PROPOSITION IV: GL_2/T FAILS IWAHORI GENERATION AT BOTH LEVELS

For $X = \mathrm{GL}_2/T$ the Iwahori-generation pages decide the function level (the classical page, Proposition F: (IG) fails) and leave the sheaf level open. This note computes both Iwahori-generated parts exactly. They miss the same two closed I-orbits, so this row of Table 1 is not a divergence. The sheaf half assumes the strict level presentation (I1) for $N \geq 1$, as the pages' Proposition IV.1 does for $\mathbb{A}^n$.

1. Setting

Groups and the tree. $G = \mathrm{GL}_2 \supset T$ the diagonal torus, $X = G/T$. For sheaves $k = \overline{\mathbb{F}}_q$ (p odd, $\overline{\mathbb{Q}}_\ell$-coefficients) or $k = \mathbb{C}$ (Betti), $F = k(\!(t)\!)$, $\mathcal{O} = k[\![t]\!]$; for functions F is a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer t and residue field $\mathbb{F}_q$. $K = G(\mathcal{O})$, $I \subset K$ the matrices upper triangular modulo t, $\omega = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$, $K' = \omega K \omega^{-1}$.

$\mathcal{T}$ is the tree of $\mathrm{PGL}_2(F)$: vertices are homothety classes of $\mathcal{O}$-lattices in F^2, ends are the points of $\mathbb{P}^1(F)$, and an apartment is the geodesic between two ends [Cas, 7.2]. Put $v_0 = [\mathcal{O}^2]$, $v_1 = [\mathcal{O} \oplus t\mathcal{O}]$ and $C_0 = (v_0, v_1)$, the base edge; K fixes v_0, K' fixes v_1, I fixes C_0.

The variety as geodesics. $gT \mapsto M_{gT} = g\,\mathrm{diag}(1,-1)g^{-1}$ identifies X with the quadric $\{x^2 + yz = 1\} \subset \mathbb{A}^3$ of trace-zero involutions ($p \neq 2$), and $X(F) = G(F)/T(F)$ (Hilbert 90) with the set of oriented geodesics $\gamma = (a, b)$, $a, b \in \mathbb{P}^1(F)$ the $(+1)$- and (-1)-eigenlines of M_γ. For $n \geq 0$ put

$$Y_n = \{\gamma :\ d(v_0, \gamma) \leq n\} = LX \cap t^{-n}\mathfrak{gl}_2(\mathcal{O}), \qquad Y_0 = L^+X = X(\mathcal{O}), \qquad LX = \varinjlim_n Y_n.$$

Orbits. For $i \in \{0, 1\}$ and $m \geq 0$,

$$A_\pm = \{\gamma \supset C_0,\ \gamma \text{ runs } v_0 \to v_1\ (+) \text{ or } v_1 \to v_0\ (-)\},$$
$$O_{i,m} = \{\gamma :\ d(v_i, \gamma) = m,\ d(v_{1-i}, \gamma) = m + 1\}.$$

Categories. A *level presentation* is $LX = \varinjlim_N Y_N$, $Y_N = \varprojlim_M Y_{N,M}$, with Y_N closed and I-stable, $Y_{N,M}$ of finite type, $\pi_{N,M} \colon Y_N \to Y_{N,M}$ smooth, affine and surjective, and I acting on $Y_{N,M}$ through a finite-dimensional quotient $I/I_{N,M}$. A *piece* of level (N, M) is $\pi_{N,M}^{-1}(o)$ for an $I/I_{N,M}$-orbit o; $\mathcal{P}_{N,M}$ is the set of pieces and $\mathcal{C}_I^{(N,M)}$ the objects pulled back from $D_c^b(I/I_{N,M}\backslash Y_{N,M})$.

(I1) *Finite levels.* LX has a level presentation such that $\mathcal{C}_I = \bigcup_{N,M} \mathcal{C}_I^{(N,M)}$ and $|\mathcal{P}_{N,M}| < \infty$ for all N, M.

$\mathcal{C}_I = \mathrm{Shv}_c(I\backslash LX)$; $\mathcal{C}_K$, $\mathcal{C}_{K'}$ are the K- and K'-equivariant categories, mapped to $\mathcal{C}_I$ by forgetting equivariance. $\mathcal{H}_{c,I} = \mathrm{Shv}_c(I\backslash LG/I)$ acts by convolution $*_I$; IC_w is the intersection complex of $\overline{IwI}/I$, $w \in \widetilde{W} = \mathbb{Z}^2 \rtimes S_2$, and IC_0 the constant sheaf on L^+X. Thick is the smallest

full subcategory containing 0 and closed under isomorphisms, shifts, cofibres and retracts, and

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathrm{IC}_w *_I \mathrm{IC}_0 :\ w \in \widetilde{W}\},$$
$$\mathcal{M}_I = \{S \text{ simple} :\ S[s] \text{ is a direct summand of some } \mathrm{IC}_w *_I \mathrm{IC}_0\}.$$

$\mathrm{IC}_{0,n} = \mathrm{IC}(Y_n)$ and $\mathrm{IC}_{1,n} = \mathrm{IC}(\omega Y_n)$, normalized to be perverse; $\mathrm{IC}_{A_\pm} = \underline{k}_{A_\pm}$.

