PROPOSITION V: WHERE A DIVERGENCE CAN OCCUR

Iwahori generation has a sheaf form, $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$, and a function form, (IG). Propositions I and II of the run turn both into statements about one module and one cyclic vector. This note records what they leave possible: over $\mathbb{F}_q(\!(t)\!)$ only "sheaves generate, (IG) fails" can occur, and a positivity test can decide a row at sheaf level.

1. Setting

q is a power of an odd prime, $k = \overline{\mathbb{F}}_q$ with $\overline{\mathbb{Q}}_\ell$-coefficients, $F = \mathbb{F}_q(\!(t)\!)$ and $\mathcal{O} = \mathbb{F}_q[\![t]\!]$. $G \supseteq B \supseteq T$ is split connected reductive over $\mathbb{F}_q$ and X a smooth affine G-variety over $\mathbb{F}_q$ with a level presentation $LX = \varinjlim_N Y_N$, $Y_N = \varprojlim_M Y_{N,M}$. $K = L^+G \supseteq I$ is the Iwahori, $\widetilde{W} = X_*(T) \rtimes W$ with W the finite Weyl group, s runs over the simple affine reflections, $P_s = I \cup IsI$, and $\Omega \subseteq \widetilde{W}$ is the set of elements of length zero.

Sheaves. $\mathcal{H}_{c,I} = \mathrm{Shv}_c(I\backslash LG/I)$ with intersection complexes IC_w; $\mathcal{C}_I = \bigcup_{N,M} \mathcal{C}_I^{(N,M)}$; IC_0 is the constant sheaf on L^+X; $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathrm{IC}_w *_I \mathrm{IC}_0 : w \in \widetilde{W}\}$. $\mathcal{D}$ is the set of pairs (piece of a fine level, irreducible equivariant local system), $\mathcal{S}_I = \{\mathrm{IC}_Q : Q \in \mathcal{D}\}$ the simple objects, $\mathcal{M}_I \subseteq \mathcal{S}_I$ those that are shifted direct summands of some $\mathrm{IC}_w *_I \mathrm{IC}_0$, and $\mathcal{M}_I^+ \supseteq \mathcal{M}_I$ those that are perverse constituents of one.

Functions. $\mathcal{H}_I = C_c(I\backslash G(F)/I)$ with $\mathrm{vol}\, I = 1$ and $T_w = \mathbf{1}_{IwI}$, so $(T_s - q)(T_s + 1) = 0$; $C_I = C_c^\infty(X(F))^I \supseteq C_K = C_c^\infty(X(F))^K$; $\varphi_0 = \mathbf{1}_{X(\mathcal{O})}$. Iwahori and spherical generation are

$$\text{(IG)} \quad \mathcal{H}_I * \varphi_0 = C_I, \qquad\qquad \text{(SG)} \quad \mathcal{H}_K * \varphi_0 = C_K.$$

Weil classes [CY, §8.2]. $A = \overline{\mathbb{Q}}_\ell^\times/\mu$ with μ the roots of unity. $\mathbb{M}$ is the Grothendieck group of Weil objects of $\mathcal{C}_I$ whose Frobenius eigenvalues lie in $q^{\mathbb{Z}/2} \subset A$, and $\mathbb{H}$ the same for $\mathcal{H}_{c,I}$; $\mathbb{H}$ is an algebra acting on $\mathbb{M}$ by $[\mathcal{G}] \cdot [\mathcal{F}] = [\mathcal{G} *_I \mathcal{F}]$ [CY, (27), Prop. 70]. Write $v[\mathcal{F}] = [\mathcal{F}[1](\frac{1}{2})]$; then $\mathbb{M}$ is a free $\mathbb{Z}[v^{\pm 1}]$-module on the standard classes $\mathbf{1}_Q = [j_{Q!}\mathcal{L}_Q]$ ([CY, Thm. 1], where $q^{1/2} = -v^{-1}$). Put

