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SINGULARITIES OF ORBIT CLOSURES IN LOOP SPACES OF SYMMETRIC VARIETIES

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Abstract. We study the singularities of closures of Iwahori orbits on loop spaces of symmetric varieties extending the celebrated work of Lusztig-Vogan to the affine setting. We show that the IC-complexes of orbit closures (with possible non-trivial coefficients) are pointwise pure and satisfy a parity vanishing property. We apply those geometric results to study the affine Lusztig-Vogan modules and obtain fundational results about them including the positivity properties of the affine Kazhdan-Lusztig-Vogan polynomials. Along the way, we construct conical transversal slices inside loop spaces of symmetric varieties generalizing the work of Mars-Springer in the finite dimensional setting. Our results answer a question of Lusztig.

We deduce results for singularities of spherical orbit closures and provide applications to relative Langlands duality including the positivity for the relative Kostka-Foulkes polynomials and the formality conjecture.

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1. Introduction

Let G be a connected reductive group over $k = \overline{\mathbb{F}}_p$ of characteristic $p \neq 2$, and $\theta : G \to G$ an involution. Let $K = G^\theta$ be the symmetric subgroup of fixed point set of θ. In their celebrated work [LV83], Lusztig-Vogan explore connections between the geometry of K-orbits on the flag variety of G and representation theory of real groups. They establish deep results on singularities of K-orbit closures on flag varieties, along the way, they introduce several fundamental objects in representation theory, such as Lusztig-Vogan modules and Kazhdan-Lusztig-Vogan polynomials, providing representation theoretic and combinatorial interpretation for the singularities.

In [Lus21, page 27], Lusztig asked a question about extending the work [LV83] to the affine setting, namely, one considers the LK-orbits on the affine flag variety $\mathcal{F}\ell = LG/I_0$, where

$LK = K(\!(t)\!)$ (resp. $LG = G(\!(t)\!)$) is the loop group of K (resp. G) and $I_0 \subset L^+G = G[\![t]\!]$ is an Iwahori subgorup. The main results of the paper provide an answer to Lusztig's question.[1]

We will be working with I_0-orbits on the loop space $LX = X(\!(t)\!)$ of the symmetric variety $X = G/K$ instead of LK-orbits on $\mathcal{F}\ell$.[2] Let $\mathscr{D}$ be the set of pairs $(\mathcal{L}, \mathcal{O})$ where $\mathcal{L}$ is an I_0-equivariant local system on an I_0-orbit $\mathcal{O}$ in LX. We show that the free $\mathbb{Z}[q, q^{-1}]$-module M (here q is an indeterminate) with basis indexed by $\mathscr{D}$ has a canonical module structure over the affine Hecke algebra of G, to be called the *affine Lusztig-Vogan module*. We show that the IC-complexes for closures of I_0-orbits on LX give rise to another basis of M, called the *Kazhdan-Lusztig basis*, with several remarkable properties. The entries of the transition matrix between the standard basis and the Kazhdan-Lusztig basis are polynomials in q, to be called the *affine Kazhdan-Lusztig-Vogan polynomials*, and we provide an algorithm to compute them. We further show that the IC-complexes of orbit closures are pointwise pure and satisfy a parity vanishing property and we deduce from it the positivity of the affine Kazhdan-Lusztig-Vogan polynomials.

The proof strategy follows the one in [LV83] but there are several new difficulties in the affine setting and hence requires new ideas. First, the combinatorics and geometry of Iwahori orbits are much more complicated. For example, Iwahori orbits and their closures are all semi-infinite (dimension and codimension are all infinite) and the theory of perverse sheaves and Verdier duality on them are delicate. To this end, we prove several fundational results on Iwahori orbits including orbit parametrizations, characterization of closed orbits, and the placidness of orbit closures. The latter enables us to reduce many technical difficulties in sheaf theory to the finite dimensional setting. Secondly, the proof of parity vanishing of IC-complexes in *loc. cit.* uses representation theory of real groups, which is not available in the affine setting. In [MS98], Mars-Springer found a more geometric argument for the parity vanishing which bypass the use of real groups.[3] We extend Mars-Springer's argument to the affine setting. A key geometric ingredient here is a construction of certain conical transversal slices in LX generalizing the construction due to Mars-Springer in the case of X.