Functions. $C_I = C_c^\infty(X(F))^I$, and C_K, $C_{K'}$ are its K- and K'-invariant subspaces. $\mathcal{H}_I = C_c(I\backslash G(F)/I)$ with $\mathrm{vol}\, I = 1$, $T_w = \mathbf{1}_{IwI}$, $(T_s - q)(T_s + 1) = 0$, $(f * \varphi)(\gamma) = \int f(g)\varphi(g^{-1}\gamma)\, dg$; s_1 is the finite simple reflection (the panel of C_0 at v_0), s_0 the affine one (the panel at v_1); $\varphi_0 = \mathbf{1}_{X(\mathcal{O})} = \mathbf{1}_{Y_0}$.

$$f_n(\gamma) = \max\big(0,\, n + 1 - d(v_0, \gamma)\big), \qquad f_n^{(1)} = T_\omega f_n, \qquad f_{-1} = f_{-1}^{(1)} = 0,$$

$$\lambda(f) = f(A_+) - f(A_-), \qquad \psi(f) = \tfrac{1}{2}\big(f(A_+) + f(A_-)\big) + \sum_{i \in \{0,1\},\, m \geq 0} (-1)^{m+1} f(O_{i,m}),$$

where $f(O)$ is the value of f on the orbit O; $\varepsilon_\pm$ are the characters of $\mathcal{H}_I$ with $\varepsilon_\pm(T_{s_0}) = \varepsilon_\pm(T_{s_1}) = -1$, $\varepsilon_\pm(T_\omega) = \pm 1$.

Weil classes. Over $\overline{\mathbb{F}}_q$, with all orbits defined over $\mathbb{F}_q$: $\mathcal{M}_A$ is the Grothendieck group of Weil objects of $\mathcal{C}_I$ [CY, 8.2], a module over $\mathbb{Z}[A]$, $A = \overline{\mathbb{Q}}_\ell^\times/\mu_\infty$, with $\mathbb{Z}[v^{\pm 1}] \subset \mathbb{Z}[A]$ by $v \mapsto -[q^{-1/2}]$, and over the affine Hecke algebra $\mathcal{H}$: $[\mathcal{G} *_I \mathcal{F}] = [\mathcal{G}] \cdot [\mathcal{F}]$. Write $\mathbf{1}_O = [j_{O!}\underline{k}_O]$, $H_s = vT_s = [j_{s!}k[1](\frac{1}{2})]$, $\underline{H}_s = H_s + v = [\mathrm{IC}_s]$ with $\mathrm{IC}_s = k_{\overline{IsI}/I}[1](\frac{1}{2})$; λ and ψ extend $\mathbb{Z}[A]$-linearly by their values on the $\mathbf{1}_O$.

Proposition IV (GL_2/T: both levels fail)**.** *Let $X = \mathrm{GL}_2/T$, with the closed I-orbits $A_\pm$ and $K' = \omega K \omega^{-1}$.*

(a) Functions, over every non-archimedean local field F:

$$\mathcal{H}_I * \varphi_0 = \mathrm{span}\{f_n, f_n^{(1)} : n \geq 0\} = C_K + C_{K'}, \qquad C_I = (C_K + C_{K'}) \oplus \mathbb{C}\mathbf{1}_{A_+} \oplus \mathbb{C}\mathbf{1}_{A_-},$$

and $(\lambda, \psi)\colon C_I/\mathcal{H}_I * \varphi_0 \xrightarrow{\sim} \varepsilon_- \oplus \varepsilon_+$ as $\mathcal{H}_I$-modules.

(b) Sheaves, over $\overline{\mathbb{F}}_q$ (p odd) and over $\mathbb{C}$, assuming (I1)*:*

$$\mathcal{M}_I = \{\mathrm{IC}(Y_n), \mathrm{IC}(\omega Y_n) : n \geq 0\},$$
$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}(\mathcal{C}_K \cup \mathcal{C}_{K'}) \subsetneq \mathcal{C}_I = \mathrm{Thick}\big(\mathcal{C}_I^{\mathrm{Iwa}} \cup \{\underline{k}_{A_+}, \underline{k}_{A_-}\}\big),$$

and the simple objects outside $\mathcal{C}_I^{\mathrm{Iwa}}$ are exactly $\underline{k}_{A_+}$ and $\underline{k}_{A_-}$.