$$T_w = [j_{w!}k], \qquad C'_w = [\mathrm{IC}_w], \qquad \underline{H}_s = C'_s = v(T_s + 1), \qquad m_0 = [\mathrm{IC}_0], \qquad c_Q = [\mathrm{IC}_Q],$$

each intersection complex with its Weil structure pure of weight 0. The c_Q form a $\mathbb{Z}[v^{\pm 1}]$-basis of $\mathbb{M}$, unitriangular against the $\mathbf{1}_Q$ [CY, Thms. 3, 4]. For $x \in \mathbb{M}$ write $x = \sum_Q \kappa_Q(x)\, c_Q$ and $\mathrm{supp}_c(x) = \{Q : \kappa_Q(x) \neq 0\}$. The specialization $\mathrm{sp}_q \colon \mathbb{Z}[v^{\pm 1}] \to \overline{\mathbb{Q}}_\ell$, $v \mapsto -q^{-1/2}$, gives $\mathbb{M}_q = \mathbb{M} \otimes_{\mathrm{sp}_q} \overline{\mathbb{Q}}_\ell$.

A $\mathbb{Z}[v^{\pm 1}]$-linear $\Lambda \colon \mathbb{M} \to \mathbb{Z}[v^{\pm 1}]$ is of *sign type* if $\Lambda(\underline{H}_s x) = 0$ for every simple affine s and $\Lambda(T_\omega x) = \pm\Lambda(x)$ for every $\omega \in \Omega$; equivalently $\Lambda(T_s x) = -\Lambda(x)$. A *positive certificate* for $Q \in \mathcal{D}$ is a linear Λ with $\Lambda(C'_w m_0) = 0$ for all w, $\Lambda(c_{Q'}) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ for all Q', and $\Lambda(c_Q) \neq 0$.

Hypotheses. (Lev), I1, I2, (DT) and (Dict) are those of Propositions I and II: the level axioms, finitely many pieces per level, connected stabilizers at each level, the equivariant decomposition of every $\mathrm{IC}_w *_I \mathrm{IC}_0$, and the trace dictionary over $\mathbb{F}_q$. In addition:

(Wt) *Weights.* X and the level presentation are defined over $\mathbb{F}_q$; every simple object IC_Q is Frobenius-stable and carries its weight-0 Weil structure $j_{!*}\mathcal{L}_Q[\delta](\frac{\delta}{2})$; IC_w and IC_0 carry Weil structures pure of weight 0; and every $\mathrm{IC}_w *_I \mathrm{IC}_0$ lies in $\mathcal{C}_I$ and is the proper pushforward of a pure complex on a finite-type model.

(Sym) *Symmetric setting* [CY, §8.2]. $X = G/G^\theta$ for an involution θ over $\mathbb{F}_q$, $p \neq 2$, G^θ connected, (B_0, T_0) a θ-stable Borel pair over $\mathbb{F}_q$ with T_0 split, and every I-orbit on LX Frobenius-stable with connected stabilizer; so $\mathcal{D}$ is the set of I-orbits O, with IC_O of constant coefficients.

(Dict) and (DT) give (Wt) [BBD, 5.1.14]. For the symmetric rows, I1 in its strict form is assumed, not proved.

Proposition V (where a divergence can occur)**.** *In this setting:*

(a) (Divergence type.) *Assume* (Lev), I1, I2, (Dict) *and* (DT). *If* (IG) *holds over F, then $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ over k. So if the two levels differ, sheaves generate and* (IG) *fails over F. In terms of $\mathbb{M}$:*

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I \iff \bigcup_{w \in \widetilde{W}} \mathrm{supp}_c(C'_w m_0) = \mathcal{D}, \qquad \text{(IG)} \iff \mathrm{span}_{\overline{\mathbb{Q}}_\ell}\{\mathrm{sp}_q(C'_w m_0)\}_{w \in \widetilde{W}} = \mathbb{M}_q;$$

span implies support, and for general families of vectors support does not imply span.

