PROPOSITION III: THE FIRST ROWS OF TABLE 1, A SECOND PROOF

Rows 1–2 of Table 1 of the Iwahori-generation pages (the group, where Iwahori generation fails for sheaves and for functions; $\mathbb{A}^n$ under GL_n, where both hold) are proved again here, together with $\mathbb{G}_m$, by the two general results of the run: the Iwahori summand criterion (Proposition I) and functions $\Rightarrow$ sheaves over $\mathbb{F}_q(\!(t)\!)$ (Proposition II). No affine Whittaker functor and no dévissage by triangularity (I3) is used.

1. Setting

k is $\overline{\mathbb{F}}_q$ (ℓ-adic sheaves) or $\mathbb{C}$ (Betti sheaves). $G \supseteq B \supseteq T$ is split connected reductive, $K = L^+G \supseteq I$ the arc group and an Iwahori subgroup, $\widetilde{W} = X_*(T) \rtimes W$, ℓ the length, $\leq$ the Bruhat order, $\Omega \subseteq \widetilde{W}$ the elements of length zero, so $\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$. $\mathrm{Fl} = LG/I$, $\mathrm{Fl}_w = IwI/I \cong \mathbb{A}^{\ell(w)}$, $\Delta_w = j_{w!}\underline{k}$ (unshifted), $\mathrm{IC}_w = j_{w!*}\underline{k}[\ell(w)]$ (perverse). For X a G-variety with a level presentation $LX = \varinjlim_N Y_N$, $Y_N = \varprojlim_M Y_{N,M}$,

$$\mathcal{C}_I = \mathrm{Shv}_c(I\backslash LX) = \bigcup_{N,M} \mathcal{C}_I^{(N,M)}, \qquad \mathcal{H}_{c,I} = \mathrm{Shv}_c(I\backslash LG/I),$$

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}_{\mathcal{C}_I}\{\mathcal{G} *_I \mathrm{IC}_0 : \mathcal{G} \in \mathcal{H}_{c,I}\},$$

IC_0 the constant sheaf on L^+X, Thick the closure under shifts, cofibres and retracts, Tri the closure under shifts and cofibres. $\mathcal{S}_I$ is the set of simple objects of the perverse heart of $\mathcal{C}_I$ (up to shift); $\mathcal{M}_I \subseteq \mathcal{S}_I$ those that are shifted direct summands of some generator $\mathrm{IC}_w *_I \mathrm{IC}_0$, and $\mathcal{M}_I^+ \supseteq \mathcal{M}_I$ those that are perverse constituents of one. For $\Sigma \subseteq \mathcal{S}_I$, $\mathcal{D}_\Sigma$ is the full subcategory of objects all of whose perverse cohomology objects have constituents in Σ.

Functions. F is a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer t and residue field $\mathbb{F}_q$; $\mathcal{H}_I = C_c(I\backslash G(F)/I)$ with $\mathrm{vol}(I) = 1$ and basis $T_w = \mathbf{1}_{IwI}$, so $(T_s - q)(T_s + 1) = 0$; $C_I = C_c^\infty(X(F))^I$, $\varphi_0 = \mathbf{1}_{X(\mathcal{O})}$, and

$$(\mathrm{IG}): \qquad \mathcal{H}_I * \varphi_0 = C_I.$$

Over $F_m = \mathbb{F}_{q^m}(\!(t)\!)$, tr_m is the trace of Frobenius of a Weil object at the $\mathbb{F}_{q^m}$-points, and $\mathrm{tr}_m(K[1]) = -\mathrm{tr}_m K$.

Inputs. Proposition I needs (Lev) (level axioms), I1 (finitely many pieces per level) and I2 (connected stabilizers) for the constituent half $\mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathrm{Thick}(\mathcal{M}_I^+)$, and (DT) (every generator splits in the equivariant category) for $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}(\mathcal{M}_I)$. Proposition II needs in addition (Dict) (the trace dictionary over $\mathbb{F}_q$). They are verified below for each row. Over $\mathbb{C}$ the group statement also uses

(C) *Comparison with* $\mathbb{C}$. For fixed x, y, the dimensions of $\mathrm{Hom}(\mathrm{IC}_\omega, {}^pH^n(\mathrm{IC}_x * \mathrm{IC}_0 * \mathrm{IC}_y))$ for Betti sheaves over $\mathbb{C}$ and for ℓ-adic sheaves over $\overline{\mathbb{F}}_p$ agree for almost all p [BBD, 6.1.9–6.1.10], [BF, Prop. 5].