(c) Second proof of (b), over $\overline{\mathbb{F}}_q$, assuming (I1)*: 2ψ vanishes on every $[\mathrm{IC}_w *_I \mathrm{IC}_0]$, is ≥ 0 on the class of every simple object and equals 1 on $\underline{k}_{A_\pm}$; by Proposition V(b) of the run, $\underline{k}_{A_\pm} \notin \mathcal{C}_I^{\mathrm{Iwa}}$. The Steinberg functional λ also vanishes on every generator class but takes the values ± 1 on $\underline{k}_{A_\pm}$, so it cannot see a summand $\underline{k}_{A_+} \oplus \underline{k}_{A_-}$.*

Status. (a) is unconditional. (b) and (c) are conditional on (I1) for $N \geq 1$; its finiteness half and its case $N = 0$ are proved in Step 1. Both levels fail at $A_\pm$, and the spherical level holds (Remark 2.2).

2. Proof

Step 1 (tree model and I-orbits). PGL_2 acts simply transitively modulo T on ordered pairs of distinct points of $\mathbb{P}^1$, so $X \cong (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$. Since $2 \in \mathcal{O}^\times$, $M_\gamma L = L$ iff $L = (L \cap a) \oplus (L \cap b)$ iff $[L] \in \gamma$ [Cas, 7.2]; for $\gamma = (\infty, t^{-m})$ one computes $\operatorname{ord}_t M_\gamma = -m = -d(v_0, \gamma)$, and K is transitive on $\{d(v_0,\gamma) = m\}$, so $-\operatorname{ord}_t M_\gamma = d(v_0,\gamma)$ and Y_n is as stated.

The map $gI \mapsto gC_0$ from $G(F)/I$ onto the oriented edges is a torsor under $Z(F)/Z(\mathcal{O}) \cong \mathbb{Z}$. As $Z \subset T$, $IgT \mapsto Tg^{-1}I$ identifies $I\backslash X(F)$ with the $T(F)$-orbits of oriented edges, T the stabilizer of $\gamma_0 = (0, \infty)$. $T(F)$ translates γ_0, with two orbits of edges on it ($A_\pm$); $T(\mathcal{O})/Z(\mathcal{O}) \cong \mathcal{O}^\times$ acts simply transitively on the $(q-1)q^{m-1}$ vertices at distance $m \geq 1$ above a vertex of γ_0, with stabilizer $1 + t^m\mathcal{O}$ [Cas, 7.4]. An edge off γ_0 has endpoint distances $(m, m+1)$ or $(m+1, m)$, which gives the orbits $O_{0,m}$, $O_{1,m}$. Stabilizers are

$$\mathrm{Stab}_I(\gamma) = Z(\mathcal{O}) \cdot \{\mathrm{diag}_\gamma(a, 1) : a \in 1 + t^{m'}\mathcal{O}\}, \qquad m' = \max_i d(v_i, \gamma),$$

connected; if $\pi_{N,M}$ has connected fibres, the stabilizers of $I/I_{N,M}$ on $Y_{N,M}$ are connected too, so every equivariant local system on an orbit is constant. T_s is self-adjoint for an I-invariant measure, which gives $\mu(O_{i,m}) = (q-1)q^m\mu(A_\pm)$, the point count of $\mathbb{G}_m \times \mathbb{A}^m$: so $\delta(A_\pm) = 0$, $\delta(O_{i,m}) = m + 1$ is a dimension theory [CY, Prop. 41].

Y_n is closed and K-stable, with K-orbits $O_m = \{d(v_0, \cdot) = m\} = O_{0,m} \sqcup O_{1,m-1}$. The curve $s \mapsto \left(\begin{smallmatrix} 1 & -2(st^{-n} + t^{-m}) \\ 0 & -1 \end{smallmatrix}\right)$ lies in O_n for $s \neq 0$ and in O_m at $s = 0$, so $O_m \subset \overline{O_n}$ for $m < n$; O_n is irreducible and $O_{0,n}$ is its open I-orbit, so $\overline{O_{0,n}} = Y_n$, and $\overline{O_{1,n}} = \omega Y_n$. $A_\pm$ are the preimages of the closed B-orbits of $X(k)$ in the closed L^+X, hence closed: the closed orbits of [CY, Lemma 31]. So Y_N is the union of the $2N + 3$ orbits $A_\pm$, $O_{0,m}$ $(m \leq N)$, $O_{1,m}$ $(m < N)$; each closure is cut out by finitely many coefficient conditions, so at large M the pieces of Y_N are its orbits: the finiteness half of (I1). At $N = 0$, $L^+X = \varprojlim_M X(\mathcal{O}/t^M)$ with affine-space transitions, K acting through $K/(1 + t^M\mathfrak{gl}_2(\mathcal{O}))$: the strict form. Only the tree is used, so the orbit list holds over every non-archimedean F.

