M}$ on $Y_N(\mathbb{F}_q)$, $M \geq 0$, form a basis of the topology of $Y_N(\mathbb{F}_q)$.

(c) *Rational orbits.* Every $I/I_{N,M}$-orbit on $Y_{N,M}$ is Fr-stable.

(d) *Bounded convolution.* For $w \in \widetilde{W}$ and N there is N' with $\overline{IwI} \cdot Y_N \subseteq Y_{N'}$, and for each M an M'' such that $(g, x) \mapsto \pi_{N,M}(g^{-1}x)$, on $\{(g, x) \in \overline{IwI} \times Y_{N'} : g^{-1}x \in Y_N\}$, factors through $\overline{IwI} \times Y_{N',M''}$.

(Dict) is used with three cited theorems: Lang's theorem $H^1(\mathbb{F}_q, H) = 1$ for connected H [Mil, Cor. 27.56], with Mittag-Leffler for pro-algebraic groups; proper base change and the Grothendieck–Lefschetz formula [Lau, (1.1.1.3)]; and purity: IC_w is pure of weight $\ell(w)$ [BBD, Cor. 5.3.2], IC_0 of weight 0, twisted products of pure objects are pure, and proper push-forward preserves purity [BBD, 5.1.14], so each Weil object G_w, computed on the finite-type model of (d), is pure of weight $\ell(w)$. With (d) and purity, (DT) holds over k by the decomposition theorem on the quotient stack $[Y_{N',M''}/(I/I_{N',M''})]$, which has affine stabilizers [Sun, Thm. 1.2, 3.11–3.12]; a splitting in $D^b_c(Y_{N',M''})$ alone does not suffice (Remark 5.4).

Proposition I (Iwahori summand criterion). Under (Lev), I1, I2 and (DT), $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{S : S \in \mathcal{M}_I\}$, $S \in \mathcal{C}_I^{\mathrm{Iwa}} \iff S \in \mathcal{M}_I$ for $S \in \mathcal{S}_I$, and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I \iff \mathcal{M}_I = \mathcal{S}_I$.

2. Statement

Proposition II. *Assume* (Lev), I1, I2, (Dict) *and* (DT).

(a) *Function generation $\Rightarrow$ ℓ-adic sheaf generation.*

$$\mathcal{H}_I * \varphi_0 = C_I \quad \Longrightarrow \quad \mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I \quad \text{over } \overline{\mathbb{F}}_q.$$

(b) *One good prime: ℓ-adic $\Leftrightarrow$ Betti $\Leftrightarrow$ $\mathcal{D}$-modules. Spread the data out over a finitely generated $R_0 \subset \mathbb{C}$, $R_0 \supseteq \mathbb{Z}[1/N]$, and let x be a closed point of* $\operatorname{Spec} R_0$. *Assume* (Lev), I1, I2, (DT) *over $\mathbb{C}$ and over $\overline{\mathbb{F}}_x$ (in place of $\overline{\mathbb{F}}_q$), and* (BS), (UP_x), ($\psi\mathrm{IC}_x$), ($\mathrm{I1}^{\mathcal{D}}$) *of Section 8. Then*

$$\mathcal{C}_I^{\mathrm{Iwa}}(\overline{\mathbb{F}}_x) = \mathcal{C}_I(\overline{\mathbb{F}}_x) \iff \mathcal{C}_I^{\mathrm{Iwa}}(\mathbb{C}) = \mathcal{C}_I(\mathbb{C}) \iff \mathcal{C}_I^{\mathcal{D},\mathrm{Iwa}} = \mathcal{C}_I^{\mathcal{D}}.$$

Without ($\psi\mathrm{IC}_x$), Betti generation still implies ℓ-adic generation at x; the second equivalence uses only ($\mathrm{I1}^{\mathcal{D}}$). For $\mathbb{A}^n$ under GL_n, $R_0 = \mathbb{Z}$, the hypotheses hold at every prime, and every odd prime is good.

Proposition II(a) is part (b)(ii) of the following complements, proved in the rest of this note.

Theorem 2.1 (Complements)**.** *Assume* (Lev), I1, I2, (Dict) *and* (DT). *The trace sends Weil objects of* $\mathcal{C}_I$ *to* $C_I \otimes \overline{\mathbb{Q}}_\ell$.

(a) *For Weil objects* $\mathcal{G} \in \mathcal{H}_{c,I}$ *and* $M \in \mathcal{C}_I$, $\operatorname{tr}(\mathcal{G} *_I M) = \operatorname{tr}\mathcal{G} * \operatorname{tr} M$, *with* $\operatorname{vol} I = 1$. *The functions* $\operatorname{tr}\mathrm{IC}_w$, $w \in \widetilde{W}$, *form a basis of* $\mathcal{H}_I \otimes \overline{\mathbb{Q}}_\ell$, *and* $\mathcal{H}_I * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell = \operatorname{span}\{\operatorname{tr} G_w : w \in \widetilde{W}\}$.

(b) (i) *The traces* $\operatorname{tr} S$, $S \in \mathcal{S}_I$, *with canonical Weil structures, form a basis of* $C_I \otimes \overline{\mathbb{Q}}_\ell$, *and*

$$\operatorname{tr} G_w = \sum_S m_{S,w} \operatorname{tr} S, \qquad m_{S,w} = \sum_i (-1)^i \operatorname{tr} A_{S,i}(w),$$

with $m_{S,w} = 0$ *unless* S *is a shifted summand of* G_w. (IG) *holds if and only if the columns* $(m_{S,w})_S$ *span* $\bigoplus_{\mathcal{S}_I} \overline{\mathbb{Q}}_\ell$.

(ii) *If* (IG) *holds over* $F = \mathbb{F}_q(\!(t)\!)$, *then* $\mathcal{M}_I = \mathcal{S}_I$, *and*

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I \qquad \text{over } \overline{\mathbb{F}}_q.$$

(iii) *(Pointwise form.) If* $\mathbf{1}_{P(\mathbb{F}_q)} \in \mathcal{H}_I * \varphi_0$ *for one piece* P, *then* $\mathrm{IC}(P) \in \mathcal{M}_I \subseteq \mathcal{C}_I^{\mathrm{Iwa}}$.

(iv) *(Witness.) If* $S \in \mathcal{S}_I \setminus \mathcal{M}_I$, *the coordinate form* κ_S *of* $\operatorname{tr} S$ *vanishes on* $\mathcal{H}_I * \varphi_0$ *and* $\kappa_S(\operatorname{tr} S) = 1$*; so* (IG) *fails.*

(c) *(Disconnected stabilizers.) Drop* I2 *and* (Dict)*(c), and let* $\mathcal{S}_I$ *consist of the* $\mathrm{IC}(P, \xi)$. *Then* (IG) *over* $\mathbb{F}_q(\!(t)\!)$ *puts every* Fr*-stable simple object in* $\mathcal{M}_I$, *and if* (IG) *holds over* $\mathbb{F}_{q^n}(\!(t)\!)$ *for every* $n \geq 1$, *then* $\mathcal{M}_I = \mathcal{S}_I$ *and* $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ *over* $\overline{\mathbb{F}}_q$. *A simple object that is not* Fr*-stable contributes* 0 *to every trace over* $\mathbb{F}_q$.