Proposition III. *(a) The group. Let $X = H$ under $G = H \times H$, $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$, with H split and finite Weyl group $W \neq 1$. Over $\overline{\mathbb{F}}_q$, for all $x, y \in \widetilde{W}$, $\omega \in \Omega$, $n \in \mathbb{Z}$,*

$$\operatorname{Hom}\big(\Delta_\omega, {}^pH^n(\mathrm{IC}_x * \mathrm{IC}_0 * \mathrm{IC}_y)\big) = 0, \qquad \textit{hence} \qquad \Delta_e \notin \mathcal{C}_I^{\mathrm{Iwa}} \subseteq \operatorname{Thick}\{\mathrm{IC}_z : z \notin \Omega\} \subsetneq \mathcal{C}_I.$$

(IG) fails over every F: the sign character $\varepsilon(T_w) = (-1)^{\ell(w)}$ kills $\mathcal{H}_I * \varphi_0$ and not T_e; over $\mathbb{F}_q(\!(t)\!)$ it fails also by Proposition II from the sheaf statement. Identifying IC_z with z: $\mathcal{M}_I \supseteq \{xw_0y : \ \ell(xw_0y) = \ell(x) + \ell(w_0) + \ell(y)\}$, $\mathcal{M}_I = \widetilde{W} \setminus \Omega$ if the roots of H are of type A_1, and $\mathcal{M}_I = c_0\Omega$ in general, c_0 the lowest two-sided cell (cited). Over $\mathbb{C}$ the sheaf statements hold given (C).

(b) $\mathbb{A}^n$ under GL_n, over $k = \overline{\mathbb{F}}_q$ and $k = \mathbb{C}$. Let $\tau = t^{\varepsilon_n}\sigma$ ($\tau e_1 = te_n$, $\tau e_i = e_{i-1}$), a generator of Ω, and $\Lambda_{m,j} = t^{\lambda_{m,j}}\mathcal{O}^n$ with $\lambda_{m,j} = (m^{(j)}, (m+1)^{(n-j)})$. Then $\Lambda_{m,j} = \tau^{nm+n-j}\mathcal{O}^n$, the simple objects of $\mathcal{C}_I$ are $S_k = \underline{k}_{\tau^k\mathcal{O}^n}[-k]$ ($k \in \mathbb{Z}$), and

$$\Delta_{\tau^k} *_I \mathrm{IC}_0 = S_k[k], \qquad \mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I = \operatorname{Tri}\{\Delta_{\tau^k} *_I \mathrm{IC}_0 : k \in \mathbb{Z}\}.$$

Over every F, $T_{\tau^k} * \varphi_0 = \mathbf{1}_{\tau^k\mathcal{O}^n}$; these form a basis of C_I, so $h \mapsto h * \varphi_0$ is an isomorphism $\mathbb{C}[\Omega] \to C_I$ and (IG) *holds. Over $\mathbb{F}_q(\!(t)\!)$,* (IG) *with Proposition II gives the sheaf statement over $\overline{\mathbb{F}}_q$ again.*

(c) $\mathbb{G}_m = \mathrm{GL}_2/\mathrm{Sp}_2$ ($g \cdot x = \det(g)x$), over $\overline{\mathbb{F}}_q$, $\mathbb{C}$ and every F: the I-orbits are the $t^m\mathcal{O}^\times$, $\mathcal{S}_I = \{\underline{k}_{t^m\mathcal{O}^\times}\} = \Omega \cdot \mathrm{IC}_0$, so $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$; and $T_{\omega^m} * \varphi_0 = \mathbf{1}_{t^m\mathcal{O}^\times}$ span C_I, so (IG) *holds.*

Status. (a) is proved over $\overline{\mathbb{F}}_q$ for sheaves and over every F for functions; over $\mathbb{C}$ it is conditional on (C) (the pages' Proposition II proves it there directly); the description of $\mathcal{M}_I$ beyond type A_1 is cited. (b) and (c) are proved over $\overline{\mathbb{F}}_q$, $\mathbb{C}$ and every F.

2. Proof

Step 1 (strategy). By Proposition I, a simple object that is no perverse constituent of any generator is not in $\mathcal{C}_I^{\mathrm{Iwa}}$, and every simple object that is a generator up to shift is. For the group we show that no IC_ω, $\omega \in \Omega$, is a constituent (Steps 2–10); for $\mathbb{A}^n$ and $\mathbb{G}_m$ we show that every simple object is a generator up to shift (Steps 12–18).

