PROPOSITION I: THE IWAHORI SUMMAND CRITERION

The spherical run's Proposition I decides $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ by the set of orbit labels whose intersection complex is a summand of a Satake object. This note proves the Iwahori analogue. Orbit closures in LX may contain infinitely many I-orbits (as for $\mathbb{A}^n$), so every finiteness is used one finite level of LX at a time.

1. Setting

$k = \overline{\mathbb{F}}_q$ with $\overline{\mathbb{Q}}_\ell$-coefficients, or $k = \mathbb{C}$ with Betti coefficients. $G \supseteq B \supseteq T$ is split connected reductive, $K = L^+G \supseteq I$ the arc group and the Iwahori subgroup, $\widetilde{W} = X_*(T) \rtimes W$, $\Omega \subseteq \widetilde{W}$ the elements of length zero, $\mathrm{Fl} = LG/I$. $\mathcal{H}_{c,I} = \mathrm{Shv}_c(I\backslash LG/I)$, and $\mathrm{IC}_w = \mathrm{IC}(\overline{IwI}/I)$ is perverse. X is a G-variety, $\mathrm{IC}_0 = \underline{k}_{L^+X}$.

Levels and pieces. A level presentation is

$$LX = \varinjlim_N Y_N, \qquad Y_N = \varprojlim_M Y_{N,M}, \qquad \pi_{N,M} \colon Y_N \to Y_{N,M},$$

with $Y_{N,M}$ of finite type and I acting on $Y_{N,M}$ through $H_{N,M} := I/I_{N,M}$. A *piece* of level (N,M) is $P = \pi_{N,M}^{-1}(o)$, o an $H_{N,M}$-orbit; $\mathcal{P}_{N,M}$ is the set of pieces, and $\overline{P} = \pi_{N,M}^{-1}(\overline{o})$. Put

$$\mathcal{C}_I^{(N,M)} := \pi_{N,M}^* D_c^b(H_{N,M}\backslash Y_{N,M}) \text{ (extended by zero)}, \qquad \mathcal{C}_I := \bigcup_{N,M} \mathcal{C}_I^{(N,M)}.$$

$\mathcal{H}_{c,I}$ acts by $\mathcal{G} *_I F = m_!(\mathcal{G} \,\widetilde{\boxtimes}\, F)$, $m \colon LG \times^I LX \to LX$, and

$$\mathcal{C}_I^{\mathrm{Iwa}} := \mathrm{Thick}_{\mathcal{C}_I}\{\mathcal{G} *_I \mathrm{IC}_0 : \mathcal{G} \in \mathcal{H}_{c,I}\}, \qquad G_w := \mathrm{IC}_w *_I \mathrm{IC}_0 \quad (w \in \widetilde{W}).$$

For the group $X = H$ under $H \times H$ the generators are two-sided, $G_{w_1,w_2} = \mathrm{IC}_{w_1} * c * \mathrm{IC}_{w_2}$ with $c = \mathrm{IC}_0 = \underline{k}_{K_H/I_H}$.

Thick spans. $\mathrm{Thick}(\mathcal{S})$ is the smallest full subcategory containing $\mathcal{S}$ and 0 and closed under isomorphisms, shifts, cofibres and retracts. Two properties are used: (T1) if $A \to B \to C \xrightarrow{+1}$ is exact and $A, C \in \mathrm{Thick}(\mathcal{S})$ then $B \in \mathrm{Thick}(\mathcal{S})$; (T2) an exact functor F satisfies $F(\mathrm{Thick}(\mathcal{S})) \subseteq \mathrm{Thick}(F(\mathcal{S}))$.

Hypotheses.