(d) *(The converse.)*

(i) *If* $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$, *then for every* $S \in \mathcal{S}_I$ *there are* w *and* $n \leq \dim \bigoplus_i E_{S,i}(w)$ *with* $m_{S,w}^{(n)} = \sum_i (-1)^i \operatorname{tr} A_{S,i}(w)^n \neq 0$. *Equivalently,* $S \in \mathcal{M}_I$ *if and only if* $\kappa_S^{(n)}(\mathcal{H}_{I_n} * \varphi_0) \neq 0$ *for some* n.

(ii) *Nonzero rows do not give* (IG), *which asks that the columns of* $m = m^{(1)}$ *span.*

(iii) (WT) *(Definition 7.2) implies* (IG).

Corollary 2.2. *Under the hypotheses of Proposition II, "*(IG) *holds over* $\mathbb{F}_q(\!(t)\!)$ *but* $\mathcal{C}_I^{\mathrm{Iwa}} \neq \mathcal{C}_I$ *over* $\overline{\mathbb{F}}_q$*" does not occur: over* $\mathbb{F}_q(\!(t)\!)$ *a divergence of the two levels can only be "sheaves generate, functions do not".*

Remark 2.3 (Status). Part (a): proved over $\overline{\mathbb{F}}_q$ for sheaves and $\mathbb{F}_q(\!(t)\!)$ for functions. Part (b): proved under its hypotheses and audited in-run (round-2 auditors of [runb]); the separate audit pass has not been run. The existence of a good point is a hypothesis (Remark 8.1). The gap in (d)(ii) is real at spherical level: for E_6/F_4 every row is nonzero while the columns do not span [sat, Prop. IX]; no Iwahori-level instance is established.

3. The Weil heart and functions at one level

Lemma 3.1 (Weil heart)**.** *Assume* (Lev), I1, I2, (Dict)*(a),(c). With* $H_{N,M} = I/I_{N,M}$, *let*

$$\Phi_{N,M} = \pi_{N,M}^*[e_M] \colon D_c^b\big([Y_{N,M}/H_{N,M}]\big) \longrightarrow \mathcal{C}_I, \qquad K \longmapsto \pi_{N,M}^* K[e_M],$$

the fully faithful level functor (pull back, shift, extend by zero), with $e_M = e_{M'} + d(M, M')$. *The perverse t-structures on* $D^b_c(I/I_{N,M}\backslash Y_{N,M})$, *transported by* $\Phi_{N,M}$, *glue to a bounded t-structure on* $\mathcal{C}_I$ *whose heart* $\mathcal{P}_I$ *is the increasing union of finite-length hearts. Its simple objects are the* $\mathrm{IC}(P) = \Phi_{N,M}\mathrm{IC}(\bar{o})$, $P = \pi^{-1}_{N,M}(o)$, *with* $\operatorname{End}\mathrm{IC}(P) = \overline{\mathbb{Q}}_\ell$ *and no maps between non-isomorphic ones; at a finer level* $\mathrm{IC}(P) = \mathrm{IC}(P^\circ)$, P° *the open piece of* P. *With* $\sim$ *the equivalence relation on* $\bigsqcup_{N,M} \mathcal{P}_{N,M}$ *generated by* $P \sim P^\circ$,

$$\mathcal{P}_\infty = \Big(\bigsqcup_{N,M} \mathcal{P}_{N,M}\Big)\Big/\sim \xrightarrow{\sim} \mathcal{S}_I, \qquad [P] \longmapsto \mathrm{IC}(P).$$

Every $S \in \mathcal{S}_I$ *has* $\mathrm{Fr}^* S \simeq S$ *and a unique Weil structure* θ_S *that is the identity on* $S|_P = \overline{\mathbb{Q}}_\ell[\delta(P)]$, $\delta(P) = \dim o + e_M$.

Proof. One level. $I/I_{N,M}$ is connected with finitely many orbits on $Y_{N,M}$ ((Lev), I1), so the equivariant perverse heart has finite length with simple objects $\mathrm{IC}(\bar{o}, \xi)$ [BBD, Thm. 4.3.1]; by I2 an equivariant local system on $o = H/S_y$ is a representation of $\pi_0 S_y = 1$, so ξ is trivial. *Change of* M. $Y_{N,M'} \to Y_{N,M}$ is smooth, surjective, with affine-space fibres of rank $d = d(M, M')$; pullback shifted by d is t-exact and fully faithful and sends $\mathrm{IC}(\bar{o})$ to IC of the preimage, which is irreducible with a unique open orbit; the kernel $I_{N,M}/I_{N,M'}$ is unipotent, so the equivariant categories agree. Since d does not depend on N ((Dict)(a)), one choice $e_M = e_{M'} + d(M, M')$ serves all N. *Change of* N. At a common M, $Y_N \subseteq Y_{N'}$ is a closed embedding; closed push-forward is t-exact, fully faithful and preserves simple objects. *Gluing.* $\mathcal{C}_I$ is the filtered union of the $\mathcal{C}_I^{(N,M)}$ along t-exact fully faithful functors (I1), so the t-structures glue; boundedness and finite length are checked at one level. *Frobenius.* $\mathrm{Fr}^*\mathrm{IC}(P) \simeq \mathrm{IC}(\mathrm{Fr}^{-1}P) = \mathrm{IC}(P)$ by (Dict)(c). Restriction to P identifies $\operatorname{Hom}(\mathrm{Fr}^* S, S)$ with $\operatorname{End}(S) = \overline{\mathbb{Q}}_\ell$, so the normalized θ_S exists and is unique; it is I-equivariant because $\mathrm{Perv}_H \to \mathrm{Perv}$ is fully faithful for connected H. On the open piece of a finer level, $\delta = \dim o^\circ + e_{M'} = \dim o + d + e_{M'} = \delta(P)$, so θ_S and $\operatorname{tr} S$ do not depend on the level. □

Lemma 3.2 (Lang count)**.** *Assume* (Lev), I2, (Dict)*(a)–(c).*

(a) $C_I = \bigcup_{N,M} C_I^{(N,M)}$, *a directed union.*

(b) $\pi_{N,M}$ *is surjective on* $\mathbb{F}_q$*-points and* $I(\mathbb{F}_q) \to (I/I_{N,M})(\mathbb{F}_q)$ *is surjective, so* $C_I^{(N,M)} \simeq \mathbb{C}[Y_{N,M}(\mathbb{F}_q)]^{(I/I_{N,M})(\mathbb{F}_q)}$.