(b) (Positive certificates.) *Over k, assume* (Lev), I1 *and* (Wt), *and let $\Lambda\colon \mathbb{M} \to \mathbb{Z}[v^{\pm 1}]$ be linear.*

(i) If Λ is of sign type, and $W \neq 1$ or $\Lambda(m_0) = 0$, then $\Lambda(C'_w m_0) = 0$ for every w.

(ii) If $\Lambda(C'_w m_0) = 0$ for every w and $\Lambda(c_{Q'}) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ for every $Q' \in \mathcal{D}$, then $\Lambda(c_Q) \neq 0$ implies $\mathrm{IC}_Q \notin \mathcal{M}_I^+$, hence $\mathrm{IC}_Q \notin \mathcal{C}_I^{\mathrm{Iwa}}$.

(iii) $\mathrm{IC}_Q \notin \mathcal{M}_I^+$ if and only if a positive certificate for Q exists.

(c) (Closed-orbit test.) *Assume* (Sym) *and $W \neq 1$, and let $\psi\colon \mathbb{M} \to \mathbb{Z}[v^{\pm 1}]$ be of sign type.*

(i) $\psi(c_O) = 0$ for every I-orbit O that is not closed.

(ii) (Sheaf half.) *Under the hypotheses of (b), if $\psi(c_O) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ for every closed orbit O and $\psi(c_{O_1}) \neq 0$ for one, then $\mathrm{IC}_{O_1} \notin \mathcal{C}_I^{\mathrm{Iwa}}$: sheaves fail.*

(iii) (Function half.) *Under* (Dict), *if $\mathrm{sp}_q\psi(c_O) \neq 0$ for some closed orbit O, then* (IG) *fails over F. Positivity does not give this: $v+2v^2 \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ and $\mathrm{sp}_4(v+2v^2) = 0$.*

Status. (a) is proved over F given the inputs of Proposition II; (b) and (c) over k, with I1 among the hypotheses. No row of Table 1 is decided here.

2. Proof

Step 1 (two questions about one module). The levels agree when both hold or both fail; a divergence is one of the two mixed cases:

	(IG) holds	(IG) fails
$\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$	agree	divergence
$\mathcal{C}_I^{\mathrm{Iwa}} \neq \mathcal{C}_I$	divergence	agree

The proof reads both rows in $\mathbb{M}$ (Step 3), excludes the lower-left case (Step 4), shows the argument cannot exclude the upper-right one (Step 5), and gives tests for the lower row (Steps 7–12).

Step 2 (multiplicities are nonnegative). Assume (Lev) and (Wt). For $w \in \widetilde{W}$ let $\mathcal{F} = \mathrm{IC}_w *_I \mathrm{IC}_0$, $V_Q^n = \mathrm{Hom}(\mathrm{IC}_Q, {}^pH^n\mathcal{F})$ and $d_Q^n = \dim V_Q^n$. Then

$$C'_w m_0 = \sum_Q m_{Q,w}\, c_Q, \qquad m_{Q,w} = \sum_n d_Q^n\, v^{-n} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}],$$

and $m_{Q,w} \neq 0$ if and only if IC_Q is a perverse constituent of $\mathcal{F}$.

Proof. $\mathcal{F}$ is a proper pushforward of a pure complex of weight 0, hence pure of weight 0 [BBD, 5.1.14]. So ${}^pH^n\mathcal{F}$ is pure of weight n [BBD, 5.4.1] and geometrically semisimple [BBD, 5.3.8]; since $I/I_{N,M}$ is connected, it is semisimple in the equivariant heart. As $\mathrm{End}\,\mathrm{IC}_Q = \overline{\mathbb{Q}}_\ell$ and IC_Q is Frobenius-stable, the isotypic decomposition ${}^pH^n\mathcal{F} \simeq \bigoplus_Q \mathrm{IC}_Q \otimes V_Q^n$ is one of Weil objects, and V_Q^n is pure of weight n. Its eigenvalues lie in $q^{\mathbb{Z}/2}$ modulo μ: they occur on stalks of ${}^pH^n\mathcal{F}$, hence (the perverse spectral sequence degenerates [BBD, 5.4.5]) on stalks of $\mathcal{F}$, and $[\mathcal{F}] = C'_w m_0 \in \mathbb{M}$. So $[V_Q^n] = d_Q^n[q^{n/2}] = d_Q^n(-1)^n v^{-n}$, and the truncation triangles give $[\mathcal{F}] = \sum_n (-1)^n [{}^pH^n\mathcal{F}] = \sum_Q \big(\sum_n d_Q^n v^{-n}\big) c_Q$. □