We deduce results for singularities of spherical orbit closures in LX and provide applications to the relative Langlands duality including the positivity for the relative Kostka-Foulkes polynomials, a semisimplicity criterion for the (abelian) relative Satake category, and the formality conjecture.

The recent works [CN24], [MT24], [BZSV] relate the singularities of orbit closures in LX with harmonic analysis, real groups, and quantum groups. We mention some results in these directions.

We now describe the paper in more details.

1.1. **Main results.** To simplify the discussions, we will assume G is simply connected in the introduction. We fix a θ-stable Borel pair (B_0, T_0) and let $I_0 \subset L^+G$ be the corresponding

[1] In *loc. cit.*, Lusztig remarked that, in 1993, he communicated the question to M. Finkelberg as a research problem.

[2] The reason for working with I_0-orbits on LX is two-fold: first, there is a canonical injection $LK\backslash\mathcal{F}\ell \cong I_0\backslash LG/LK \to I_0\backslash LX$ from the set of LK-orbits on $\mathcal{F}\ell$ to the set of I_0-orbits on LX, which is a bijection if K is connected. Thus we are dealing with a more general setting. Secondly, as we shall see, the geometry of I_0-orbits closures on LX behaves much nicer than LK-orbit closures on $\mathcal{F}\ell$, see Section 5.

[3] That being said, the use of real groups in [LV83] should be viewed as an important feature of symmetric varieties since it provides representation interpretations of the singularities. See Section 1.2.3 for relevant discussions in the affine setting.

Iwahori subgroup. Let $\mathrm{W}_{\mathrm{aff}}$ be the affine Weyl group of G. Let H be the affine Hecke algebra associated to $\mathrm{W}_{\mathrm{aff}}$ over $\mathbb{Z}[q, q^{-1}]$ with standard basis elements $T_w, w \in \mathrm{W}_{\mathrm{aff}}$.

Recall the free $\mathbb{Z}[q, q^{-1}]$-module M with basis indexed by $(\mathcal{L}, \mathcal{O}) \in \mathscr{D}$.

Theorem 1. *The module M has a unique module structure over the affine Hecke algebra H such that T_w acts according to the formula* (29)*.*

Theorem 1 is proved in Proposition 74 and Lemma 75. Along the way, we obtain a combinatoric parameterization of I_0-orbits on LX and a characterization of closed orbits (see Lemma 23 and Lemma 31). The module M is an affine version of the Lusztig-Vogan module in [LV83, Proposition 1.7], to be called the *affine Lusztig-Vogan module*.

To state the next result, we recall some geometric facts on I_0-orbits in LX. The I_0-orbits on LX in general are infinite dimensional. However, the placidness results in Section 5 imply that, for a pair of I_0-orbits $\mathcal{O}_u, \mathcal{O}_v \subset LX$ such that $\mathcal{O}_u \subset \overline{\mathcal{O}}_v$, there is well-defined notion of codimension $\mathrm{codim}_{\overline{\mathcal{O}}_v}(\overline{\mathcal{O}}_u) \in \mathbb{Z}_{\geq 0}$ between their orbit closures. Moreover, in Proposition 63 we show that there exists a dimension theory of LX, that is, one can assign an integer $\delta(v) \in \mathbb{Z}$ for each orbit $\mathcal{O}_v$ such that

$$\delta(v) - \delta(u) = \mathrm{codim}_{\overline{\mathcal{O}}_v}(\overline{\mathcal{O}}_u) \tag{1}$$

if $\mathcal{O}_u$ is in the closure of $\mathcal{O}_v$. The integers $\delta(v)$ can be thought as a replacement of the dimension of $\mathcal{O}_v$.