(c) *For every piece* $P = \pi^{-1}_{N,M}(o)$, $o \subseteq Y_{N,M}$ *is a locally closed subvariety defined over* $\mathbb{F}_q$ *((Dict)(c)), and its set of* $\mathbb{F}_q$*-points* $o(\mathbb{F}_q) = \{y \in o : \mathrm{Fr}(y) = y\}$ *is nonempty and is one orbit of the finite group* $(I/I_{N,M})(\mathbb{F}_q)$*:* $o(\mathbb{F}_q) = (I/I_{N,M})(\mathbb{F}_q) \cdot y$ *for any* $y \in o(\mathbb{F}_q)$. *Hence the* $\mathbf{1}_{P(\mathbb{F}_q)}$, $P \in \mathcal{P}_{N,M}$, *form a* $\mathbb{Z}$*-basis of the* $\mathbb{Z}$*-valued functions in* $C_I^{(N,M)}$.

Proof. (a) A compactly supported $\varphi \in C_I$ is supported in one $Y_N(\mathbb{F}_q)$ ((Dict)(b)); it is locally constant on the compact $Y_N(\mathbb{F}_q)$, whose topology has the nested basis of $\pi_{N,M}$-fibres, so finitely many fibres of one $\pi_{N,M}$ suffice. (b) The transitions are affine-space bundles over $\mathbb{F}_q$, so a rational fibre has rational points, and an inverse limit of surjections of finite sets surjects onto each term. $I_{N,M}$ is pro-unipotent ((Lev)), so $H^1(\mathbb{F}_q, I_{N,M}) = 1$ [Mil, Cor. 27.56] with Mittag-Leffler, and $I(\mathbb{F}_q) \to (I/I_{N,M})(\mathbb{F}_q)$ is surjective. The congruence choice matters: for the group SL_2 the full kernel contains $\Delta\mu_2$ and $H^1(\mathbb{F}_q, \mu_2) = \mathbb{Z}/2$ [run, A7-5]. (c) o is homogeneous under the connected $H = I/I_{N,M}$ and Fr-stable ((Dict)(c)), so $o(\mathbb{F}_q) \neq \varnothing$ by Lang's theorem. For $y \in o(\mathbb{F}_q)$,

the $H(\mathbb{F}_q)$-orbits on $o(\mathbb{F}_q)$ correspond to $\ker\big(H^1(\mathbb{F}_q,S_y)\to H^1(\mathbb{F}_q,H)\big)=H^1(\mathbb{F}_q,\pi_0S_y)=1$ by I2. The orbit map $H/S_{y,\mathrm{red}}\to o$ may be purely inseparable in characteristic p; a radicial surjective morphism of $\mathbb{F}_q$-varieties is bijective on $\mathbb{F}_q$-points, since $\mathbb{F}_q$ is perfect, so the count is unaffected [run, A7-6]. The sets $o(\mathbb{F}_q)$ partition $Y_{N,M}(\mathbb{F}_q)$ into $H(\mathbb{F}_q)$-orbits; pull back along the surjection of (b). *Example.* For $\mathbb{A}^1$, $o_j(\mathbb{F}_q)=\{\sum_{i=j}^{M-1}a_it^i : a_j\neq 0\}$ has $(q-1)q^{M-j-1}=(q-1)q^{N+M-1}/q^{N+j}$ elements: the order of $(\mathbb{F}_q[t]/t^{N+M})^\times$ divided by that of the stabilizer $1+t^{M-j}\mathbb{F}_q[t]$ of t^j. □

Lemma 3.3 (Generation by simple objects)**.** *Assume* (Lev), I1, I2. *For every $\mathcal{F}\in\mathcal{C}_I$ there is a tower $0=\mathcal{F}_0\to\mathcal{F}_1\to\cdots\to\mathcal{F}_r=\mathcal{F}$ with distinguished triangles $\mathcal{F}_{k-1}\to\mathcal{F}_k\to S_k[n_k]\xrightarrow{+1}$, $S_k\in\mathcal{S}_I$, $n_k\in\mathbb{Z}$. Hence $\mathcal{C}_I=\mathrm{Thick}(\mathcal{S}_I)$.*

Proof. By I1(i), $\mathcal{F}\simeq\Phi_{N,M}(\mathcal{F}_0)$ at one level, where the perverse t-structure is bounded: ${}^p\mathcal{H}^i(\mathcal{F})=0$ outside $[a,b]$. Induct on $b-a$ with $\tau^{\le b-1}\mathcal{F}\to\mathcal{F}\to{}^p\mathcal{H}^b(\mathcal{F})[-b]\xrightarrow{+1}$. By I1(ii) the heart at that level has finite length [BBD, Thm. 4.3.1]; a Jordan–Hölder step $0\to Q'\to Q\to S\to 0$ is a triangle $Q'\to Q\to S\xrightarrow{+1}$. The octahedral axiom concatenates the towers, with $r=\sum_i\mathrm{length}\,{}^p\mathcal{H}^i(\mathcal{F})$; level transitions are t-exact and fully faithful (Lemma 3.1). *Example.* For $\mathbb{A}^1$ at level $(0,1)$, with $e_1=-1$, $\mathrm{IC}(\mathcal{O}^\times)=\underline{k}_{\mathcal{O}}=\mathrm{IC}_0$ and $\mathrm{IC}(t\mathcal{O})=\underline{k}_{t\mathcal{O}}[-1]$, and $\mathrm{IC}(t\mathcal{O})\to j_!\underline{k}_{\mathcal{O}^\times}\to\mathrm{IC}(\mathcal{O}^\times)\xrightarrow{+1}$ is a tower with $r=2$. □

4. Proof of (a): traces of the generators

Lemma 4.1 (Trace of convolution; standard)**.** *For Weil objects $\mathcal{G}\in\mathcal{H}_{c,I}$ supported on $\mathrm{Fl}_{\le w}$ and $M\in\mathcal{C}_I$, and $x\in X(F)$,*

$$\mathrm{tr}(\mathcal{G}*_I M)(x)=\sum_{gI\in G(F)/I}\mathrm{tr}(\mathcal{G})(g)\,\mathrm{tr}(M)(g^{-1}x)=(\mathrm{tr}\,\mathcal{G}*\mathrm{tr}\,M)(x),\qquad \mathrm{tr}\,\mathrm{IC}_0=\varphi_0.$$