Step 2 (Hecke action on orbit indicators). For $m \geq 1$,

$$T_{s_1}\mathbf{1}_{A_\pm} = \mathbf{1}_{A_\mp} + \mathbf{1}_{O_{0,0}}, \quad T_{s_1}\mathbf{1}_{O_{0,0}} = (q-1)(\mathbf{1}_{A_+} + \mathbf{1}_{A_-}) + (q-2)\mathbf{1}_{O_{0,0}},$$
$$T_{s_1}\mathbf{1}_{O_{1,m-1}} = \mathbf{1}_{O_{0,m}}, \qquad T_{s_1}\mathbf{1}_{O_{0,m}} = q\,\mathbf{1}_{O_{1,m-1}} + (q-1)\mathbf{1}_{O_{0,m}};$$

T_{s_0} is obtained by exchanging the indices $0 \leftrightarrow 1$ of the orbits (keeping $A_\pm$), $T_\omega \mathbf{1}_O = \mathbf{1}_{\omega O}$, $T_\omega^2 = 1$ on C_I, and $\varphi_0 = \mathbf{1}_{A_+} + \mathbf{1}_{A_-} + \mathbf{1}_{O_{0,0}}$ satisfies $T_{s_1}\varphi_0 = q\varphi_0$. Indeed $(T_s\mathbf{1}_O)(\gamma)$ counts the chambers $C' \neq C_0$ of the s-panel of C_0 in relative position O to γ. If $\gamma \ni v_0$, the panel at v_0 splits as: the neighbour of v_0 towards b (giving A_+), towards a (A_-), the other $q-1$ ($O_{0,0}$); if $d(v_0,\gamma) = m \geq 1$: the neighbour towards γ ($O_{1,m-1}$), the other q ($O_{0,m}$). Over $\overline{\mathbb{F}}_q$ the panel is cut into points, affine lines and punctured affine lines, with Tate classes $1, q, q-1$; so the same matrices with $q = v^{-2}$ give $[\Delta_s *_I \mathcal{F}] = H_s[\mathcal{F}]$ in $\mathcal{M}_A$. This is [CY, Prop. 74, (29)], with types IIIa ($A_\pm$), IIIb ($O_{0,0}$), IIa, IIb for s_1. The coefficients are polynomials of degree at most one in q.

Step 3 (generators by words). Cells $IwI/I \cong \mathbb{A}^{\ell(w)}$ have connected stabilizers, so $\mathcal{H}_{c,I} = \mathrm{Thick}\{\Delta_w\}$, and the cone of $\Delta_w \to \mathrm{IC}_w[-\ell(w)]$ lives on lower cells, so $\mathcal{H}_{c,I} = \mathrm{Thick}\{\mathrm{IC}_w\}$ by induction on $\ell(w)$. As $- *_I \mathrm{IC}_0$ is exact, $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathrm{IC}_w *_I \mathrm{IC}_0\}$ by (T2). For a reduced

word $w = s_{i_1} \cdots s_{i_r} \omega^j$ put

$$B_w = \mathrm{IC}_{s_{i_1}} * \cdots * \mathrm{IC}_{s_{i_r}} * \mathrm{IC}_{\omega^j} * \mathrm{IC}_0 \in \mathcal{C}_I^{\mathrm{Iwa}}.$$

The Bott–Samelson object $\mathrm{IC}_{s_{i_1}} * \cdots * \mathrm{IC}_{\omega^j}$ is a direct sum of shifted IC_y containing IC_w once: decomposition theorem for the proper Bott–Samelson map of finite-type quotient stacks, in the equivariant form ([Sun, Thm. 1.2] over $\overline{\mathbb{F}}_q$; [BL] over $\mathbb{C}$; the run's audit A7-3). Hence $\mathrm{IC}_w * \mathrm{IC}_0$ is a retract of B_w, and $B_w \simeq \bigoplus_y \mathrm{IC}_y * \mathrm{IC}_0[s_y]$: the generating words.