To simplify notations, we will write $(\xi, v) = (\mathcal{L}_{\xi,v}, \mathcal{O}_v) \in \mathscr{D}$ and $m_{\xi,v} \in M$ for the corresponding basis element. We write $u \leq v$ if $\mathcal{O}_u \subset \overline{\mathcal{O}}_v$ and $u < v$ if $u \leq v$ and $\mathcal{O}_u \neq \mathcal{O}_v$.

Theorem 2. *A choice of a dimension theory $\delta = \{\delta(v)\}$ of LX gives rise to a unique $\mathbb{Z}$-linear map $D_\delta : M \to M$ such that*

(i) *$D_\delta(qm) = q^{-1} D_\delta(m)$ for $m \in M$.*
(ii) *$D_\delta((T_{s_\alpha} + 1)m) = q^{-1}(T_{s_\alpha} + 1) D_\delta(m)$.*
(iii) *For any $(\xi, v) \in \mathscr{D}$, we have*

$$D_\delta(m_{\xi,v}) = q^{-\delta(v)} m_{\xi,v} + \sum_{u<v} b_{\eta,u,\xi,v} m_{\eta,\mu}$$

where $q^{\delta(v)} b_{\eta,u,\xi,v}$ are polynomials in q of degree at most $\delta(v) - \delta(u)$. Moreover, there is an algorithm to compute $b_{\eta,u,\xi,v}$.

Theorem 2 is proved in Section 8.4 and Section 8.5. The existence of D_δ is a combinatorial problem, but we have to use sheaf theory on LX (namely, the Verdier duality) to solve it. The polynomial $q^{\delta(v)} b_{\eta,u,\xi,v}$ is an affine analogy of the polynomial $R_{\gamma,\delta}$ in [LV83, Theorem 1.10].

Theorem 3. *There is a unique family of polynomials $P_{\eta,u;\xi,v} \in \mathbb{Z}_{\geq 0}[q]$ in q with non-negative integer coefficients, indexed by pairs $(\eta, u), (\xi, v) \in \mathscr{D}$, satisfying the following conditions:*

(i) *$P_{\xi,v;\xi,v} = 1$.*
(ii) *If $u \neq v$, $\deg(P_{\eta,u;\xi,v}) \leq \frac{1}{2}(\delta(v) - \delta(u) - 1)$.*
(iii) *$C_{\xi,v} := \sum_{\eta,u} P_{\eta,u;\xi,v} m_{\eta,u} \in M$ satisfies $D_\delta C_{\xi,v} = q^{-\delta(v)} C_{-\xi,v}$.*

Theorem 3 is restated in Theorem 84. In fact in *loc. cit.* we prove a slightly stronger uniqueness result: the family of polynomials $P_{\eta,u;\xi,v}$ are unique over the larger ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$.

Note that part (i) and (ii) imply that the family of elements $C_{\xi,v}$, $(\xi, v) \in \mathscr{D}$ forms a basis of M to be called the Kazhdan-Lusztig basis of M and the entries of the transition matrix between the standard basis $m_{\xi,v}$ and Kazhdan-Lusztig basis $C_{\xi,v}$ are given by the family of polynomials $P_{\eta,u;\xi,v}$. We will call the family $P_{\eta,u;\xi,v}$ the *affine Kazhdan-Lusztig-Vogan polynomials* associated to X since they include the Kazhdan-Lusztig-Vogan polynomials $P_{\gamma,\delta}$ in [LV83, Theorem 1.11] (see Lemma 60). Theorem 2 (iii) provides an algorithm to compute them.