Step 3 (sheaves read the support, functions the span). (i) Assume (Lev), I1, (Wt) and (DT). By Step 2, $Q \in \mathrm{supp}_c(C'_w m_0)$ iff IC_Q is a perverse constituent of $\mathrm{IC}_w *_I \mathrm{IC}_0$, iff (by (DT)) it is a shifted summand. So $\mathcal{M}_I = \mathcal{M}_I^+ = \{\mathrm{IC}_Q : Q \in \bigcup_w \mathrm{supp}_c(C'_w m_0)\}$, and the criterion $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I \iff \mathcal{M}_I = \mathcal{S}_I$ (Proposition I) gives the first equivalence of (a).

(ii) Under (Dict), the trace of Frobenius identifies $\mathbb{M}_q$ with $C_I \otimes \overline{\mathbb{Q}}_\ell$ as an $\mathcal{H}_I$-module: $\mathbf{1}_Q \mapsto \mathbf{1}_{Q(\mathbb{F}_q)}$, $T_w \mapsto T_w$, and m_0 goes to a nonzero multiple of φ_0 (the coefficients of [CY, (29)] are point counts). The class $C'_w = v^{\ell(w)} T_w + \sum_{y<w} p_{y,w} T_y$ specializes to an element of $\mathcal{H}_I$ unitriangular in the T_y. Hence $\mathrm{span}\{\mathrm{sp}_q(C'_w m_0)\} = \mathcal{H}_I * \varphi_0 \otimes \overline{\mathbb{Q}}_\ell$, and (IG) over $\mathbb{C}$ is equivalent to (IG) over $\overline{\mathbb{Q}}_\ell$ (a rank condition on integral matrices). □

Step 4 (span implies support: functions generate, so sheaves generate). Assume the hypotheses of (a). If (IG) holds, the $\mathrm{sp}_q(C'_w m_0)$ span $\mathbb{M}_q$ (Step 3(ii)). The $\mathrm{sp}_q(c_Q)$ are a basis of $\mathbb{M}_q$ (unitriangular), and the Q-coordinate of $\mathrm{sp}_q(C'_w m_0)$ is $\mathrm{sp}_q(m_{Q,w})$. A spanning family is not killed by the coordinate form, so for each Q some w has $\mathrm{sp}_q(m_{Q,w}) \neq 0$, hence $m_{Q,w} \neq 0$ and $Q \in \mathrm{supp}_c(C'_w m_0)$. Step 3(i) gives $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$. This is Proposition II; if $\mathrm{IC}_Q \notin \mathcal{M}_I$, its coordinate form is a linear form that kills $\mathcal{H}_I * \varphi_0$ and not $\mathrm{tr}\,\mathrm{IC}_Q$, so (IG) fails. □

Step 5 (support does not imply span). Two simple objects c_1, c_2 and columns $c_1 + c_2$, $2c_1 + 2c_2$: every row is nonzero, the columns span a line. A single n can also cancel across degrees ($2a - b = 0$ with $|b| = q^{1/2}|a|$ at $q = 4$); traces over all $\mathbb{F}_{q^n}$ determine a class, one n does not [Lau, 1.1.2]. At spherical level the gap occurs: for E_6/F_4 sheaves generate and $C_K = \mathcal{H}_K * \varphi_0 \oplus \mathbb{C}\mathbf{1}_{O_{3\omega_1^\vee}}$ over every $\mathbb{F}_{q^n}(\!(t)\!)$ (Proposition IX of the satake run). No Iwahori-level instance is known. A common sufficient condition for both levels is weighted triangularity: for every piece P a Weil object $\mathcal{G} *_I Y$, $Y \in \{\mathrm{IC}_0\} \cup \{\Delta_{P'} : P' < P\}$, supported on $\overline{P}$ with nonzero trace on $P(\mathbb{F}_q)$ gives (IG) by induction on the finite poset of pieces. This completes (a). □