Proof. The shearing map $LG\times^I LX\to\mathrm{Fl}\times LX$, $[g,y]\mapsto(gI,gy)$, with inverse $(gI,x)\mapsto[g,g^{-1}x]$ (well defined by I-equivariance), turns the action map into the second projection: $\mathcal{G}*_I M=\mathrm{pr}_{2!}K$ with $K_{(gI,x)}\simeq\mathcal{G}_g\otimes M_{g^{-1}x}$, compatibly with Frobenius at rational points. By (Dict)(d), K is pulled back from the finite-type $\mathrm{Fl}_{\le w}\times Y_{N',M''}$, on which pr_2 is proper. Proper base change and the Grothendieck–Lefschetz formula [Lau, (1.1.1.3)], with $\mathrm{tr}(A\otimes B)=\mathrm{tr}\,A\cdot\mathrm{tr}\,B$, give the sum over $\mathrm{Fl}_{\le w}(\mathbb{F}_q)$. Since $H^1(\mathbb{F}_q,I)=1$, $\mathrm{Fl}(\mathbb{F}_q)=G(F)/I(\mathbb{F}_q)$. The summand $g\mapsto\mathrm{tr}(\mathcal{G})(g)\,\mathrm{tr}(M)(g^{-1}x)$ is right I-invariant, so the sum is the integral over $G(F)$ for $\mathrm{vol}\,I=1$. IC_0 is unshifted with trivial Weil structure, so $\mathrm{tr}\,\mathrm{IC}_0=\mathbf{1}_{L^+X(\mathbb{F}_q)}=\mathbf{1}_{X(\mathcal{O})}$. For the group $X=H$, $G=H\times H$, the right factor acts through g_2^{-1} and the formula reads $f_1*\varphi*\check{f}_2$. With $\mathrm{vol}\,K=1$ instead, the right side acquires $[K:I]^{-1}$; spans are unaffected. This is the standard function–sheaf dictionary [Lau, §1.1]; for symmetric X see [CY, Lemma 73(ii)]. It is an input, not a step of the main argument. □

Lemma 4.2 (Hecke basis)**.** *With $P_{v,w}$ the Kazhdan–Lusztig polynomials of $\widetilde{W}$,*

$$\mathrm{tr}\,\mathrm{IC}_w=\sum_{v\le w}c_{vw}T_v,\qquad c_{vv}=(-1)^{\ell(w)}P_{v,w}(q),\qquad c_{ww}=(-1)^{\ell(w)}.$$

On each finite Bruhat lower set $[\le w]$ the matrix $(c_{vu})_{v,u\in[\le w]}$ is upper triangular for the Bruhat order ($c_{vu}=0$ unless $v\le u$) with diagonal ±1, so $\{\operatorname{tr}\mathrm{IC}_w\}_{w\in\widetilde{W}}$ is a basis of $\mathcal{H}_I\otimes\overline{\mathbb{Q}}_\ell$; equivalently $\operatorname{tr}\mathrm{IC}_w=(-1)^{\ell(w)}q^{\ell(w)/2}C'_w$, the Kazhdan–Lusztig basis $C'_w=q^{-\ell(w)/2}\sum_{y\le w}P_{y,w}(q)T_y$, unitriangular in $\{T_y\}$ [KL, §1].

Proof. $\operatorname{tr}\mathrm{IC}_w$ is I-bi-invariant and supported on $\mathrm{Fl}_{\le w}(\mathbb{F}_q)=\bigsqcup_{v\le w}IvI/I$ (Lang's theorem for the connected $I\cap vIv^{-1}$). On the open cell $\mathrm{IC}_w=\overline{\mathbb{Q}}_\ell[\ell(w)]$ with Frobenius 1, value $(-1)^{\ell(w)}$. Bruhat lower sets in $W_{\mathrm{aff}}\rtimes\Omega$ are finite, so the transition matrix to $\{T_v\}$ is triangular with diagonal ±1 on every finite lower set. The value: at $x\in IvI/I$, $\mathcal{H}^{-\ell(w)+2i}(\mathrm{IC}_w)_x$ has dimension the coefficient of q^i in $P_{v,w}$ and Frobenius eigenvalue q^i, and odd degrees vanish [DCM, Thm. 4.4.7] (finite flag varieties), [CY, §8.2.3] (for $\widetilde{W}$); only the support and the diagonal are used. *Example.* For $G=\mathrm{SL}_2$ and an affine simple reflection s, $\mathrm{Fl}_{\le s}\cong\mathbb{P}^1$ and $\mathrm{IC}_s=\overline{\mathbb{Q}}_\ell[1]$, so $\operatorname{tr}\mathrm{IC}_s=-(T_e+T_s)$ and the matrix on $[\le s]=\{e,s\}$ is $\left(\begin{smallmatrix}1&-1\\0&-1\end{smallmatrix}\right)$. □

Proof of Theorem 2.1(a). Lemma 4.1 gives the first claim, Lemma 4.2 the second. Here a span is the set of finite $\overline{\mathbb{Q}}_\ell$-linear combinations, and $\mathcal{H}_I*\varphi_0$ is the image of $f\mapsto f*\varphi_0$, spanned by the $T_w*\varphi_0$. By Lemma 4.2, invertible on each lower set, $\mathcal{H}_I*\varphi_0\otimes\overline{\mathbb{Q}}_\ell$ is spanned by the $\operatorname{tr}\mathrm{IC}_w*\varphi_0$, which are the $\operatorname{tr}G_w$ by Lemma 4.1 with $M=\mathrm{IC}_0$. Moreover $T_w*\varphi_0(x)=\#\{gI\in IwI/I : g^{-1}x\in X(\mathcal{O})\}\in\mathbb{Z}_{\ge0}$. □

5. Proof of (b): the IC-trace basis and the matrix m

Lemma 5.1 (IC-trace basis)**.** *For $S=\mathrm{IC}(P)$, $P\in\mathcal{P}_{N,M}$, and $x_{P'}\in P'(\mathbb{F}_q)$,*

$$\operatorname{tr}S=(-1)^{\delta(P)}\mathbf{1}_{P(\mathbb{F}_q)}+\sum_{P'<P}c_{P'P}\mathbf{1}_{P'(\mathbb{F}_q)},\qquad c_{P'P}=\operatorname{tr}(S)(x_{P'}).$$

For each level, $\{\operatorname{tr}\mathrm{IC}(P):P\in\mathcal{P}_{N,M}\}$ is a basis of $C_I^{(N,M)}\otimes\overline{\mathbb{Q}}_\ell$, and $\{\operatorname{tr}S\}_{S\in\mathcal{S}_I}$ is a basis of $C_I\otimes\overline{\mathbb{Q}}_\ell$. The coordinate map $\kappa\colon C_I\otimes\overline{\mathbb{Q}}_\ell\to\bigoplus_{\mathcal{S}_I}\overline{\mathbb{Q}}_\ell$ is an isomorphism; κ_S is its S-th component.