Step 2 (group: levels, pieces, simple objects, generators). $I = I_H \times I_H$ and $I\backslash LX = I_H\backslash LH/I_H$; the I-orbits are the double cosets $I_H z I_H$, $z \in \widetilde{W}$ [IM, Thm. 2.16], with $\overline{I_H z I_H} = \bigsqcup_{v \le z} I_H v I_H$. Fix finite subsets $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ exhausting Ω and put

$$Y_N = \bigcup\{\overline{IwI} : \ \ell(w) \le N, \ w \in W_{\mathrm{aff}}\Omega_N\}, \qquad Y_{N,M} = Y_N/I_M,$$

$$I_M = \ker\big(I_H \to H(\mathcal{O}/t^M)\big).$$

W_{aff} has finitely many elements of each length and Ω_N is finite, so each level has finitely many pieces, the double cosets $I_H v I_H \subseteq Y_N$ (I1); the transitions are torsors under vector groups, trivial on affine opens, which gives (Lev). (With $Y_N = \bigcup_{\ell(w)\le N} \overline{IwI}$ the level $Y_0 \supseteq \bigsqcup_{\omega\in\Omega} I\omega I$ has infinitely many pieces as soon as Ω is infinite, e.g. $H = \mathrm{GL}_n$.) The stabilizer of vI_M is the image of $I_H \cap vI_Hv^{-1}$, connected (I2; [CY, Lemma 71(i)]). Hence the heart on each $\mathrm{Fl}_{\le u}$ has finite length with simple objects IC_z, $\operatorname{End}\mathrm{IC}_z = k$, $\mathcal{S}_I = \{\mathrm{IC}_z : z \in \widetilde{W}\}$, $\mathrm{IC}_\omega = \Delta_\omega$ for $\omega \in \Omega$, and $\mathcal{C}_I = \operatorname{Thick}\{\mathrm{IC}_z\}$. Next, $\mathcal{H}_{c,I} = \operatorname{Thick}\{\mathrm{IC}_x \boxtimes \mathrm{IC}_y\}$, $(\mathcal{A} \boxtimes \mathcal{B}) *_I \mathcal{M} = \mathcal{A} * \mathcal{M} * \iota^*\mathcal{B}$ with $\iota^*\mathrm{IC}_y = \mathrm{IC}_{y^{-1}}$, and $\mathrm{IC}_0 = \underline{k}_{K_H/I_H} = \mathrm{IC}_{w_0}[-\ell(w_0)]$ [BY, 3.2.7]; by (T2)

$$\mathcal{C}_I^{\mathrm{Iwa}} = \operatorname{Thick}\{M_{x,y} : x, y \in \widetilde{W}\}, \qquad M_{x,y} := \mathrm{IC}_x * \mathrm{IC}_{w_0} * \mathrm{IC}_y.$$

Finally $M_{x,y} = m_!(\mathrm{IC}_x \mathbin{\widetilde{\boxtimes}} \mathrm{IC}_{w_0} \mathbin{\widetilde{\boxtimes}} \mathrm{IC}_y)$ for the proper map m from a projective twisted product. Over $\overline{\mathbb{F}}_q$, with the Weil structures that are trivial on open cells, $M_{x,y}$ is pure of weight $w = \ell(x) + \ell(w_0) + \ell(y)$ and geometrically $M_{x,y} \simeq \bigoplus_n P_n[-n]$, $P_n = {}^pH^n(M_{x,y}) \simeq \bigoplus_z \mathrm{IC}_z \otimes E_{z,n}$, $E_{z,n} = \mathrm{Hom}(\mathrm{IC}_z, P_n)$ [BBD, 5.4.5]; the splitting holds in the equivariant category by [Sun, Thm. 1.2] on the finite-type quotient stacks, which gives (DT).