Step 4 (intersection complexes of the closures). Let $\mathrm{Gr}_{\le n} = \overline{Kt^{(n,0)}K}/K$ (vertices u with $d(v_0,u) \le n$, $d(v_0,u) \equiv n \bmod 2$), $LG_{\le n}$ its preimage in LG, and $\nu_n \colon \widetilde{Y}_n = LG_{\le n} \times^K L^+X \to LX$, $[g,x] \mapsto gx$. Then $\nu_n[g,x] = \gamma$ iff $\gamma \ni gv_0$, so $\nu_n^{-1}(\gamma)$ is the set of vertices of γ in $\mathrm{Gr}_{\le n}$, of cardinality $\max(0, n+1-d(v_0,\gamma))$; ν_n is proper and quasi-finite, hence finite, with image Y_n, and a universal homeomorphism over the open K-orbit O_n. $\mathrm{Gr}_{\le n}$ is rationally smooth (the Kazhdan–Lusztig polynomials of the infinite dihedral group are 1), so $\mathrm{IC}_{\widetilde{Y}_n} = k[n+1]$. A finite map is t-exact and commutes with duality, so it carries the support and cosupport conditions of $k[n+1]$ to those of an intersection complex [BBD, 1.4.24], [dCM, 4.2.7]:

$$\mathrm{IC}_{0,n} \cong \nu_{n*} k[n+1], \qquad \mathcal{H}^{-n-1}(\mathrm{IC}_{0,n})_\gamma = k^{\,f_n(\gamma)}, \qquad [\mathrm{IC}_{0,n}] = v^{n+1} f_n, \qquad \mathrm{IC}_{1,n} = \omega_* \mathrm{IC}_{0,n}.$$

Stalks are pointwise pure Tate. Since $\nu_{n*}(k_{\mathrm{Gr}_{\le n}}[n] \widetilde{\boxtimes} k_{L^+X}[1])$ is by definition the Satake action, $\mathrm{IC}(Y_n) \cong \mathrm{IC}^{\mathrm{Gr}}_{(n,0)} *_K \mathrm{IC}_0[1]$; this step uses (I1) only through finite-type models of ν_n.

Step 5 (convolution rules). For $n \ge 0$, with $\mathrm{IC}_{i,-1} = 0$, up to Tate twists:

$$\mathrm{IC}_{s_0} *_I \mathrm{IC}_{0,n} \cong \mathrm{IC}_{1,n+1} \oplus \mathrm{IC}_{1,n-1}, \qquad \mathrm{IC}_{s_1} *_I \mathrm{IC}_{0,n} \cong \mathrm{IC}_{0,n}[1] \oplus \mathrm{IC}_{0,n}[-1],$$
$$\mathrm{IC}_{\omega} *_I \mathrm{IC}_{i,n} \cong \mathrm{IC}_{1-i,n},$$

and the same with $0 \leftrightarrow 1$. By Step 4, $\mathrm{IC}_{s_0} * \mathrm{IC}_{0,n} = \nu'_* k[n+2]$ for $\nu' \colon P_{s_0} \times^I \widetilde{Y}_n \to LX$, a $\widetilde{Y}_n$-bundle over $P_{s_0}/I = \mathbb{P}^1$. Fix $u \in \gamma$ with $r = d(v_1,u)$; the neighbours u' of v_1 are at distance $r \pm 1$ from u, so the condition $d(u',u) \le n$, $d(u',u) \equiv n$ forces $r \equiv n+1$ and holds for all u' if $r \le n-1$, for one u' if $r = n+1$, for none otherwise:

$$\nu'^{-1}(\gamma) = \bigsqcup_{r(u) \le n-1} \mathbb{P}^1 \;\sqcup \bigsqcup_{r(u)=n+1} \mathrm{pt}, \qquad H^*(\nu'^{-1}\gamma) = k^{f^{(1)}_{n+1}(\gamma)} \oplus k^{f^{(1)}_{n-1}(\gamma)}[-2].$$

k is pure and ν' proper, so $\nu'_* k[n+2]$ is pure [BBD, 5.1.14] and an equivariant direct sum of shifted intersection complexes of orbits with constant coefficients ([Sun, Thm. 1.2]; over $\mathbb{C}$, [BL]). On $O_{1,n+1}$ the stalk is $k[n+2]$, so $\mathrm{IC}_{1,n+1}$ occurs once, unshifted; the remainder has stalks $k^{f^{(1)}_{n-1}}[n]$, so $\mathrm{IC}_{1,n-1}$ occurs once and the rest has zero stalks. For the second rule, $\mathrm{IC}_{0,n}$ is P_{s_1}-equivariant ($P_{s_1} = K$), so $P_{s_1} \times^I LX \cong \mathbb{P}^1 \times LX$ and $\mathrm{IC}_{s_1} * \mathrm{IC}_{0,n} \cong R\Gamma(\mathbb{P}^1,k)[1] \otimes \mathrm{IC}_{0,n}$. For the third, $I\omega I = \omega I$.