Step 6 (E_6/F_4: (IG) fails). Applying the idempotent $e_K = [K : I]^{-1}\mathbf{1}_K$ gives $C_K = e_K * C_I$ and $e_K * \mathcal{H}_I * \varphi_0 = \mathcal{H}_K * \varphi_0$ [HKP, §4.1], so (IG) implies (SG). For E_6/F_4, (SG) fails over $\mathbb{F}_q(\!(t)\!)$ and over p-adic F, p odd (satake run, Propositions VI, VII, IX(a); [Sak, Thm. 8.0.2]). So (IG) fails, and E_6/F_4 is a divergence if and only if $\mathcal{M}_I = \mathcal{S}_I$; its Iwahori sheaf level is open. □

Step 7 (exclusion in trace form). Assume the inputs of (a) and write $F_n = \mathbb{F}_{q^n}((t))$. (i) $\mathrm{IC}_Q \in \mathcal{M}_I$ iff for some n the coordinate form $\kappa_Q^{(n)}$ of $\mathrm{tr}_n \mathrm{IC}_Q$ is nonzero on $\mathcal{H}_{I_n} * \varphi_0$: weights separate degrees and a Vandermonde determinant bounds n. (ii) For $W \neq 1$, a form λ_n on C_{I_n} with $\lambda_n(T_s * f) = -\lambda_n(f)$ kills $\mathcal{H}_{I_n} * \varphi_0$ and $\mathrm{tr}_n S$ for every P_s-equivariant S; if for every n it vanishes on the other non-equivariant simple objects and not on S_0, then $S_0 \notin \mathcal{M}_I$. (iii) With local systems allowed (I2 dropped), let $\lambda_r(h * f) = \varepsilon_r(h)\lambda_r(f)$ for all $r \geq 1$, with $\lambda_r(\mathrm{tr}_r S) = 0$ for every F^r-stable simple $S \not\simeq S_0$ (for one, equivalently every, F^r-Weil structure) and $\lambda_r(\mathrm{tr}_r S_0) = c\beta^r$, $c \neq 0$, S_0 pure. Then $S_0 \notin \mathcal{M}_I^+$. The trace of a perverse cohomology object is expanded over Frobenius orbits of constituents: an orbit of length d contributes 0 to tr_r unless $d \mid r$, and then a sum of traces of F^r-stable simple objects. Each test needs control at every n; positivity replaces it by one condition on the classes c_Q. □

Step 8 (sign maps kill the generators). Let Λ be of sign type and $s \in W$ a finite simple reflection. IC_0 is K-equivariant, so $\mathrm{IC}_s *_I \mathrm{IC}_0 \simeq \mathrm{IC}_0[1](\frac{1}{2}) \oplus \mathrm{IC}_0[-1](-\frac{1}{2})$ ($P_s/I \simeq \mathbb{P}^1$), i.e. $\underline{H}_s m_0 = (v + v^{-1})m_0$. Then $(v + v^{-1})\Lambda(m_0) = 0$, and $\mathbb{Z}[v^{\pm 1}]$ is a domain, so $\Lambda(m_0) = 0$. The $\underline{H}_s$ and T_ω generate $\mathbb{H}$ over $\mathbb{Z}[v^{\pm 1}]$; write C'_w as a combination of monomials $\underline{H}_{s_1} \cdots \underline{H}_{s_r} T_\omega$. A monomial with $r \geq 1$ is killed by the first factor, and $\Lambda(T_\omega m_0) = \pm\Lambda(m_0) = 0$. As $[\mathrm{IC}_w *_I \mathrm{IC}_0] = C'_w m_0$, this is (b)(i); for $W = 1$ the hypothesis $\Lambda(m_0) = 0$ replaces the first part. □