Step 3 (the constituent criterion). For $\Sigma \subseteq \widetilde{W}$ let $\mathcal{D}_\Sigma$ be the objects $\mathcal{M}$ such that every constituent of every ${}^pH^i(\mathcal{M})$ is some IC_z, $z \in \Sigma$. It is thick: the constituents of ${}^pH^i$ of the middle term of an exact triangle are among those of the outer terms, and those of a retract among those of the object [Stacks, Lemma 13.6.3]. By Step 2, every generator lies in $\mathcal{D}_{\mathcal{M}_I^+}$, so $\mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathcal{D}_{\mathcal{M}_I^+}$; if $\mathrm{IC}_z \notin \mathcal{M}_I^+$, then ${}^pH^0(\mathrm{IC}_z) = \mathrm{IC}_z$ is its own unique constituent and $\mathrm{IC}_z \notin \mathcal{C}_I^{\mathrm{Iwa}}$. Also $\mathcal{D}_\Sigma = \mathrm{Thick}\{\mathrm{IC}_z : z \in \Sigma\}$ (truncation triangles and composition series). No decomposition is used. With (DT), $\mathcal{M}_I = \mathcal{M}_I^+$ and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathrm{IC}_z : \mathrm{IC}_z \in \mathcal{M}_I\}$.

Step 4 (lemma: the sign character). For every non-archimedean F, $\varepsilon\colon \mathcal{H}_I(H) \to \mathbb{Z}$, $\varepsilon(T_w) = (-1)^{\ell(w)}$, is a ring homomorphism with $\varepsilon(T_\omega) = 1$, and $\varepsilon(\mathbf{1}_{K_H}) = 0$. Indeed, by the Iwahori–Matsumoto relations [HKP, §7.2], $T_xT_s = T_{xs}$ if $x < xs$, $T_xT_s = qT_{xs} + (q-1)T_x$ if $xs < x$, $T_xT_\sigma = T_{x\sigma}$ ($\sigma \in \Omega$), and $q(-1)^{\ell(x)-1} + (q-1)(-1)^{\ell(x)} = (-1)^{\ell(x)+1}$; induction on a reduced word gives multiplicativity (this is Borel's special character [Bo, 3.5]). Next $K_H = \bigsqcup_{w \in W} I_H w I_H$ [IM, Prop. 2.4], so $\mathbf{1}_{K_H} = \sum_{w \in W} T_w$, and $w \mapsto ws$ for a finite simple reflection s is a fixed-point-free involution of W changing the parity of ℓ; hence $\varepsilon(\mathbf{1}_{K_H}) = \sum_W (-1)^{\ell(w)} = 0$ because $W \neq 1$.

Step 5 (lemma: traces of convolutions). For Weil objects $\mathcal{A}, \mathcal{B}$ of $\mathcal{H}_{c,I_H}$ and $m \geq 1$, $\mathrm{tr}_m(\mathcal{A} * \mathcal{B}) = \mathrm{tr}_m \mathcal{A} * \mathrm{tr}_m \mathcal{B}$ in $\mathcal{H}_{I,m} = C_c(I_H \backslash H(F_m)/I_H)$: proper base change and the Grothendieck–Lefschetz formula on the fibres of the convolution map [La, (1.1.1.3)], [Zhu, Lemma 5.6.1], the $\mathbb{F}_{q^m}$-points of $LH \times^I \mathrm{Fl}$ being $H(F_m) \times^{I(\mathcal{O}_m)} \mathrm{Fl}(\mathbb{F}_{q^m})$ by Lang's theorem for the connected groups I/I_n [Mi, Cor. 27.56] and a limit over n. Values: $\mathrm{tr}_m \mathrm{IC}_\omega = T_\omega$ (a rational point), $\mathrm{tr}_m \mathrm{IC}_{w_0} = (-1)^{\ell(w_0)} \mathbf{1}_{K_H}$ (K_H/I_H is smooth), $\mathrm{tr}_m \mathrm{IC}_s = -(T_e + T_s)$ ($\overline{\mathrm{Fl}_s} \cong \mathbb{P}^1$).

Step 6 (the sign character kills the traces off Ω). For $z \in \widetilde{W} \setminus \Omega$ and $m \geq 1$, $\varepsilon(\mathrm{tr}_m \mathrm{IC}_z) = 0$. Write $z = \omega z'$ with $z' \in W_{\mathrm{aff}}$ of length $\ell(z) > 0$ and choose a simple affine reflection s with $zs < z$. Then $\overline{IzI}$ is right P_s-stable, so IC_z is pulled back, with its Weil structure, along the $\mathbb{P}^1$-bundle $\mathrm{Fl} \to LH/P_s$, and convolution with $\underline{k}_{P_s/I}$ is $\pi_s^*\pi_{s*}$ [BY, 3.2.7]. So $f = \mathrm{tr}_m \mathrm{IC}_z$ is right $P_s(\mathcal{O}_m)$-invariant and $f * T_s = q^m f$. Applying ε: $-\varepsilon(f) = q^m \varepsilon(f)$, so $\varepsilon(f) = 0$. For $z = \omega \in \Omega$, $\varepsilon(\mathrm{tr}_m \mathrm{IC}_\omega) = \varepsilon(T_\omega) = 1$; and $\varepsilon(\mathrm{tr}_m \mathrm{IC}_{w_0}) = (-1)^{\ell(w_0)} \varepsilon(\mathbf{1}_{K_H}) = 0$ by Step 4.