Step 6 (constituent subcategories are thick). On each finite level the perverse heart has finite length [BBD, 4.3.1]; smooth pullback shifted by the relative dimension and closed pushforward $Y_N \hookrightarrow Y_{N+1}$ are t-exact and preserve intersection complexes, so $\mathcal{C}_I$ has a perverse t-structure whose simple objects, by Step 1, are the $\mathrm{IC}_{i,n}$ and $\underline{k}_{A_\pm}$. For a set Σ of simple objects let $\mathcal{S}_\Sigma$ be the objects all of whose ${}^pH^i$ have Jordan–Hölder factors in Σ. It is closed

under shifts and retracts, and under cones: in the long exact sequence, ${}^pH^i(C)$ is an extension of a subobject of ${}^pH^{i+1}(A)$ by a quotient of ${}^pH^i(B)$. So $\mathcal{S}_\Sigma$ is thick [Stacks, Lemma 13.6.3].

Step 7 (sheaves: proof of (b)). By Step 5 and induction on r, starting from $\mathrm{IC}_0 = \mathrm{IC}_{0,0}[-1]$, every B_w lies in $\mathrm{add}\{\mathrm{IC}_{i,n}[m]\}$. At a finite level the category has finite-dimensional Hom spaces and split idempotents, so it is Krull–Schmidt, and $\mathrm{End}\,\mathrm{IC}_{i,n} = k$; so every summand of the retract $\mathrm{IC}_w{*}\mathrm{IC}_0$ of B_w (Step 3) is some $\mathrm{IC}_{i,n}[m]$, and $\mathcal{M}_I \subseteq \{\mathrm{IC}_{i,n}\}$. Conversely, the alternating words give $B_{s_0} = \mathrm{IC}_{1,1}[-1]$, $B_{s_1 s_0} = (\mathrm{IC}_{0,2} \oplus \mathrm{IC}_{0,0})[-1]$, and in general $\mathrm{IC}_{i,n}$ as a summand of some $B_w \simeq \bigoplus_y \mathrm{IC}_y * \mathrm{IC}_0[s_y]$, hence, by Krull–Schmidt, of one $\mathrm{IC}_y * \mathrm{IC}_0$; ω gives the other index. So $\mathcal{M}_I = \{\mathrm{IC}_{i,n}\}$ and

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathrm{IC}_w * \mathrm{IC}_0\} = \mathrm{Thick}\{\mathrm{IC}_{i,n}\} = \mathrm{Thick}(\mathcal{C}_K \cup \mathcal{C}_{K'}),$$

the last equality because $\mathcal{C}_K = \mathrm{Thick}\{\mathrm{IC}(Y_n)\}$ by dévissage over the finitely many K-orbits of a level with connected stabilizers, and $\omega_* \mathcal{C}_K = \mathcal{C}_{K'}$. By Step 6, $\mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathcal{S}_{\{\mathrm{IC}_{i,n}\}}$; $\underline{k}_{A_\pm}$ is simple with support $A_\pm \neq Y_n, \omega Y_n$, so $\underline{k}_{A_\pm} \notin \mathcal{C}_I^{\mathrm{Iwa}}$. Finally $\mathcal{C}_I = \mathrm{Thick}(\mathcal{C}_I^{\mathrm{Iwa}} \cup \{\underline{k}_{A_+}, \underline{k}_{A_-}\})$ by dévissage over the $2N+3$ orbits of each level. This proves (b). The exclusion uses only the perverse constituents of the generators, for which the non-equivariant splitting of [BBD, 5.4.5] suffices.

Step 8 (functions: proof of (a)). *Shadow identities.* $(T_{s_0}+1)f_n(\gamma) = \sum_{u \sim v_1} \max(0, n+1-d(u,\gamma))$. If $r_1 = d(v_1,\gamma) \geq 1$, one neighbour is at distance $r_1 - 1$ and q at $r_1 + 1$; if $r_1 = 0$, two are at distance 0 and $q-1$ at distance 1. In both cases

$$(T_{s_0}+1)f_n = f_{n+1}^{(1)} + q\, f_{n-1}^{(1)}, \qquad (T_{s_1}+1)f_n = (q+1)f_n, \qquad T_\omega f_n = f_n^{(1)}.$$

Generation. By [IM, Thm. 3.5], $\mathcal{H}_I$ is generated by T_{s_0}, T_{s_1} and $T_\omega^{\pm 1}$. By the identities the span $V = \mathrm{span}\{f_n, f_n^{(1)}\}$ is $\mathcal{H}_I$-stable and contains $f_0 = \varphi_0$, so $\mathcal{H}_I * \varphi_0 \subseteq V$; conversely, by induction on n,

$$f_0^{(1)} = T_\omega \varphi_0, \qquad f_{n+1}^{(1)} = (T_{s_0}+1)f_n - q\, f_{n-1}^{(1)}, \qquad f_{n+1} = T_\omega f_{n+1}^{(1)}.$$

Since $f_n - f_{n-1} = \mathbf{1}_{Y_n}$ and the $\mathbf{1}_{Y_n} - \mathbf{1}_{Y_{n-1}}$ form a basis of C_K, $\mathrm{span}\{f_n\} = C_K$; apply ω.