Proof. $\operatorname{tr}S$ is supported on $\overline{P}(\mathbb{F}_q)$, constant on $\pi_{N,M}$-fibres, and $I(\mathbb{F}_q)$-invariant because θ_S is equivariant (Lemma 3.1); so it lies in $C_I^{(N,M)}\otimes\overline{\mathbb{Q}}_\ell$ and is constant on each $P'(\mathbb{F}_q)$, one orbit by Lemma 3.2(c). $\overline{P}=\pi_{N,M}^{-1}(\bar o)$ is a union of pieces $P'\le P$, and on P the value is $(-1)^{\delta(P)}$. So the matrix from $\{\operatorname{tr}\mathrm{IC}(P)\}$ to $\{\mathbf{1}_{P(\mathbb{F}_q)}\}$ is triangular for the closure order with diagonal ±1; by I2 each piece supports exactly one simple object. Across levels: $C_I=\bigcup C_I^{(N,M)}$ (Lemma 3.2(a)); finitely many simple objects lie at a common level, where they are IC's of distinct pieces, and $\operatorname{tr}S$ does not depend on the level (Lemma 3.1). This is the trace form of the bijection of the two bases of the affine Lusztig–Vogan module [CY, §8.2.3], [MS, 3.1.2]. □

Definition 5.2 (Multiplicity spaces)**.** Let $P_i={}^p\mathcal{H}^i(G_w)$, with the Weil structure θ_i induced by the Fr-stable t-structure. The multiplicity space of S in degree i is $E_{S,i}(w)=\mathrm{Hom}(S,P_i)$, and its Frobenius is

$$A_{S,i}(w)\colon E_{S,i}(w)\to E_{S,i}(w),\qquad f\longmapsto\theta_i\circ\mathrm{Fr}^*(f)\circ\theta_S^{-1}.$$

Lemma 5.3 (Expansion)**.** $\operatorname{tr}G_w=\sum_S m_{S,w}\operatorname{tr}S$ *with* $m_{S,w}=\sum_i(-1)^i\operatorname{tr}A_{S,i}(w)$, *a finite sum. If $E_{S,i}(w)=0$ for all i, then $m_{S,w}=0$; and $E_{S,i}(w)\neq0$ if and only if $S[-i]$ is a direct summand of G_w in $\mathcal{C}_I$.*

Proof. By (DT) in $\mathcal{C}_I$, over k, $G_w \simeq \bigoplus_i P_i[-i]$ with each P_i semisimple, and evaluation $\bigoplus_S S \otimes E_{S,i}(w) \to P_i$ is an isomorphism ($\operatorname{End} S = \overline{\mathbb{Q}}_\ell$, no maps between distinct simple objects). Under it $\theta_i = \bigoplus_S \theta_S \otimes A_{S,i}(w)$, by functoriality of evaluation; on stalks $\operatorname{tr}(\theta_S \otimes A) = \operatorname{tr}\theta_S \cdot \operatorname{tr} A$, so $\operatorname{tr} P_i = \sum_S \operatorname{tr} A_{S,i}(w) \operatorname{tr} S$. The trace is additive on the Fr-compatible perverse truncation triangles and $\operatorname{tr}(P[-i]) = (-1)^i \operatorname{tr} P$ [Lau, (1.1.1.1)]. The splitting is in $\mathcal{C}_I$, so $E_{S,i}(w) \neq 0$ makes $S[-i]$ a summand there. A constituent with $\mathrm{Fr}^* S \not\simeq S$ would be permuted off the diagonal and contribute 0; by (Dict)(c) none occurs. □

Remark 5.4 (The equivariant splitting is needed). Let $\mathbb{G}_m$ act trivially on a point, $c \in H^2(B\mathbb{G}_m)$ a generator and $K = \operatorname{Cone}(c\colon k \to k[2])$. The underlying complex of K is $k[2] \oplus k[1]$, with simple perverse cohomology, but k is not a retract of K in $D_{\mathbb{G}_m}(\mathrm{pt})$: the objects A with $\sum_n \dim \operatorname{Hom}(k, A[n]) < \infty$ form a thick subcategory containing K and not k. So a splitting after forgetting equivariance (as in [BBD, Thm. 5.4.5]) does not show $S \in \mathcal{M}_I$; the stack version [Sun, Thm. 3.12] does [run, A7-3]. The converse of the second sentence of Lemma 5.3 fails: eigenvalues $\alpha, -\alpha$ on one $E_{S,i}(w)$ cancel. It is never used.

Lemma 5.5 (Columns)**.** *$\kappa(\mathcal{H}_I * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell)$ is the span of the columns $(m_{S,w})_{S \in \mathcal{S}_I}$, $w \in \widetilde{W}$. Hence* (IG) *holds if and only if the columns span $\bigoplus_{\mathcal{S}_I} \overline{\mathbb{Q}}_\ell$, and* (IG) *does not depend on the coefficient field of characteristic* 0*. If the columns span, every row $(m_{S,w})_w$ is nonzero.*

Proof. $\kappa(\operatorname{tr} G_w)$ is column w (Lemma 5.3), and the $\operatorname{tr} G_w$ span $\mathcal{H}_I * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell$ by (a); κ is an isomorphism (Lemma 5.1). For the coefficients: an indicator $\mathbf{1}_{P(\mathbb{F}_q)}$ and finitely many $T_w * \varphi_0$ are $\mathbb{Z}$-valued (Lemma 3.2(c), (a)); writing the first as a combination of the others is a linear system with an integer matrix, solvable over a field $E \supseteq \mathbb{Q}$ exactly when the matrix and the augmented matrix have equal rank, which does not depend on E. If row S were zero, the nonzero form κ_S would vanish on every column, hence on their span. For the converse failure take $m = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$. □

Proof of Theorem 2.1(b). (i) is Lemmas 5.1, 5.3 and 5.5. (ii) is the chain

$$\text{(IG)} \xRightarrow{5.5} \forall S\ \exists w:\ m_{S,w} \neq 0 \xRightarrow{5.3} \exists i:\ E_{S,i}(w) \neq 0 \xRightarrow{5.3} \mathcal{M}_I = \mathcal{S}_I \xRightarrow{3.3} \mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I.$$

Only "zero multiplicity spaces ⇒ zero coefficient" is used: no positivity and no purity beyond (DT); cancellation inside $m_{S,w}$ can only lose information. By (Dict)(c) and I2 every simple object is Fr-stable and occurs in the basis of Lemma 5.1.

(iii) By Lemma 5.3, $m_{S,w} = 0$ for $S \notin \mathcal{M}_I$, so by (a) $\mathcal{H}_I * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell \subseteq \operatorname{span}\{\operatorname{tr} S : S \in \mathcal{M}_I\}$. Write $\mathbf{1}_{P(\mathbb{F}_q)} = \sum_Q c_Q \operatorname{tr} \mathrm{IC}(Q)$ over the pieces Q of a common finer level; uniqueness of coordinates gives $c_Q \neq 0 \Rightarrow \mathrm{IC}(Q) \in \mathcal{M}_I$. Let Q_0 be maximal with $c_{Q_0} \neq 0$; evaluating at $Q_0(\mathbb{F}_q)$, only Q_0 contributes (triangularity), so $Q_0 \subseteq P$. Evaluating at the open piece P° of P gives 1, so some $Q \geq P^\circ$ has $c_Q \neq 0$; a maximal $Q_1 \geq Q$ with $c_{Q_1} \neq 0$ is maximal overall, lies in P, and has $P^\circ \subseteq \overline{Q_1}$. A piece of P whose closure contains P° is P°, since otherwise it lies in the closed set $P \setminus P^\circ$. So $\mathrm{IC}(P) = \mathrm{IC}(P^\circ) \in \mathcal{M}_I$, and $\mathrm{IC}(P) \in \mathcal{C}_I^{\mathrm{Iwa}}$ by Proposition I.