Step 7 (purity fixes the weight of each multiplicity space). Fix x, y, $w = \ell(x) + \ell(w_0) + \ell(y)$, and let $A_{z,n}$ be the Frobenius of $E_{z,n}$, $f \mapsto \Phi_n \circ \mathrm{Fr}^* f \circ \Phi_z^{-1}$. For every $m \geq 1$:

$$\mathrm{tr}_m M_{x,y} = \mathrm{tr}_m \mathrm{IC}_x * \mathrm{tr}_m \mathrm{IC}_{w_0} * \mathrm{tr}_m \mathrm{IC}_y = \sum_{z,n} (-1)^n \,\mathrm{tr}\big(A_{z,n}^m\big)\, \mathrm{tr}_m \mathrm{IC}_z,$$

and every eigenvalue α of $A_{z,n}$ satisfies $|\iota\alpha| = q^{(w+n-\ell(z))/2}$ for every $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. The first equality is Step 5. For the second, the evaluation map $\bigoplus_z \mathrm{IC}_z \otimes E_{z,n} \to P_n$ is an isomorphism of Weil perverse sheaves (geometric semisimplicity [BBD, 5.3.8], $\mathrm{End}\,\mathrm{IC}_z = \overline{\mathbb{Q}}_\ell$), and the trace is additive on the Frobenius-stable perverse truncation triangles with $\mathrm{tr}(P[-n]) = (-1)^n \,\mathrm{tr}\, P$ [La, (1.1.1.1)]. For the weights, P_n is pure of weight $w + n$ [BBD, 5.4.1–5.4.4]; its isotypic part

$\mathrm{IC}_z \otimes (E_{z,n}, A_{z,n})$ is a Weil direct summand, hence pure of weight $w+n$; on the open cell it is the shifted lisse sheaf $E_{z,n} \otimes \overline{\mathbb{Q}}_\ell[\ell(z)]$ with Frobenius $A_{z,n}$, so $A_{z,n}$ is pure of weight $w+n-\ell(z)$. (Equivalently: the stalk at $\dot{z}I$ bounds the weights from above, Verdier duality from below.)

Step 8 (ruled out: the trace alone). $\varepsilon \circ \mathrm{tr}_m$ is additive on exact triangles, so it vanishes on Tri of the generators, but not on retracts: $\Delta_e \oplus \Delta_e[1]$ has ε-trace 0 at every m and contains Δ_e. The vanishing of a linear functional on generator traces cannot exclude a summand; Step 9 uses the weights of Step 7 to fix the sign of every possible contribution.

Step 9 (no cancellation; no Δ_ω). By Steps 4–6,

$$\varepsilon(\mathrm{tr}_m M_{x,y}) = \varepsilon(\mathrm{tr}_m \mathrm{IC}_x)\, \varepsilon(\mathrm{tr}_m \mathrm{IC}_{w_0})\, \varepsilon(\mathrm{tr}_m \mathrm{IC}_y) = 0,$$

and by Steps 6–7, $\varepsilon(\mathrm{tr}_m M_{x,y}) = \sum_{\omega,n} (-1)^n \,\mathrm{tr}(A_{\omega,n}^m)$. So

$$\sum_{\omega \in \Omega} \sum_n (-1)^n \,\mathrm{tr}\big(A_{\omega,n}^m\big) = 0 \qquad (m \geq 1).$$