The most difficult part in the proof of Theorem 3 is the positivity property of $P_{\eta,u;\xi,v}$. It is a consequence of the following pointwise purity and parity vanishing results for IC-complexes for Iwahori orbit closures. For any $(\xi, v) \in \mathscr{D}$, let $\mathrm{IC}_{\xi,v} = (j_v)_{!*}(\mathcal{L}_{\xi,v})[\delta(v)]$ be the IC-extension of the local system $\mathcal{L}_{\xi,v}$ along the inclusion $j_v : \mathcal{O}_v \to \overline{\mathcal{O}}_v$. For any pair $(\eta, u), (\xi, v) \in \mathscr{D}$ we write $[\mathcal{L}_{\eta,u} : \mathscr{H}^i(\mathrm{IC}_{\xi,v})|_{\mathcal{O}_u}]$ for the multiplicity of $\mathcal{L}_{\eta,u}$ in the Jordan-Hölder series of $\mathscr{H}^i(\mathrm{IC}_{\xi,v})|_{\mathcal{O}_u}$.

Theorem 4. *For any pair $(\eta, u), (\xi, v) \in \mathscr{D}$ we have*

(i) $\mathrm{IC}_{\xi,v}$ *is pointwise pure of weight $\delta(v)$.*
(ii) $\mathscr{H}^i(\mathrm{IC}_{\xi,v}) = 0$ *if $i + \delta(v)$ is odd.*
(iii) $P_{\eta,u,\xi,v} = \sum_i [\mathcal{L}_{\eta,u} : \mathscr{H}^{2i-\delta(v)}(\mathrm{IC}_{\xi,v})|_{\mathcal{O}_u}]q^i$.

Theorem 4 is a combination of Theorem 81, Theorem 82, and Theorem 84. Part (ii) extends the parity vanishing results in [LV83, Theorem 1.12] to the affine setting. The proof in *loc. cit.* uses the theory of weights and a representation theoretic interpretation of the multiplicity. In the affine setting, a representation theoretic interpretation of the multiplicity $[\mathcal{L}_{\eta,u} : \mathscr{H}^{2i-\delta(v)}(\mathrm{IC}_{\xi,v})|_{\mathcal{O}_u}]$ is not available at the moment (but it would be very interesting to find one, see Section 1.2.3 for some related discussions). Thus we follow the approach in [MS98] where Mars-Springer exploit the pointwise purity property in (i). The key geometric ingredient in [MS98] is the construction of a conical transversal slices for B_0-orbits on X, to be called the Mars-Springer slices, where the pointwise purity property is a direct consequence.

We generalize Mars-Springer's construction to the affine setting.

Theorem 5. *There exists conical transversal slices to the L^+G-orbits and I_0-orbits on LX.*

Theorem 5 is stated in Corollary 49 and Corollary 59. We refer to Section 6 for a more detailed explanation of the statement including the identification of the spherical orbits slices with certain Lagrangian subvarieties of the affine Grassmannian slices (see Proposition 68).

Note that since the orbits are in general semi-infinite, the meaning of transversal slices and pointwise purity require some treatment (see Section A.8).

1.2. **Applications.** We deduce the pointwise purity property and parity vanishing for spherical orbits in LX from the results for Iwahori orbits.[4] The singularities of spherical orbit closures in LX are related to the so called relative Langlands duality [BZSV] and we discuss applications of our main results to the subject. We also discuss connections with quantum groups and real groups, providing (possible) representation theoretic meaning of the singularities.

[4]In fact, one can give a direct proof of pointwise purity property using the transversal slices for spherical orbits. However, we do not know a proof of parity vanishing without appealing to the results for Iwahori orbits (see Remark 15).

1.2.1. *Formality conjecture.* One application of the pointwise purity for spherical orbits is the formality conjecture in relative Langlands duality. In [BZSV, Conjecture 8.1.8], the authors conjecture a version of derived Satake equivalence for a spherical variety X satisfying some technical assumptions. Their conjecture implies that the so called de-equivariantized Ext algebra

$$A_X := R\mathrm{Hom}_{D(L^+G\backslash LX)}(\mathrm{e}_{L^+X}, \mathrm{IC}_{\mathcal{O}(\check{G})} \star \mathrm{e}_{L^+X})$$

(a.k.a the Plancherel algebra) for the derived relative Satake category $D(L^+G\backslash LX)$ of X is formal. A general approach to proving formality is to prove the purity of the Ext algebra $H^\bullet(A_X)$. We confirm this conjecture in the case of symmetric varieties.