Step 9 (positive certificates exclude, and are sharp). (ii) By Step 2, $0 = \Lambda(C'_w m_0) = \sum_Q m_{Q,w}\Lambda(c_Q)$ with every term in $\mathbb{Z}_{\geq 0}[v^{\pm 1}]$. A sum of Laurent polynomials with nonnegative coefficients vanishes only if each does, and $\mathbb{Z}[v^{\pm 1}]$ is a domain, so $\Lambda(c_Q) \neq 0$ forces $m_{Q,w} = 0$ for every w: $\mathrm{IC}_Q \notin \mathcal{M}_I^+$. Let $\mathcal{E} \subseteq \mathcal{C}_I$ be the objects all of whose perverse cohomology objects have their constituents in $\mathcal{M}_I^+$; it is thick [Stacks, Lemma 13.6.3] and contains every generator, hence $\mathcal{C}_I^{\mathrm{Iwa}}$, and $\mathrm{IC}_Q \notin \mathcal{E}$. So $\mathrm{IC}_Q \notin \mathcal{C}_I^{\mathrm{Iwa}}$.

(iii) ($\Leftarrow$) is (ii). ($\Rightarrow$) Take $\Lambda = \kappa_Q$: $\kappa_Q(C'_w m_0) = m_{Q,w} = 0$ by Step 2, $\kappa_Q(c_{Q'}) \in \{0, 1\}$, and $\kappa_Q(c_Q) = 1$. The coordinate form is not of sign type, so this gives no test computable from [CY, (29)] alone. □

Step 10 (orbits that are not closed are invisible). By [CY, Lemma 31] an I-orbit O is closed iff it is not of type IIb, IIIb or IVb for any simple affine s; by [CY, Lemma 28] an orbit of one of these types is open in $P_s O$. So for O not closed, $\overline{O} = \overline{P_s O}$ is P_s-stable for some s, and IC_O, of constant coefficients by (Sym), is P_s-equivariant (P_s acts at a finite level through a connected group). As in Step 8, $\underline{H}_s c_O = (v + v^{-1})c_O$, so $(v + v^{-1})\psi(c_O) = 0$ and $\psi(c_O) = 0$. This is (c)(i). □

Step 11 (closed-orbit test: sheaf half). By Step 10, $\psi(c_O) = 0$ on the orbits that are not closed; on the closed ones $\psi(c_O) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ by hypothesis. Under (Sym) these are all the simple objects. By Step 8 ($W \neq 1$) ψ kills every $C'_w m_0$, so it is a positive certificate for O_1 and Step 9 gives $\mathrm{IC}_{O_1} \notin \mathcal{C}_I^{\mathrm{Iwa}}$. The same holds if a unit multiple $\pm v^j \psi$ satisfies the hypothesis: a row of this kind can be a divergence of type "sheaves generate" only if no nonzero sign-type ψ has a unit multiple that is nonnegative on all closed-orbit classes. □

Step 12 (closed-orbit test: function half). Under (Dict) put $\psi_q = \mathrm{sp}_q \circ \psi$ on $\mathbb{M}_q = C_I \otimes \overline{\mathbb{Q}}_\ell$. From $\underline{H}_s = v(T_s + 1)$ and v a unit, $\psi_q(T_s * f) = -\psi_q(f)$ and $\psi_q(T_\omega * f) = \pm\psi_q(f)$: ψ_q is an $\mathcal{H}_I$-map to a character with $T_s \mapsto -1$, which respects $(T_s - q)(T_s + 1) = 0$. $\psi_q(\varphi_0) = 0$ because

$\psi(m_0) = 0$ (Step 8). If $\mathrm{sp}_q\psi(c_O) \neq 0$ for a closed O, then $\psi_q \neq 0$ and $\mathcal{H}_I{*}\varphi_0 \subseteq \ker\psi_q \subsetneq C_I \otimes \overline{\mathbb{Q}}_\ell$: (IG) fails.