Let $\Lambda \subset \overline{\mathbb{Q}}_\ell^\times$ be the finite set of eigenvalues of the nonzero $A_{\omega,n}$, $\mu_{\omega,n}(\alpha)$ the algebraic multiplicities, and $c_\alpha = \sum_{\omega,n} (-1)^n \mu_{\omega,n}(\alpha)$. Then $\sum_\alpha c_\alpha \alpha^m = 0$ for $m = 1, \ldots, |\Lambda|$, and the Vandermonde matrix is invertible, so every $c_\alpha = 0$. By Step 7 with $\ell(\omega) = 0$, every eigenvalue of $A_{\omega,n}$ has weight $w+n$; so α determines $n(\alpha) = \mathrm{wt}(\alpha) - w$, and $c_\alpha = (-1)^{n(\alpha)} \sum_\omega \mu_{\omega,n(\alpha)}(\alpha)$ is a sum of non-negative integers with one sign. Hence all $\mu_{\omega,n} = 0$ and $E_{\omega,n} = 0$: $\Omega \cap \mathcal{M}_I^+ = \emptyset$. By Step 3, $\Delta_\omega = \mathrm{IC}_\omega \notin \mathcal{C}_I^{\mathrm{Iwa}}$ for every ω, in particular $\Delta_e \notin \mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathcal{D}_{\widetilde{W} \setminus \Omega} = \mathrm{Thick}\{\mathrm{IC}_z : z \notin \Omega\}$. Over $\mathbb{C}$, (C) transports $\dim \mathrm{Hom}(\mathrm{IC}_\omega, {}^pH^n M_{x,y}) = 0$ from $\overline{\mathbb{F}}_p$, $p \gg 0$, and Step 3 applies.

Step 10 ((IG) fails, by the character and by Proposition II). Here $C_I = \mathcal{H}_{I_H}$ and $\mathcal{H}_I = \mathcal{H}_{I_H} \otimes \mathcal{H}_{I_H}^{\mathrm{op}}$ acts by $(f_1 \otimes f_2) \cdot \varphi = f_1 * \varphi * f_2^\vee$, $f^\vee(g) = f(g^{-1})$, $\varepsilon(f^\vee) = \varepsilon(f)$. With $\varphi_0 = \mathbf{1}_{K_H}$, $\mathcal{H}_I * \varphi_0 = \mathcal{H}_{I_H} \mathbf{1}_{K_H} \mathcal{H}_{I_H} \subseteq \ker \varepsilon$ by Step 4, while $\varepsilon(T_e) = 1$: (IG) fails over every F (the pages' Proposition E). Over $\mathbb{F}_q(\!(t)\!)$, (Dict) holds for the group (rational cells and Lang for D1, Step 5 for D2, the unitriangular IC-trace basis for D3, Step 7 for D4), so if (IG) held, Proposition II would put every IC_z, in particular Δ_e, in $\mathcal{C}_I^{\mathrm{Iwa}}$, against Step 9. The converse inference is false: Step 8.

Step 11 (the summand set of the group). (i) If $\ell(xw_0y) = \ell(x) + \ell(w_0) + \ell(y)$, then IC_{xw_0y} occurs in $M_{x,y}$ exactly once, in ${}^pH^0$: the support of $M_{x,y}$ is $\overline{\mathrm{Fl}_{xw_0y}}$ and m is an isomorphism over Fl_{xw_0y} [BY, Lemma 3.2.2]. (ii) $\Omega \cap \mathcal{M}_I = \emptyset$ (Step 9). (iii) In type A_1, $w_0 = s_1$, every $z \in W_{\mathrm{aff}} \setminus \{e, s_0\}$ has a reduced word through s_1, so (i) applies; Ω-translates follow from $M_{x,y\omega} = M_{x,y} * \Delta_\omega$; and IC_{s_0} is a summand of $M_{s_0 s_1, s_1 s_0}$ because the Bott–Samelson map for $s_0 s_1 s_0$ is semismall with relevant strata $\mathrm{Fl}_{s_0 s_1 s_0}$ and Fl_{s_0} [dCM, Thm. 4.2.7]. So $\mathcal{M}_I = \widetilde{W} \setminus \Omega$ and, with (DT), $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathrm{IC}_z : \ell(z) > 0\}$. (iv) In general $\mathcal{M}_I = \{z : z \leq_{LR} w_0\} = c_0\Omega$ (positivity [BY, Prop. 3.2.5] and the lowest two-sided cell of Lusztig and Shi, cited); then $\mathrm{IC}_s \notin \mathcal{C}_I^{\mathrm{Iwa}}$ for H simple of rank ≥ 2.