Codimension two. Each I-orbit is compact and open, so the $\mathbf{1}_O$ form a basis of C_I, and $\mathbf{1}_{O_{0,0}} = \mathbf{1}_{Y_0} - \mathbf{1}_{A_+} - \mathbf{1}_{A_-}$, $\mathbf{1}_{O_{0,m}} = \mathbf{1}_{O_m} - \mathbf{1}_{O_{1,m-1}}$ give $C_I = C_K + C_{K'} + \mathbb{C}\mathbf{1}_{A_+} + \mathbb{C}\mathbf{1}_{A_-}$. A functional μ kills $C_K + C_{K'}$ iff $\mu(O_{0,0}) = \mu(O_{1,0}) = -\mu(A_+) - \mu(A_-)$ and $\mu(O_{i,m}) = -\mu(O_{1-i,m-1})$: a 2-dimensional space, free in $(\mu(A_+), \mu(A_-))$, spanned by λ, ψ. The pairing of $\{\lambda, \psi\}$ with $\{\mathbf{1}_{A_+}, \mathbf{1}_{A_-}\}$ has determinant $1 \cdot \frac{1}{2} - (-1) \cdot \frac{1}{2} = 1$, so the sum is direct.

The quotient. On the blocks of Step 2, $\lambda \circ T_s = -\lambda$, $\psi \circ T_s = -\psi$ $(s = s_0, s_1)$, $\lambda \circ T_\omega = -\lambda$, $\psi \circ T_\omega = \psi$; for instance $\psi(T_{s_1}\mathbf{1}_{O_{0,0}}) = (q-1) - (q-2) = 1 = -\psi(\mathbf{1}_{O_{0,0}})$ and $\psi(T_{s_1}\mathbf{1}_{O_{0,m}}) = q(-1)^m + (q-1)(-1)^{m+1} = -\psi(\mathbf{1}_{O_{0,m}})$. So $(\lambda, \psi)\colon C_I \to \varepsilon_- \oplus \varepsilon_+$ is $\mathcal{H}_I$-linear, onto, with kernel $C_K + C_{K'}$. Moreover λ is, up to $\mathrm{vol}(A_+)$, the restriction to C_I of the Steinberg map $\mathcal{S}(X) \to \mathrm{St}$ given by the T-invariant functional $h \mapsto h([1:0]) - h([0:1])$, and $\mathrm{St}^I = \varepsilon_-$, a special character in the sense of [Bor, 3.5]. This proves (a), with no hypothesis on F and no use of (I1).

Step 9 (positive certificates). Over $\overline{\mathbb{F}}_q$, assume (I1), finitely many orbits per level, every simple object Frobenius-stable with its weight-0 structure $j_{!*}k[\delta](\delta/2)$, and IC_w, IC_0 pure of weight 0. Let $\Lambda\colon \mathcal{M}_A \to \mathbb{Z}[A]$ be $\mathbb{Z}[A]$-linear.

(i) If $\Lambda \circ \underline{H}_s = 0$ for every simple affine s and $\Lambda \circ T_\omega = \pm\Lambda$, and $W \neq 1$, then Λ kills every $[\mathrm{IC}_w * \mathrm{IC}_0]$. For a finite simple s, IC_0 is P_s-equivariant, so $\underline{H}_s[\mathrm{IC}_0] = (v + v^{-1})[\mathrm{IC}_0]$ and $(v + v^{-1})\Lambda([\mathrm{IC}_0]) = 0$; $\mathbb{Z}[A]$ is a domain (A is torsion-free), so $\Lambda([\mathrm{IC}_0]) = 0$. Writing $[\mathrm{IC}_w]$ as a $\mathbb{Z}[v^{\pm1}]$-combination of monomials $\underline{H}_{s_{i_1}} \cdots \underline{H}_{s_{i_r}} T_\omega^j$ finishes (i).

(ii) If Λ kills every generator class and $\Lambda([S]) \in \mathbb{Z}_{\geq 0}[v^{\pm1}]$ for every simple S, then $\Lambda([S]) \neq 0$ implies $S \notin \mathcal{M}_I$ and $S \notin \mathcal{C}_I^{\mathrm{Iwa}}$. Indeed $\mathcal{F} = \mathrm{IC}_w * \mathrm{IC}_0$ is pure of weight 0 [BBD, 5.1.14], so ${}^pH^n(\mathcal{F})$ is pure of weight n and $\mathcal{F} \simeq \bigoplus_n {}^pH^n(\mathcal{F})[-n]$ with ${}^pH^n(\mathcal{F}) \cong \bigoplus_S S \otimes V_S^n$ [BBD, 5.4.5, 5.4.6]; the eigenvalues of V_S^n lie in $q^{\mathbb{Z}/2}$ modulo μ_∞ ($[\mathcal{F}]$ lies in the module of [CY, Thm. 1]) and have weight n, so they are $q^{n/2}$, so $[\mathcal{F}] = \sum_S m_S[S]$ with $m_S = \sum_n d_S^n v^{-n}$, $d_S^n \geq 0$. Then $0 = \sum_S m_S \Lambda([S])$ is a sum of Laurent polynomials with nonnegative coefficients, so each term vanishes, and $\Lambda([S]) \neq 0$ forces $d_S^n = 0$ for all n: S is not a perverse constituent of any generator, and $S \notin \mathcal{C}_I^{\mathrm{Iwa}}$ by Step 6.