Corollary 6. *For X a symmetric variety, A_X is formal.*

Theorem 6 is stated in Theorem 93. We refer to Section 10.2 for a more detailed explanation of the statement.

Remark 7. In [BZSV], the authors consider a more general setting of spherical varieties X with the assumption that the cotangent bundles $M = T^*X$ are the so called *hyperspherical varieties*. Corollary 6 implies that the formality conjecture holds true for *all* symmetric varieties X without the hyperspherical assumption.

Remark 8. Corollary 6 was previously known in the case of symmetric varieties of splitting ranks (see [CMNO22] and [CO24]). The proof in *loc. cit.* uses some special features of splitting rank symmetric varieties which are not available for the general cases. Our proof gives a uniform argument for all cases.

1.2.2. *Semisimplicity criterion and relative Satake equivalences.* A basic result about perverse sheaves on affine Grassmannian $\mathrm{Gr} = LG/L^+G$ is the semisimplicity of the Satake category $\mathrm{Perv}(L^+G\backslash\mathrm{Gr})$ of G(see, e.g., [Zhu17, Proposition 5.1.1.]). The proof relies on the parity vanishing result for spherical orbits on Gr due to Lusztig [Lus83, Theorem 11.c]. Using the parity vanishing for spherical orbits in Theorem 82 and [CN24], we prove the following semisimplicity criterion and Langlands dual description of the relative Satake categories:

Corollary 9. *Assume the codimensions of L^+G-orbits in the same connected component of LX are even.*

(i) *The relative Satake category* $\mathrm{Perv}(L^+G\backslash LX)$ *of L^+G-equivariant perverse sheaves on LX is semisimple.*

(ii) *Assume further that the L^+G-stabilizers on LX are connected. Then there is an equivalence of abelian categories*

$$\mathrm{Perv}(L^+G\backslash LX) \cong \mathrm{Rep}(\check{G}_X)$$

where the $\mathrm{Rep}(\check{G}_X)$ *is the category of finite dimensional complex representations of the dual group $\check{G}_X$ of X* [GN10].

Corollary 9 is restated in Theorem 87. In Proposition 64 we proved a codimension formula of spherical orbits (assuming LX is connected) and one can use it to find symmetric varieties satisfying the assumptions in (i) and (ii) above. Interesting examples include the splitting rank symmetric varieties $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}, \mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}$, $\mathrm{E}_6/\mathrm{F}_4$. The corresponding relative dual groups $\check{G}_X$ are $\mathrm{PGL}_{2n}, \mathrm{PGL}_2, \mathrm{PGL}_3$. In particular, we obtain new instances of relative Satake equivalence, see Example 88.

Remark 10. We suspect that the converse of Corollary 9 (i) might be also true.

Remark 11. The proof of the codimension formula Proposition 64 is quite involved, it uses real groups, quasi-maps, and a spreading out argument. It would be nice to find a more direct argument.

Remark 12. Using the real-symmetric equivalence in [CN24], one can show that the equivalence in (ii) is compatible with the fusion product on $\mathrm{Perv}(L^+G\backslash LX)$ and the tensor product on $\mathrm{Rep}(\check{G}_X)$, and hence admits an upgrade to an equivalence of symmetric monoidal abelian categories.

Remark 13. The assumptions in the corollary are necessary. In fact, the work [CN24] shows that $\mathrm{Perv}(L^+G\backslash LX)$ always contains the semisimple abelian category $\mathrm{Rep}(\check{G}_X)$ as a full subcategory but if there are orbits with odd codimensions or having disconnected stabilizers, then $\mathrm{Perv}(L^+G\backslash LX)$ might not be semisimple or contain IC-complexes with non-trivial coefficients not belonging to $\mathrm{Rep}(\check{G}_X)$ (see Remark 89 for the discussion when $X = \mathrm{SL}_2/\mathrm{SO}_2$).