- (Lev) Each $Y_{N,M'} \to Y_{N,M}$ $(M' \geq M)$ is a Zariski-locally trivial affine-space fibration, equivariant for $H_{N,M'} \to H_{N,M}$; each $Y_N \subseteq Y_{N+1}$ is a closed embedding with $Y_N = \pi_{N+1,M}^{-1}(Y_{N,M})$ for $M \gg 0$; each $H_{N,M}$ is a connected linear algebraic group and $I_{N,M}$ is pro-unipotent; and $\pi_{N,M}^* \colon D_c^b(H_{N,M}\backslash Y_{N,M}) \to \mathcal{C}_I$ is fully faithful with image $\mathcal{C}_I^{(N,M)}$.
- (I1) $|\mathcal{P}_{N,M}| < \infty$ for all N, M.
- (I2) Every stabilizer of $H_{N,M}$ on $Y_{N,M}$ is connected.
- (I2′) (Variant.) No condition on stabilizers; local systems are counted at every level.
- (DT) Every generator is, in the equivariant category, a finite direct sum of shifted simple objects: $G_w \simeq \bigoplus_b S_b[s_b]$ in $\mathcal{C}_I$, $S_b \in \mathcal{S}_I$.

Simple objects, summands, constituents. A *simple object* of $\mathcal{C}_I$ is an object isomorphic to $\pi_{N,M}^*\bar{S}$ for a level and a simple object $\bar{S}$ of the equivariant perverse heart $\mathrm{Perv}_{H_{N,M}}(Y_{N,M})$; $\mathcal{S}_I$ is the set of simple objects up to isomorphism and shift. Perverse cohomology ${}^pH^i$ is taken at any level containing the object (Step 3 shows independence). Put

$$\mathcal{M}_I := \{S \in \mathcal{S}_I : \ S[s] \text{ is a direct summand of some } G_w\},$$
$$\mathcal{M}_I^+ := \{S \in \mathcal{S}_I : \ S \text{ is a constituent of some } {}^pH^i(G_w)\},$$

and, for $\Sigma \subseteq \mathcal{S}_I$, the constituent subcategory

$$\mathcal{D}_\Sigma := \{F \in \mathcal{C}_I : \ \text{every constituent of every } {}^pH^i(F) \text{ lies in } \Sigma\}.$$

A summand $S[s]$ of G_w gives a summand S of ${}^pH^{-s}(G_w)$, so $\mathcal{M}_I \subseteq \mathcal{M}_I^+$.

Finite-type model (FM_w): there are a level (N, M) with $\overline{IwI}\cdot L^+X \subseteq Y_N$, a variety Z of finite type with an action of $H'' = I/I''$ ($I'' \subseteq I_{N,M}$ normal, $I_{N,M}/I''$ unipotent), an H''-equivariant proper morphism $p\colon Z \to Y_{N,M}$ and a smooth morphism $r\colon Z \to \mathrm{Fl}_w := \overline{IwI}/I$, defined over $\mathbb{F}_q$ when $k = \overline{\mathbb{F}}_q$, with $\overline{IwI} \times^I L^+X \simeq Z \times_{Y_{N,M}} Y_N$ over Y_N and $\mathrm{IC}_w \,\widetilde{\boxtimes}\, \underline{k}_{L^+X} \leftrightarrow r^*\mathrm{IC}_w$.

Proposition I (Iwahori summand criterion)**.** *Let $k = \overline{\mathbb{F}}_q$ or $\mathbb{C}$. Assume* (Lev), (I1) *and* (I2) (*or* (I2′)), *and that every G_w lies in $\mathcal{C}_I$.*

(a) $\mathcal{C}_I = \mathrm{Thick}(\mathcal{S}_I)$, and $\mathcal{D}_\Sigma = \mathrm{Thick}(\Sigma)$ for every $\Sigma \subseteq \mathcal{S}_I$.