Step 10 (the Steinberg functional is not a certificate). λ satisfies (i), so it kills every generator class; $\lambda([\mathrm{IC}_{i,n}]) = 0$ (the argument of Step 11), but $\lambda([\underline{k}_{A_\pm}]) = \pm 1$ has mixed sign. From $0 = \sum_S m_S \lambda([S])$ one gets only $m_{A_+} = m_{A_-}$: a summand $\underline{k}_{A_+} \oplus \underline{k}_{A_-}$ has λ-value 0 and is invisible to λ. At function level λ suffices, since the classical page's Corollary D needs no positivity: this is Proposition F there.

Step 11 (second proof: proof of (c)). By Step 8, $2\psi \circ \underline{H}_s = v(2\psi \circ T_s + 2\psi) = 0$ and $2\psi \circ T_\omega = 2\psi$; the identities are polynomial in q, so they hold in $\mathcal{M}_A$ with $q = v^{-2}$, and Step 9(i) applies. Every orbit other than $A_\pm$ has closure Y_n or ωY_n, stable under K or K' (Step 1), so $\mathrm{IC}_{i,n}$ is P_{s_1}- or P_{s_0}-equivariant, $\underline{H}_s[\mathrm{IC}_{i,n}] = (v + v^{-1})[\mathrm{IC}_{i,n}]$ for that s, and $2\psi([\mathrm{IC}_{i,n}]) = 0$. $\underline{k}_{A_\pm} = j_{A_\pm !}k$ has class $\mathbf{1}_{A_\pm}$ and $2\psi = 1$. These are all the simple objects, so Step 9(ii) gives $\underline{k}_{A_\pm} \notin \mathcal{M}_I$ and $\underline{k}_{A_\pm} \notin \mathcal{C}_I^{\mathrm{Iwa}}$, without Steps 4 and 5. Among maps Λ with $\Lambda \circ \underline{H}_s = 0$ and $\Lambda \circ T_\omega = \pm\Lambda$ the positive ones are the nonnegative multiples of ψ; $\lambda + 2\psi$ and $2\psi - \lambda$ are certificates for one closed orbit each.

Step 12 (conclusion: the same two orbits). Over $\mathbb{F}_q(\!(t)\!)$, given (I1): a Weil object whose underlying object lies in $\mathcal{C}_I^{\mathrm{Iwa}}$ has class in the span of the $[\mathrm{IC}_{i,n} \otimes V]$, $\dim V = 1$ (Step 7), whose traces are multiples of f_n, $f_n^{(1)}$ (Step 4); traces are additive [Lau, 1.1.1]. So the trace of $\mathcal{C}_I^{\mathrm{Iwa}}$ is $\mathcal{H}_I * \varphi_0$, and $\operatorname{tr} \underline{k}_{A_\pm} = \mathbf{1}_{A_\pm}$ spans the complement by (a). Both levels fail, at the same two orbits; the row is not a divergence, and functions $\Rightarrow$ sheaves is vacuous here because (IG) fails. One mechanism: every generator is a sum of objects equivariant for K or K' (Step 5), and $A_\pm$ are the only orbits whose closure is stable under neither P_{s_0} nor P_{s_1} (type IIIa for both reflections). □

Remark 2.1. The classical page's Proposition F proves $\mathcal{H}_I * \varphi_0 \subsetneq C_I$ because the Steinberg representation is T-distinguished; (a) refines "proper" to "codimension two" and names the complement. The pages' summary lists the sheaf cell of this row as open; (b) decides it as *fails*, under the condition (I1) that Proposition IV.1 also assumes.

Remark 2.2. The spherical level holds: the K-orbit closures are the Y_n, with connected stabilizers, and $\mathrm{IC}(Y_n) \cong \mathrm{IC}^{\mathrm{Gr}}_{(n,0)} *_K \mathrm{IC}_0[1]$ (Step 4), so every simple object of $\mathcal{C}_K$ is a Satake object and $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$; at function level, spherical generation holds by Proposition F.

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