Step 12 ($\mathbb{A}^n$: the lattice chain is one Ω-orbit). $\Lambda_{m,j}$ is spanned by $t^m e_1, \ldots, t^m e_j$ and $t^{m+1} e_{j+1}, \ldots, t^{m+1} e_n$, and applying τ gives $\tau \Lambda_{m,j} = \Lambda_{m,j-1}$; since $\Lambda_{m,0} = \Lambda_{m+1,n}$, induction gives $\Lambda_{m,j} = \tau^{nm+n-j} \mathcal{O}^n$, and $(m,j) \mapsto nm + n - j$ is a bijection $\mathbb{Z} \times \{1, \ldots, n\} \to \mathbb{Z}$. I is the stabilizer of the chain $\{\Lambda_{m,j}\}$, which τ permutes, so $\tau I \tau^{-1} = I$, $I\tau I = \tau I$ and $\ell(\tau) = 0$ [IM,

Prop. 3.2]; $\operatorname{val}\det\tau = 1$, so τ generates $\Omega \cong \mathbb{Z}$ [HKP, §7.1]. The I-orbits on F^n are $\{0\}$ and $O_{m,j} = \tau^k\mathcal{O}^n \setminus \tau^{k+1}\mathcal{O}^n$, $k = nm + n - j$. With $Y_N = t^{-N}\mathcal{O}^n$,

$$V_{N,M} = t^{-N}\mathcal{O}^n/t^M\mathcal{O}^n, \qquad I_{N,M} = 1 + t^{N+M}M_n(\mathcal{O}),$$

the level (N, M) has the $n(N + M) + 1$ orbits o_k $(-nN \le k < nM)$ and $\{0\}$, closures $\tau^k\mathcal{O}^n/t^M\mathcal{O}^n$ linear of dimension $nM - k$, and connected stabilizers (open subsets of affine-linear spaces): (Lev), I1, I2.

Step 13 ($\mathbb{A}^n$: the simple objects S_k). Declare $A \in \mathcal{C}_{N,M}$ perverse if $A[nM]$ is perverse. The transitions q^* (smooth with fibres $\mathbb{A}^n$) and i_* (closed) are then t-exact and fully faithful with images closed under subquotients, so the heart of $\mathcal{C}_I$ is the union of the finite-length level hearts. The orbit closures are smooth, so $\mathrm{IC}(\tau^k\mathcal{O}^n/t^M\mathcal{O}^n) = \underline{k}[nM - k]$, which is $S_k = \underline{k}_{\tau^k\mathcal{O}^n}[-k]$ after the shift [BBD, Thm. 4.3.1]; $S_0 = \mathrm{IC}_0$; every object is an iterated extension of shifted S_k [Stacks, Lemma 13.35.7], so $\mathcal{C}_I = \mathrm{Tri}\{S_k\}$; and no object is supported on $\{0\}$ (the finite-level skyscraper at 0 is S_{nM}).

Step 14 ($\mathbb{A}^n$: length-zero translates). $I\tau^kI/I$ is a point, so $\mathrm{IC}_{\tau^k} = \Delta_{\tau^k}$, and $\tau^kI \times^I LX \to LX$, $[\tau^k i, y] \mapsto \tau^k iy$, is an isomorphism; hence $\Delta_{\tau^k} *_I \mathcal{A} \simeq (\tau^k)_!\mathcal{A}$ and

$$\Delta_{\tau^k} *_I \mathrm{IC}_0 = \underline{k}_{\tau^k\mathcal{O}^n} = S_k[k], \qquad (\tau^k)_!S_l = S_{k+l}[k].$$

No decomposition theorem and no weights enter.

Step 15 ($\mathbb{A}^n$: sheaves generate). Each simple S_k is a shift of the generator $\Delta_{\tau^k} *_I \mathrm{IC}_0$, so $\mathcal{M}_I = \mathcal{S}_I$, and by Step 13 $\mathcal{C}_I = \mathrm{Tri}\{S_k\} \subseteq \mathrm{Tri}\{\Delta_{\tau^k} *_I \mathrm{IC}_0\} \subseteq \mathcal{C}_I^{\mathrm{Iwa}}$. Over $\overline{\mathbb{F}}_q$ and $\mathbb{C}$ alike.