(iv) κ_S kills every $\operatorname{tr} G_w$ when $S \notin \mathcal{M}_I$, hence $\mathcal{H}_I * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell$, and $\kappa_S(\operatorname{tr} S) = 1$; the integrality of Lemma 5.5 transports the failure to $\mathbb{C}$-valued functions. □

Remark 5.6 (An abstract form). The proof of (b)(ii) uses only: (W) every simple object is Fr-stable; (P) each G_w is pure; (Lin) the functions $\operatorname{tr} S$ are linearly independent; (Tr) $\operatorname{tr} \mathrm{IC}_0 = \varphi_0$, $\operatorname{tr}(\mathcal{G} *_I A) = \operatorname{tr}\mathcal{G} * \operatorname{tr} A$, additivity and $\operatorname{tr}(A[1]) = -\operatorname{tr} A$ [run, A5-7]. Then (P) yields (DT)

only through the stack decomposition theorem, and the clause for all $\mathbb{F}_{q^n}(\!(t)\!)$ needs (Lin) over every $\mathbb{F}_{q^n}$ [run, A7-9]. Lemmas 3.1–5.1 prove (W), (Lin) and (Tr) from (Dict).

6. Proof of (c): disconnected stabilizers

Lemma 6.1 (Twisted characters)**.** *Let A be a finite group, σ an automorphism of order d, $\Gamma = A \rtimes \langle\sigma\rangle$, and for each σ-stable $\xi \in \mathrm{Irr}(A)$ let $\tilde\xi$ be an extension to Γ. Then the functions $a \mapsto \tilde\xi(a\sigma)$, $\xi \in \mathrm{Irr}(A)^\sigma$, form a basis of the functions on A constant on twisted classes $a \sim ba\sigma(b)^{-1}$.*

Proof. Extensions exist because Γ/A is cyclic [Isa, Cor. 11.22]. Γ-classes in the coset $A\sigma$ are twisted classes, as $b^{-1}(a\sigma)b = b^{-1}a\sigma(b)\sigma$. *Spanning.* $\mathrm{Irr}(\Gamma)$ spans the class functions on Γ. By Clifford theory with cyclic Γ/A, an irreducible χ either restricts irreducibly to A, and then $\chi = \tilde\xi\lambda$ with $\lambda \in \mathrm{Irr}(\Gamma/A)$ [Isa, Cor. 6.17], so $\chi|_{A\sigma} = \lambda(\sigma)\tilde\xi|_{A\sigma}$; or it is induced from the normal subgroup $A \rtimes \langle\sigma^e\rangle$, $e > 1$, which misses $A\sigma$, and $\chi|_{A\sigma} = 0$. *Independence.* For σ-stable ξ, η put $s_j = |A|^{-1}\sum_{a\in A}\tilde\xi(a\sigma^j)\overline{\tilde\eta(a\sigma^j)}$. Then $\langle\tilde\xi, \tilde\eta\lambda\rangle_\Gamma = d^{-1}\sum_j \overline{\lambda(\sigma^j)}s_j = \delta_{\xi\eta}\delta_{\lambda 1}$ for all λ, and Fourier inversion on $\mathbb{Z}/d$ gives $s_1 = \delta_{\xi\eta}$. □

Proof of Theorem 2.1(c). Points. Let $o \subseteq Y_{N,M}$ be a Fr-stable orbit of $H = I/I_{N,M}$, $y \in o(\mathbb{F}_q)$ (Lang), $A = \pi_0 S_y$ with the Frobenius σ. The $H(\mathbb{F}_q)$-orbits on $o(\mathbb{F}_q)$ are indexed by $H^1(\mathbb{F}_q, A)$, the σ-twisted classes of A: the point hy with $h^{-1}\mathrm{Fr}(h) \in S_y$ goes to the image of $h^{-1}\mathrm{Fr}(h)$. Orbits that are not Fr-stable have no rational points. *Sheaves.* The equivariant irreducible local systems on o are $\mathcal{L}_\xi$, $\xi \in \mathrm{Irr}(A)$, with $\mathrm{Fr}^*\mathcal{L}_\xi \simeq \mathcal{L}_{\xi\circ\sigma}$ [CY, Lemma 71]. For σ-stable ξ, $\tilde\xi(\sigma)$ is a Weil structure, and the trace at a point of class a is $\tilde\xi(a\sigma)$. *Basis.* By Lemma 6.1 the traces of the Fr-stable $\mathrm{IC}(P, \xi)$ are block-triangular in the indicators of the $I(\mathbb{F}_q)$-orbits of $C_I^{(N,M)}$, with invertible diagonal blocks; so they form a basis, which is (Lin) over $\mathbb{F}_q$. Lemmas 5.3, 5.5 hold with $\mathcal{S}_I$ replaced by the Fr-stable simple objects, and the chain of (b)(ii) puts each of them in $\mathcal{M}_I$. *All n.* Fr permutes the finitely many simple objects of a level, so each is Fr^n-stable for some n. Replacing q by q^n changes neither $\mathcal{C}_I$ nor $\mathcal{M}_I$, both defined over $\overline{\mathbb{F}}_q$, and the same twisted-character basis gives (Lin) over $\mathbb{F}_{q^n}$. So (IG) over every $\mathbb{F}_{q^n}(\!(t)\!)$ gives $\mathcal{M}_I = \mathcal{S}_I$, and Proposition I gives $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$. A non-Fr-stable constituent is permuted off the diagonal and contributes 0 to every trace over $\mathbb{F}_q$, so (IG) over $\mathbb{F}_q(\!(t)\!)$ alone says nothing about it. □

7. Proof of (d): what the converse gives

Lemma 7.1 (Vandermonde)**.** *If $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}_\ell^\times$ are distinct and $\sum_j c_j\alpha_j^n = 0$ for $n = 1, \dots, r$, then all $c_j = 0$.*

Proof. $\det(\alpha_j^n)_{1\le n,j\le r} = \prod_j \alpha_j \prod_{j<k}(\alpha_k - \alpha_j) \neq 0$. □

Proof of Theorem 2.1(d)(i). Assume $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ and let $S = \mathrm{IC}(P)$. Proposition I gives $S \in \mathcal{M}_I$, so $E_{S,i}(w) \neq 0$ for some w, i. With θ_S, S is pure of weight $\delta(P)$; ${}^p\mathcal{H}^i(G_w)$ is pure of weight $\ell(w) + i$ [BBD, Thm. 5.4.1], and its summand $S \otimes (E_{S,i}(w), A_{S,i}(w))$ restricts on P to $E_{S,i}(w)[\delta(P)]$. So every eigenvalue of $A_{S,i}(w)$ has absolute value $q^{(\ell(w)+i-\delta(P))/2}$, and different i give different eigenvalues [MS, 3.4.3]. Hence $m_{S,w}^{(n)} = \sum_\alpha c_\alpha\alpha^n$ over distinct α with every $c_\alpha = \pm(\text{multiplicity}) \neq 0$, and Lemma 7.1 gives $n \le \dim\bigoplus_i E_{S,i}(w)$ with $m_{S,w}^{(n)} \neq 0$. Since $\kappa_S^{(n)}(\mathrm{tr}_n G_w) = m_{S,w}^{(n)}$ (Lemma 5.3 over $\mathbb{F}_{q^n}$), this is the equivalence: if $\kappa_S^{(n)}(\mathrm{tr}_n G_w) \neq 0$, some $E_{S,i}(w) \neq 0$, so $S \in \mathcal{M}_I$. □