Thus for general symmetric varieties, the relative Satake categories $\mathrm{Perv}(L^+G\backslash LX)$ might be larger than $\mathrm{Rep}(\check{G}_X)$ and in those situations we expect the Langlands dual descriptions of them would be more complicated involving certain quantum groups (see Section 1.2.4).

1.2.3. *Relative Kostka-Foulkes polynomials.* The L^+G-orbits on LX are parametrized by the set of dominant coweights Λ_S^+ for a maximal θ-split torus S (see Proposition 17). We write $(\chi, \lambda) = (\mathcal{L}_{\chi,\lambda}, LX_\lambda)$ where $\mathcal{L}_{\chi,\lambda}$ is a L^+G-equivariant local systems on the orbits LX_λ indexed by $\lambda \in \Lambda_S^+$ and representations χ of the component groups of stabilizers, and $\mathrm{IC}_{\chi,\lambda}$ for the IC-extension of $\mathcal{L}_{\chi,\lambda}$ to the closure of $\overline{LX}_\lambda$. For any pair $(\chi', \lambda'), (\chi, \lambda)$ as above, we define

$$P_{\chi',\mu;\chi,\lambda} = \sum_i [\mathcal{L}_{\chi',\mu} : \mathscr{H}^{i-\delta(\lambda)}(\mathrm{IC}_{\chi,\lambda})|_{LX_\mu}] q^{\frac{i}{2}},$$

here $\delta(\lambda) = \delta(v_\lambda)$, where $\mathcal{O}_{v_\lambda} \subset LX_\lambda$ is the unique open I_0-orbit (see Corollary 27).

We show that the polynomials $P_{\chi',\mu;\chi,\lambda}$ are special cases of the affine Kazhdan-Lusztig-Vogan polynomials (see (35)) and deduce from Theorem 3 the following:

Corollary 14. *For any pair $(\chi', \mu), (\chi, \lambda)$, $P_{\chi',\mu;\chi,\lambda}$ is a polynomial in q with non-negative integer coefficients. It is the unique family of polynomials in $q^{\frac{1}{2}}$ satisfying the following conditions:*

(i) $P_{\chi,\lambda;\chi,\lambda} = 1$.
(ii) *If* $\mu \neq \lambda$, $\deg(P_{\chi',\mu;\chi,\lambda}) \leq \frac{1}{2}(\delta(\lambda) - \delta(\mu) - 1)$.
(iii) $C_{\chi,\lambda} := \sum_{\chi',\mu} P_{\chi',\mu;\chi,\lambda}[\mathcal{L}_{\chi',\mu}]$ *satisfies* $D_\delta C_{\chi,\lambda} = q^{-\delta(\lambda)} C_{-\chi,\lambda}$.

In particular, we have

(iv) $\mathscr{H}^i(\mathrm{IC}_{\xi,\lambda}) = 0$ *if* $i + \delta(\lambda)$ *is odd.*

Corollary 14 is restated in Theorem 82 and Theorem 86. We refer to Section 9.3 for the explanation of condition (iii). The polynomials $P_{\chi',\mu;\chi,\lambda}$ are computed explicitly in [CMNO22, Theorem 1.9] and [CO24, Theorem 1.3] for splitting rank symmetric varieties X and they coincide with the Kostka-Foulkes polynomials for the relative dual group $\check{G}_X$, generalizing the work [Lus83] in the group cases. For this reason, we will call the family of polynomials $P_{\chi',\mu;\chi,\lambda}$ the *relative Kostka-Foulkes polynomials.*

Remark 15. Proving the parity vanishing in (iv) was one of the original motivation for the work. However, due to many technical difficulties, such as the appearance of non-trivial local systems on orbits, we do not known a direct proof without appealing to the results for Iwahori orbits. In a sense, the study of Iwahori orbits closures is simpler than spherical orbit closures due to the presence of affine Hecke algebra module structures.

Remark 16. The results in [Lus83] imply, for splitting rank X, the polynomials $P_{\chi',\mu;\chi,\lambda}$ admits a representation theory interpretation as a q-analogy of weight multiplicities of the dual group $\check{G}_X$.