The hypothesis is needed. On a closed orbit $c_O = v^{\delta(O)}\mathbf{1}_O$, and the values of ψ on closed orbits are constrained only by the module relations; a value $\psi(c_{O_1}) = v + 2v^2$ is positive and specializes at $q = 4$ to $-\frac{1}{2} + 2 \cdot \frac{1}{4} = 0$. □

Step 13 (conclusion). Steps 3–5 prove (a), Steps 8–9 prove (b), Steps 10–12 prove (c). Where the inputs of (a) hold as well, (a) turns the sheaf half into a failure of (IG) over F without the specialization, through the coordinate form of Step 4; the specialization makes ψ_q itself the witness. The open rows of Table 1:

X	(IG)	what decides the row
E_6/F_4	fails	$\mathcal{M}_I = \mathcal{S}_I$? Divergence iff yes
$\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$, $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$, $n \geq 2$	fails, $\mathrm{char}\, F = 0$	(IG) over $\mathbb{F}_q(\!(t)\!)$, then $\mathcal{M}_I = \mathcal{S}_I$?
$\mathrm{GL}_{2p}/\mathrm{GL}_p^2$, $\mathrm{GL}_3/\mathrm{GL}_1 \times \mathrm{GL}_2$	fails, $\mathrm{char}\, F = 0$	(IG) over $\mathbb{F}_q(\!(t)\!)$, then $\mathcal{M}_I = \mathcal{S}_I$?
$\mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$, $q > p$, $n \geq 4$;	open	$\mathcal{M}_I = \mathcal{S}_I$? If not, (IG) fails too, given the
$\mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}$, $n \geq 4$		inputs of (a)

In every row the sheaf side is open and needs (Lev) for Chen–Yi's presentation; a nonzero sign-type ψ that is nonnegative on the closed orbits [CY, Lemma 31] decides it as "sheaves fail". □

3. Remarks

Remark 3.1 (a functional of mixed sign). For $X = \mathrm{GL}_2/T$ the I-orbits are $A_\pm$ (closed) and $O_{i,m}$. The Steinberg functional λ ($\lambda(\mathbf{1}_{A_\pm}) = \pm 1$, zero elsewhere) and its twist ψ ($2\psi(\mathbf{1}_{A_\pm}) = 1$, $2\psi(\mathbf{1}_{O_{i,m}}) = 2(-1)^{m+1}$) both satisfy $\Lambda \circ T_s = -\Lambda$, with $\lambda \circ T_\omega = -\lambda$ and $\psi \circ T_\omega = \psi$. On simple objects $\lambda(c_{A_\pm}) = \pm 1$ and $2\psi(c_{A_\pm}) = 1$, both zero on the other $c_{i,n}$. From λ one gets only $m_{A_+,w} = m_{A_-,w}$: a summand $\mathrm{IC}_{A_+} \oplus \mathrm{IC}_{A_-}$ is invisible to it. 2ψ is a positive certificate, so $\mathrm{IC}_{A_\pm} \notin \mathcal{C}_I^{\mathrm{Iwa}}$ (given I1), and $\mathrm{sp}_q(2\psi)(c_{A_+}) = 1 \neq 0$, so (IG) fails too; $\lambda + 2\psi$ and $2\psi - \lambda$ are certificates for one closed orbit each. At function level either functional witnesses the failure of (IG).

Remark 3.2 (the spherical model). At level K the same two statements are rows and columns of one matrix: (SG) over $\mathbb{F}_q(\!(t)\!)$ makes every row nonzero and gives $\mathcal{C}_K^{\mathrm{Sat}} = \mathcal{C}_K$ (Proposition II of the satake run), and the converse fails for E_6/F_4 (Proposition IX there). Part (a) is the Iwahori form of the first; whether the Iwahori form of the second holds is open, with E_6/F_4 the candidate.

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