Proof of Theorem 2.1(d)(ii). (IG) is "the columns of $m^{(1)}$ span" (Lemma 5.5). For two simple objects, $m = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ has nonzero rows and columns spanning a line. One n can also cancel across degrees, where only absolute values differ: $2a-b=0$ with $|b| = q^{1/2}|a|$ when $q=4$. Traces over all $\mathbb{F}_{q^n}$ determine a class in the Grothendieck group, and one n does not [Lau, Thm. 1.1.2]. At spherical level the gap occurs: for E_6/F_4 every IC_μ is a summand of a Weil Satake object and every row is nonzero, yet $C_K = \mathcal{H}_K * \varphi_0 \oplus \mathbb{C}\mathbf{1}_{O_{3\omega_1^\vee}}$ over every $\mathbb{F}_{q^n}((t))$ [sat, Prop. IX(a)]. □

Definition 7.2 (Weighted triangularity)**.** (WT): for every level (N,M) and $P \in \mathcal{P}_{N,M}$ there are Weil objects $\mathcal{G} \in \mathcal{H}_{c,I}$ and $Y \in \{\mathrm{IC}_0\} \cup \{\Delta_{P'} : P' < P\}$, $\Delta_{P'} = j_{P'!}\underline{k}_{P'}$ with trivial Weil structure, such that $A = \mathcal{G} *_I Y \in \mathcal{C}_I^{(N,M)}$, $\operatorname{supp} A \subseteq \overline{P}$ and $\operatorname{tr}(A)|_{P(\mathbb{F}_q)} \neq 0$.

Proof of Theorem 2.1(d)(iii). Induction on the finite poset $\mathcal{P}_{N,M}$. Assume $\mathbf{1}_{P'(\mathbb{F}_q)} \in \mathcal{H}_I * \varphi_0$ for $P' < P$. Since $\operatorname{tr}\Delta_{P'} = \mathbf{1}_{P'(\mathbb{F}_q)}$ and $\operatorname{tr}\mathrm{IC}_0 = \varphi_0$, Lemma 4.1 gives $\operatorname{tr} A = \operatorname{tr}\mathcal{G} * \operatorname{tr} Y \in \mathcal{H}_I * (\mathcal{H}_I * \varphi_0) \subseteq \mathcal{H}_I * \varphi_0$. As $A \in \mathcal{C}_I^{(N,M)}$ is supported on $\overline{P}$, Lemma 3.2(c) gives $\operatorname{tr} A = c\,\mathbf{1}_{P(\mathbb{F}_q)} + \sum_{P'<P} c_{P'}\mathbf{1}_{P'(\mathbb{F}_q)}$ with $c \neq 0$, so $\mathbf{1}_{P(\mathbb{F}_q)} \in \mathcal{H}_I * \varphi_0$; minimal pieces use $Y = \mathrm{IC}_0$. So $C_I^{(N,M)} \subseteq \mathcal{H}_I * \varphi_0$ for every level, and Lemma 3.2(a) gives (IG). If the hypothesis I3 of the Iwahori-generation pages holds with Weil structures, and in $A|_P \simeq \bigoplus_s \underline{k}_P[d_s]$ all d_s have one parity and the Frobenius eigenvalues on the multiplicity space are positive reals (NC), then $\operatorname{tr} A|_{P(\mathbb{F}_q)} = \sum_s (-1)^{d_s}\alpha_s \neq 0$: so I3 with (NC) gives (WT), hence both (IG) and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$. For $\mathbb{A}^n$, $\operatorname{tr}(\overline{\mathbb{Q}}_{\ell,gK/I} * \mathrm{IC}_0) = \left(\sum_j \dim H^{2j}(G/B)q^j\right)\mathbf{1}_{gX(\mathcal{O})}$. (WT) is a sufficient condition for both levels, not a transfer. □

8. Part (b): transfer to $\mathbb{C}$ and to $\mathcal{D}$-modules

Setting. $R_0 \subset \mathbb{C}$ is finitely generated over $\mathbb{Z}[1/N]$; $G \supseteq B \supseteq T$, X, the levels $Y_L = Y_{N,M}$ and the transitions are defined over R_0; $H_L = I/I_L$. For a closed point x, $\overline{\mathbb{F}}_x$ is an algebraic closure of its residue field, and $\mathcal{C}_I(k)$, $\mathcal{C}_I^{\mathrm{Iwa}}(k)$, $\mathcal{S}_I(k)$ are formed over $k = \mathbb{C}$ (Betti, analytic topology) or $k = \overline{\mathbb{F}}_x$ ($\overline{\mathbb{Q}}_\ell$). V is a strictly henselian trait with $R_0 \subset V \subset \mathbb{C}$, residue field $\overline{\mathbb{F}}_x$ and closed point over x [BBD, 6.1.8]; $\psi = \bar{i}^* R\bar{j}_*$ are its nearby cycles, from the geometric generic fibre $\bar{\eta}$ to the special fibre s. For a word $\underline{w} = (s_1, \dots, s_r; \omega)$ of w, $G_{\underline{w}}^{\mathrm{BS}} = m_{\underline{w}*}\underline{k}[r]$ with $m_{\underline{w}} : P_{s_1} \times^I \cdots \times^I P_{s_r} \times^I \omega L^+X \to LX$. For semisimple K, $m_k(S,K) = \sum_i \dim \mathrm{Hom}(S, {}^pH^iK)v^i$. Over $\mathbb{C}$, $\mathcal{C}_I^{\mathcal{D}} = \bigcup_L \pi_L^* D^b_{H_L,\mathrm{coh}}(\mathcal{D}_{Y_L})$ (strongly equivariant), with generators $G_w^{\mathcal{D}} = \mathrm{IC}_w^{\mathcal{D}} *_I \mathrm{IC}_0^{\mathcal{D}}$.

Hypotheses.

(BS): For every word there is a proper $p_{\underline{w}} : E_{\underline{w}} \to Y_{L,R_0}$, $E_{\underline{w}}$ smooth over R_0, with $G_{\underline{w}}^{\mathrm{BS}}(k) \simeq \pi_L^* p_{\underline{w},k*}\underline{k}[d]$ for $k = \mathbb{C}, \overline{\mathbb{F}}_x$.