(b) With no decomposition theorem,

$$\mathrm{Thick}(\mathcal{M}_I) \subseteq \mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathrm{Thick}(\mathcal{M}_I^+), \qquad S \in \mathcal{M}_I \Rightarrow S \in \mathcal{C}_I^{\mathrm{Iwa}} \Rightarrow S \in \mathcal{M}_I^+ \quad (S \in \mathcal{S}_I).$$

(c) Under (DT), *$\mathcal{M}_I = \mathcal{M}_I^+$, so $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}(\mathcal{M}_I)$, $S \in \mathcal{C}_I^{\mathrm{Iwa}} \iff S \in \mathcal{M}_I$, and*

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I \iff \mathcal{M}_I = \mathcal{S}_I.$$

The direction ($\Leftarrow$) holds without (DT).

(d) Given (FM_w), (DT) *holds for w: over $\overline{\mathbb{F}}_q$ by purity and the decomposition theorem on stacks; over $\mathbb{C}$ by the equivariant decomposition theorem of Bernstein–Lunts (cited).*

Under (I2), *$\mathcal{S}_I = \{\mathrm{IC}(Z) : Z \in \mathcal{Z}\}$, $\mathcal{Z}$ the closures of pieces of all levels, so* (c) *reads: $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ iff every $\mathrm{IC}(Z)$ is, up to shift, a summand of some $\mathrm{IC}_w *_I \mathrm{IC}_0$.*

Status. (a)–(c) are proved over $\overline{\mathbb{F}}_q$ and over $\mathbb{C}$; (d) is proved over $\overline{\mathbb{F}}_q$ and cited over $\mathbb{C}$. Run nodes A2-1–A2-5 (version 1), A5-4 (version 2), A3-5; audits A7-1–A7-4 and A7-11, whose repairs are built in: $I_{N,M}$ pro-unipotent, transitions fully faithful on D^b, local systems counted at every level, equivariant splitting.

2. Proof

Step 1 (perverse hearts glue along the levels). Fix a level $\alpha = (N, M)$, $H = H_\alpha$, $Y = Y_\alpha$. (i) $D_c^b(H\backslash Y)$ carries the perverse t-structure (an object is perverse iff its underlying complex is). H is connected, so $\mathrm{Perv}_H(Y) \to \mathrm{Perv}(Y)$ is fully faithful with image closed under subquotients [BBD, 4.2.5, 4.2.6.2] (applied to the action and projection $H \times Y \to Y$). $\mathrm{Perv}(Y)$ has finite length [BBD, 4.3.1]; a simple equivariant object is $\mathrm{IC}(\overline{o}, \xi)$, o an orbit by (I1), ξ an irreducible equivariant local system on $o = H/S$, i.e. an irreducible representation of $\pi_0(S)$; under (I2),

$\xi = k$. (ii) For $\pi = \pi_{N,M',M}$, an affine-space fibration of locally constant relative dimension d, $\pi^*[d]$ is t-exact, fully faithful on perverse sheaves with image closed under subquotients, and

$$\pi^*[d]\,\mathrm{IC}(\overline{o},\xi) \simeq \mathrm{IC}(\overline{o'},\pi^*\xi|_{o'}),$$

o' the open orbit of the irreducible set $\pi^{-1}(o)$ [BBD, 4.2.5–4.2.6.2]: $\pi^{-1}(o)$ is smooth and connected, (I1) gives a unique dense orbit, and an irreducible local system pulled back along a smooth map with connected fibres stays irreducible. (iii) For $\iota\colon Y_{N,M} \hookrightarrow Y_{N+1,M}$, ι_* is t-exact [BBD, 1.4.16], fully faithful, with image closed under subquotients (j^* is t-exact), and $\iota_*\mathrm{IC}(\overline{o},\xi) = \mathrm{IC}(\iota\overline{o},\xi)$. Hence the constituents of $\bigoplus_i {}^pH^i(\tau F)$ are the τ-images of those of $\bigoplus_i {}^pH^i(F)$, up to shift, for every transition τ.