Step 16 ($\mathbb{A}^n$: functions generate). Put $b_k = \mathbf{1}_{\tau^k\mathcal{O}^n}$. With $\mathrm{vol}(I) = 1$, $(T_{\tau^k} * f)(x) = \int_{\tau^kI} f(g^{-1}x)\,dg = f(\tau^{-k}x)$, so $T_{\tau^k} * \varphi_0 = b_k$. The b_k are independent (strictly decreasing lattices) and span C_I: $\varphi \in C_I$ is supported in some $\tau^{-nN}\mathcal{O}^n$, constant on some $\tau^{nM}\mathcal{O}^n$ and on each orbit, so $\varphi = \sum_k \varphi(o_k)(b_k - b_{k+1}) + \varphi(0)b_{nM}$. Hence $\mathbb{C}[\Omega] \to C_I$, $h \mapsto h * \varphi_0$, is an isomorphism [HKP, §7.1] and (IG) holds. (Also $T_{s_0} * \varphi_0 = b_{-1} + qb_1 - b_0$, and $\mathrm{tr}(S_k) = (-1)^k b_k$.)

Step 17 ($\mathbb{A}^n$: sheaves again, from functions). Over $\mathbb{F}_q(\!(t)\!)$ the inputs of Proposition II hold for $\mathbb{A}^n$: the levels of Step 12 are defined over $\mathbb{F}_q$ with rational orbit representatives $t^m e_j$, Lang applies to the connected $I/I_{N,M}$ [Mi, Cor. 27.56], $\mathrm{IC}_w *_I \mathrm{IC}_0 = \pi_{N,M}^* m_{w!}\mathrm{pr}_1^*\mathrm{IC}_w$ for the proper lattice bundle $m_w\colon E_w \to V_{N,M}$ and is pure, so (DT) holds [Sun, Thm. 1.2]; the simple objects are Frobenius-stable with independent traces $(-1)^k b_k$. Proposition II with (IG) (Step 16) gives $\mathcal{M}_I = \mathcal{S}_I$ and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ over $\overline{\mathbb{F}}_q$. This route needs purity; Step 15 does not.

Step 18 ($\mathbb{G}_m$). $\det(I) = \mathcal{O}^\times$, so the I-orbits on $LX = F^\times$ are the $t^m\mathcal{O}^\times$, open and closed. With $Y_{N,M} = \bigsqcup_{|m|\le N} t^m(\mathcal{O}/t^M)^\times$, I acts through the connected group $(\mathcal{O}/t^M)^\times$, simply transitively on each piece; so $\mathcal{S}_I = \{\underline{k}_{t^m\mathcal{O}^\times}\}$ up to shift. $\omega = P\,\mathrm{diag}(t, 1)$ normalizes I and $\det\omega = -t$, so $\Delta_{\omega^m} *_I \mathrm{IC}_0 = (\omega^m)_!\underline{k}_{\mathcal{O}^\times} = \underline{k}_{t^m\mathcal{O}^\times}$ and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$; and $T_{\omega^m} * \varphi_0 = \mathbf{1}_{t^m\mathcal{O}^\times}$, which span C_I.

Step 19 (conclusion). Steps 9–11 prove (a), Steps 12–17 prove (b), Step 18 proves (c). In each of the three rows the two levels agree: the group fails at both, $\mathbb{A}^n$ and $\mathbb{G}_m$ generate at both. □

3. Remarks

Against the pages' Proposition II. There $\mathcal{C}_I^{\mathrm{Iwa}} \subseteq \ker \Phi$ for the exact affine Whittaker averaging Φ, and $\Phi(\Delta_e) \neq 0$. Here $\varepsilon \circ \mathrm{tr}$ is the decategorified shadow of Φ; exactness, which lets Φ see retracts, is replaced by the weights of Step 7.

Against the pages' IV.1. There the standard objects of the pieces are reached by the triangles $\Delta_{O_{m,j}} \to \underline{k}_{\Lambda_{m,j}} \to \underline{k}_{\Lambda_{m,j-1}}$ from the translates $\Delta_{t^{\lambda_{m,j}}} *_I \mathrm{IC}_0 = \underline{k}_{\Lambda_{m,j}}[-2j(n-j)]$ (the I3 dévissage). Here the simple objects of the colimit heart are identified, each is a length-zero translate, and the criterion concludes.

Functions. Step 10's first argument is the pages' Proposition E; its second derives the same failure over $\mathbb{F}_q(\!(t)\!)$ from the sheaf statement.

Repairs from the audit. Levels must be Ω-truncated (Step 2); the statement over $\mathbb{C}$ needs (C); the lowest-cell description beyond type A_1 is cited.

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