(UP_x): At every level the orbits spread out over R_0 near x: smooth with geometrically connected fibres, fibres exactly the orbits over $\mathbb{C}$ and $\overline{\mathbb{F}}_x$, closures commuting with base change, stabilizer group schemes smooth with connected fibres. Then $\beta_x : \mathrm{IC}(\bar{o}_{\mathbb{C}}) \mapsto \mathrm{IC}(\bar{o}_x)$ is a bijection $\mathcal{S}_I(\mathbb{C}) \to \mathcal{S}_I(\overline{\mathbb{F}}_x)$.

($\psi\mathrm{IC}_x$): $\psi(\mathrm{IC}(\bar{o}_{\bar{\eta}})) \simeq \mathrm{IC}(\bar{o}_s)$ for every orbit of every level.

($\mathrm{I1}^{\mathcal{D}}$): Every coherent I-equivariant $\mathcal{D}$-module of the Iwahori-generation pages comes from a finite level.

A closed point with (UP_x) and ($\psi\mathrm{IC}_x$) is *good*.

Proof sketch of Proposition II(b). Run steps of [runb] in brackets.

1. Over each field, under (DT), generation $\iff$ every simple object is a shifted summand of some G_w (Proposition I; its proof uses no Frobenius) [A3-4]. The summand set may be computed with $G^{\mathrm{BS}}_{\underline{w}} \simeq G_w \oplus \bigoplus_{v<w} G_v \otimes V_v$ [Sun, Thm. 1.2] and non-equivariantly, H_L being connected [BBD, 4.2.5, 4.2.6.2] [A2-1, A4-1].
2. By (BS), proper base change for ψ and $\psi\Lambda = \Lambda$ on $E_{\underline{w}}$ [SGA7, XIII] give $\psi(G^{\mathrm{BS}}_{\underline{w},\bar{\eta}}) \simeq G^{\mathrm{BS}}_{\underline{w},s}$ at every closed point, uniformly in $\underline{w}$; $\bar{\eta}$ is compared with $\mathbb{C}$ by [BBD, 6.1.2] [A2-4, A4-3].
3. Under (UP_x), $\psi(S_{\bar{\eta}})$ contains $\beta_x S$ exactly once and its other summands are equivariant simple objects, since G^{BS}_s is equivariantly semisimple by (DT) at x. So $m_{\mathbb{C}}(S, G^{\mathrm{BS}}_{\underline{w}}) \leq m_x(\beta_x S, G^{\mathrm{BS}}_{\underline{w}})$ and Betti generation implies ℓ-adic generation at x [A2-5, A4-4].
4. Under $(\psi\mathrm{IC}_x)$, $G^{\mathrm{BS}}_s \simeq \bigoplus \beta_x S'[-i]^{m_{S',i}}$ with the Betti multiplicities; uniqueness of multiplicities gives equality, hence the first equivalence; for G_w the V_v are field-independent [KL80] [A2-6, A4-7].
5. Strongly equivariant coherent $\mathcal{D}$-modules on a level with finitely many orbits are regular holonomic [Kas, §1.3, §7.14] [A3-2, A5-8]. Equivariant Riemann–Hilbert [Kas, Thm. 4.6.2], compatible with the transitions and the proper push-forward realizing G_w [BBDJS, Thm. 2.18], matches the categories and generators, so thick closures correspond, given $(\mathrm{I1}^{\mathcal{D}})$ and (Dict)(d) over $\mathbb{C}$ [A3-3, A5-9].
6. $\mathbb{A}^n$: orbits, lattice closures and stabilizers are defined and smooth over $\mathbb{Z}$, closures are linear so $\psi\mathrm{IC} = \mathrm{IC}$, and $E_{\underline{w}}$ is a vector bundle over $\mathrm{BS}(\underline{w})$ [A2-8, A4-8]. □

Remark 8.1 (Why not "$p \gg 0$"). Each level L has a proper closed bad set $B_L \subset \operatorname{Spec} R_0$, independent of w, off which (UP_x) and $(\psi\mathrm{IC}_x)$ hold at L [BBD, 6.1.9–6.1.10] [A4-5]. The good points are the closed points outside $\bigcup_L B_L$, a countable union, which could contain every closed point: "on ne peut pas passer à la limite" [BBD, p. 158] [A4-6]. For $\mathbb{A}^n$ and the group, $B_L = \varnothing$. Open: (BS) for general X; dropping $(\psi\mathrm{IC}_x)$; a good point for general X. Without $(\psi\mathrm{IC}_x)$, Betti generation is equivalent to bounded witnesses on a Zariski-dense set of closed points, and failures transfer from a dense set [A1-3, A1-4, A5-3].

9. Remarks

Remark 9.1 (Fields). In part (a), functions are over $\mathbb{F}_q(\!(t)\!)$ and sheaves over $\overline{\mathbb{F}}_q$. A Betti statement from (a) alone would need, for each simple S, a witness w valid at a prime outside the bad set of the spreading-out of G_w [BBD, §6.1]; no uniformity in w is available there [run, A7-7, A7-11]. Part (b) obtains the uniformity from (BS) and the level-only conditions (UP_x), $(\psi\mathrm{IC}_x)$. The stack decomposition theorem over $\mathbb{C}$ is stated in [Sun, Thm. 3.15] with its proof deferred.

Remark 9.2 (What replaces the spherical inputs). In the spherical Proposition II: D0 (orbit geometry) is replaced by (Lev), I1 and I2, level by level, so orbit closures may contain infinitely many orbits; (R) by (Dict)(a),(c); (C) by I2; D1 by Lemma 3.1, (Dict)(d) and the shearing of Lemma 4.1; D2 by (DT); D3 by Lang's theorem for I, $I_{N,M}$ and the stabilizers. D5 is not used.

Remark 9.3 (Examples). $\mathbb{A}^n$ *under* GL_n. With $Y_N = t^{-N}\mathcal{O}^n$, $Y_{N,M} = t^{-N}\mathcal{O}^n/t^M\mathcal{O}^n$, $I_{N,M} = 1 + t^{N+M}M_n(\mathcal{O})$, the hypotheses hold: $I/I_{N,M} = \bar{A}^\times$ for a finite-dimensional algebra $\bar{A}$, and

the stabilizer of v is the nonempty open subset $\bar{A}^\times \cap (1 + \operatorname{Ann} v)$ of an affine space, hence connected [run, A3-8, A7-5]. Since (IG) holds over every non-archimedean F, (b)(ii) gives $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ over $\overline{\mathbb{F}}_q$ without I3, standard objects or a sheaf-level dévissage. $\mathbb{G}_m = \mathrm{GL}_2/\mathrm{Sp}_2$ is the same. *The group $X = H$.* The inputs hold, and (b)(ii) holds; it is vacuous, since (IG) fails over every F: the sign character ε is a ring homomorphism with $\varepsilon(\mathbf{1}_{K_H}) = \sum_{w \in W} (-1)^{\ell(w)} = 0$, so $\mathcal{H}_I * \varphi_0 = \mathcal{H}_{I_H} \mathbf{1}_{K_H} \mathcal{H}_{I_H} \subseteq \ker \varepsilon \not\ni T_e$ [run, A4-6, A8-6].

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