Step 2 (retracts stay at one level). $D^b_c(H_\alpha\backslash Y_\alpha)$ has a bounded t-structure, hence is idempotent complete [LC]. If R is a retract in $\mathcal{C}_I$ of $X \in \mathcal{C}_I^{(\alpha)}$ with idempotent e, then $e \in \mathrm{End}(X)$ lies in $\mathcal{C}_I^{(\alpha)}$ by full faithfulness in (Lev), splits there as $X \simeq X_1 \oplus X_2$, and $R \simeq X_1$. So $\mathcal{C}_I$ is idempotent complete, and a thick span is computed at one level: finitely many objects lie at a common level, closed under shifts, cofibres and retracts.

Remark 2.1 (t-exact transitions do not suffice). $\pi\colon \mathbb{G}_m \to \mathrm{pt}$ is smooth, surjective, with connected fibres, so $\pi^*[1]$ is t-exact and fully faithful on perverse sheaves [BBD, 4.2.5]; but $\mathrm{Hom}(\pi^*k, \pi^*k[1]) = H^1(\mathbb{G}_m) \neq 0 = \mathrm{Hom}(k, k[1])$. A union along such transitions is not a union of full subcategories, and summand relations could change from one level to the next. This is why (Lev) asks for full faithfulness on D^b (affine-space fibres and unipotent kernels).

Step 3 (one simple object per closure). By Step 1 a simple object stays simple, up to shift, at every finer level, so $\mathcal{S}_I$ and $\mathcal{D}_\Sigma$ do not depend on the level used. Under (I2), $S \mapsto \operatorname{supp} S$ is a bijection $\mathcal{S}_I \to \mathcal{Z}$: a simple object at level α is $\mathrm{IC}(\overline{o},k)$, with support $\pi_\alpha^{-1}(\overline{o}) = \overline{P}$; the open piece $P' \subseteq P$ of a finer level has $\overline{P'} = \overline{P}$ and carries the same object; two simple objects with the same support are compared at a common level, where the open orbit of an irreducible orbit closure is unique. Under (I2′), $\mathcal{S}_I = \{\mathrm{IC}(Z,\xi)\}$; since π_0 of a stabilizer can grow under refinement, new ξ appear at finer levels, and $\mathcal{S}_I$ is read at every level.

Step 4 (constituent subcategories are thick spans). At one level let $\mathcal{P}_\Sigma \subseteq \mathrm{Perv}_H(Y)$ be the objects with all Jordan–Hölder constituents in Σ, a Serre subcategory, q the exact quotient functor, $h = q \circ {}^pH^0$. Then $\mathcal{D}_\Sigma \cap \mathcal{C}_I^{(\alpha)} = \{F : h(F[n]) = 0\ \forall n\}$ is a saturated triangulated subcategory [Stacks, Tag 05RD]. By Step 3 these subcategories are compatible; their union $\mathcal{D}_\Sigma$ is closed under shifts and cofibres, and under retracts by Step 2: thick. Conversely $F \in \mathcal{D}_\Sigma$ has finitely many nonzero ${}^pH^i(F)$, each of finite length; the truncation triangles and Jordan–Hölder filtrations with (T1) put F in $\mathrm{Thick}(\Sigma)$ [Stacks, Tag 0FX7]. With $\Sigma = \mathcal{S}_I$: $\mathcal{C}_I = \mathcal{D}_{\mathcal{S}_I} = \mathrm{Thick}(\mathcal{S}_I)$. No weights are used. This is (a).

Step 5 (generators). An object of $\mathcal{H}_{c,I}$ lives on a finite union of Schubert varieties, on which I acts through a connected finite-dimensional quotient with finitely many orbits, the cells $IwI/I \simeq \mathbb{A}^{\ell(w)}$, with connected stabilizers [CY, Lemma 71(i)]. Steps 1 and 4 on this finite-type stack give $\mathcal{H}_{c,I} = \mathrm{Thick}\{\mathrm{IC}_w : w \in \widetilde{W}\}$. The functor $- *_I \mathrm{IC}_0$ is exact, so (T2) gives

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{G_w : w \in \widetilde{W}\}.$$

For the group, $\mathrm{IC}_{(w_1,w_2)} = \mathrm{IC}_{w_1} \boxtimes \mathrm{IC}_{w_2}$ [BBD, 4.2.7], $(A \boxtimes B) *_I F = A * F * \mathrm{inv}^*B$, $\mathrm{inv}^*\mathrm{IC}_w = \mathrm{IC}_{w^{-1}}$ and $c = \mathrm{IC}_{w_0}[-\ell(w_0)]$, so the generators are the $\mathrm{IC}_{w_1} * c * \mathrm{IC}_{w_2}$.

Step 6 (the sandwich, without (DT)). If $S \in \mathcal{M}_I$, then $S[s]$ is a retract of $G_w \in \mathcal{C}_I^{\mathrm{Iwa}}$, so $S \in \mathcal{C}_I^{\mathrm{Iwa}}$: $\mathrm{Thick}(\mathcal{M}_I) \subseteq \mathcal{C}_I^{\mathrm{Iwa}}$. Every G_w lies in $\mathcal{D}_{\mathcal{M}_I^+}$ by definition of $\mathcal{M}_I^+$ (Step 3: at any level), and $\mathcal{D}_{\mathcal{M}_I^+} = \mathrm{Thick}(\mathcal{M}_I^+)$ is thick (Step 4); minimality of $\mathrm{Thick}\{G_w\} = \mathcal{C}_I^{\mathrm{Iwa}}$ (Step 5) gives $\mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathrm{Thick}(\mathcal{M}_I^+)$. If $S \in \mathcal{C}_I^{\mathrm{Iwa}}$, take a level where S is a shifted simple perverse object: its only constituent is S, so $S \in \mathcal{M}_I^+$. This is (b).

Step 7 (under (DT), summands are constituents). Under (DT), ${}^pH^i(G_w) = \bigoplus_{s_b=-i} S_b$ is semisimple, and its constituents are the S_b, each a summand of G_w up to shift: $\mathcal{M}_I^+ = \mathcal{M}_I$. Then (b) collapses to $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}(\mathcal{M}_I)$ and $S \in \mathcal{C}_I^{\mathrm{Iwa}} \iff S \in \mathcal{M}_I$. ($\Leftarrow$) If $\mathcal{M}_I = \mathcal{S}_I$, then $\mathcal{C}_I = \mathrm{Thick}(\mathcal{S}_I) = \mathrm{Thick}(\mathcal{M}_I) \subseteq \mathcal{C}_I^{\mathrm{Iwa}}$ by (a) and (b); (DT) is not used. ($\Rightarrow$) If $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$, every $S \in \mathcal{S}_I$ lies in $\mathcal{C}_I^{\mathrm{Iwa}}$, hence in $\mathcal{M}_I^+ = \mathcal{M}_I$. A second proof of $\mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathrm{Thick}(\mathcal{M}_I)$: $\{\mathcal{G} \in \mathcal{H}_{c,I} : \mathcal{G} *_I \mathrm{IC}_0 \in \mathrm{Thick}(\mathcal{M}_I)\}$ is thick, contains every IC_w by (DT), hence is $\mathcal{H}_{c,I}$ (Step 5). Only vanishing of nonnegative Jordan–Hölder multiplicities is used, never an alternating sum. This is (c).

Remark 2.2 (a non-equivariant splitting does not suffice). Let $\mathbb{G}_m$ act trivially on a point, $c \in \mathrm{Hom}(k, k[2]) = H^2(B\mathbb{G}_m)$ a generator and $E = \mathrm{Cone}(c)$. The forgetful functor kills c, so the underlying complex of E is $k[2] \oplus k[1]$ and each ${}^pH^i(E)$ is simple. The objects A with $\sum_n \dim \mathrm{Hom}(k, A[n]) < \infty$ form a thick subcategory; it contains E (multiplication by c on $H^*(B\mathbb{G}_m) = k[c]$ is injective, onto except in degree 0) and not k. So $k \notin \mathrm{Thick}(E)$: a splitting in $D_c^b(Y)$ [BBD, 5.4.5] does not make a constituent a summand in $\mathcal{C}_I$. The groups H_α contain T and have fixed points, so the splitting in (DT) must be equivariant.

Step 8 ((DT) from a finite-type model). Assume (FM_w). The action map factors through the closed embedding $[g, y] \mapsto (gI, gy)$ of $\overline{IwI} \times^I L^+X$ into $\mathrm{Fl}_w \times Y_N$, then the projection, proper since Fl_w is projective. By (FM_w) and proper base change, $G_w \simeq \pi_{N,M}^*\, p_!\, r^*\mathrm{IC}_w$, so $G_w \in \mathcal{C}_I^{(N,M)}$. Over $\overline{\mathbb{F}}_q$, IC_w is pure of weight $\ell(w)$ [BBD, 5.3.2]; r is smooth and p proper, so $p_!r^*\mathrm{IC}_w$ is pure [Sun, Prop. 3.9]. The decomposition theorem on the finite-type quotient stacks $[H''\backslash Z] \to [H''\backslash Y_{N,M}]$, which have affine stabilizers [Sun, Thm. 1.2, 3.11, 3.12], gives $p_*r^*\mathrm{IC}_w \simeq \bigoplus_i ({}^pR^ip_*)[-i]$ with semisimple terms, equivariantly. Inflation along the unipotent $I_{N,M}/I''$ and $\pi_{N,M}^*$ are fully faithful and t-exact up to shift ((Lev), Step 1), so the splitting holds in $\mathcal{C}_I$; a semisimple object of a finite-length heart is a finite sum of simple objects. Over $\mathbb{C}$ replace the last step by the equivariant decomposition theorem [BL, §5] (stated as [Sun, Thm. 3.15] without proof). This is (d). □

3. Remarks

No finiteness of closures. The spherical hypothesis that each orbit closure is a finite union of orbits fails for $\mathbb{A}^n$ at Iwahori level: $\overline{O_{m,j}} = t^m L_j$ contains infinitely many I-orbits and $\{0\}$, which is never a piece. The proof uses finiteness only level by level (finite-length hearts and bounded t-structures), and standard objects never enter.

Examples. For $\mathbb{A}^n$ under GL_n ($Y_{N,M} = t^{-N}\mathcal{O}^n/t^M\mathcal{O}^n$, $I_{N,M} = 1 + t^{N+M}M_n(\mathcal{O})$), (I1) and (I2) hold, ($\mathrm{FM}_w$) holds (a lattice bundle; for the group a twisted product of Schubert varieties plays its role), and the simple objects are the lattice sheaves $\underline{k}_\Lambda$; with $\tau e_1 = te_n$, $\tau e_i = e_{i-1}$ in Ω, $\underline{k}_{\tau^k\mathcal{O}^n} = \Delta_{\tau^k} *_I \mathrm{IC}_0$, so $\mathcal{M}_I = \mathcal{S}_I$ and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$. For the group, with levels truncated in Ω, $\mathcal{S}_I = \{\mathrm{IC}_z : z \in \widetilde{W}\}$, $\mathrm{IC}_\omega = \Delta_\omega$ for $\omega \in \Omega$, and no Δ_ω is a constituent of a generator (over $\overline{\mathbb{F}}_q$, by weights); by (b), $\Delta_e \notin \mathcal{C}_I^{\mathrm{Iwa}}$ (this run's Proposition III).

Relation to the Iwahori-generation pages. Their Proposition III proves $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ from I1–I3 by dévissage over the pieces of one level, with standard objects; Proposition I here replaces I3 by $\mathcal{M}_I = \mathcal{S}_I$. Their Proposition I(a), through the retraction $V \to q_* q^* V$, makes $p^*(V_\lambda *_K \mathrm{IC}_0)$ a retract of G_{w_λ} up to shift, so every spherical summand IC_ν forgotten to I lies in $\mathcal{M}_I$.

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