1.2.4. *Real and quantum groups.* We comment on connections and applications to real and quantum groups. Let $G_{\mathbb{R}}$ be the real form of G corresponding to X under the Cartan bijection. The real-symmetric equivalence in [CN24] says there is an equivalence

$$\mathrm{Perv}(L^+G\backslash LX) \cong \mathrm{Perv}(L^+G_{\mathbb{R}}\backslash \mathrm{Gr}_{\mathbb{R}}) \tag{2}$$

where the right hand side is the real Satake category of equivariant perverse sheaves on the real affine Grassmannian $\mathrm{Gr}_{\mathbb{R}}$ of $G_{\mathbb{R}}$.[5] One can translate the results in the paper to the $G_{\mathbb{R}}$-side of the equivalence (2) and obtain several interesting results in Langlands duality for real groups, such as the parity vanishing for IC-complexes on $\mathrm{Gr}_{\mathbb{R}}$ and the formality conjecture for real Satake category [CN24, Corollary 1.7]. Those results are not obvious since powerful tools in algebraic geometry (e.g., Hodge theory or theory of weights) are not available in the real analytic setting.

On the other hand, in [MT24] the authors use the concrete geometry of the real affine Grassmannian to show that the real Satake category $\mathrm{Perv}(L^+\,\mathrm{PGL}_2(\mathbb{R})\backslash\mathrm{Gr}_{\mathbb{R}})$ for $G_{\mathbb{R}} = \mathrm{PGL}_2(\mathbb{R})$ is related to the representations of quantum SL_2 at primitive fourth root of unity. Through the real-symmetric equivalence (2), their results provide a representation theoretic and combinatorial interpretations of the relative Kostka-Foulkes polynomials for $X = \mathrm{PGL}_2 / \mathrm{PO}_2$. In particular, we see that, in contrast with the group cases or the splitting rank cases, the polynomials $P_{\chi',\mu;\chi,\lambda}$ for general X might be related to weight multiplicities of a certain quantum group instead of $\check{G}_X$. In the forthcoming work [CMT], we plan to explore those connections for more general symmetric varieties, real groups, and quantum groups.

1.2.5. *Spherical varieties.* We comment on generalization to general spherical varieties. A large part of the geometric approach taken in [MS98] is written uniformly for an arbitrary spherical variety. However, Mars-Springer's construction of transversal slices in symmetric varieties, and also our generalization to the affine setting, uses the involution θ in an essential way and hence does not have a direct generalization to general spherical varieties. Thus we expect that most of the results of the paper have direct generalizations to spherical varieties, except the pointwise purity and positivity results where the existence of slices is a crucial ingredient.

With that being said, there have been recent works, for examples [FGT09, BFGT21], where the authors established, among other things, the pointwise purity and formality results for a particular class of spherical varieties. In a forthcoming work, we extend the main results of the paper to the GL_{2n}-spherical variety $X = \mathrm{GL}_{2n}/\mathrm{Sp}_{2n} \times \mathbb{A}^{2n}$ where GL_{2n} acts diagonally (the so called *linear periods* cases), including a construction of transversal slices for spherical

[5] This is a remarkable new phenomenon specific to L^+G-orbits on LX.

orbits in LX and a version of the relative derived Satake equivalence. We hope to explore the situation of more general spherical varieties.

1.2.6. *Characteristic.* In this paper, we work with ℓ-adic sheaves on schemes over fields of positive characteristic where Deligne's theory of weights plays a crucial role. The proofs are all geometric and hence are applicable in the setting of D-modules on schemes over complex numbers where one uses Saito's theory of mixed Hodge modules.

1.3. **Organization.** In Section 2, we recall some basic definitions in Lie theory. In Section 3 and Section 4, we study spherical and Iwahori orbits parametrizations on LX. In Section 5, we prove the placidness of orbit closures in LX, a key result on the geometry of orbit closures, and uses it to show the existence of a dimension theory on LX. In Section 6, we construct conical transversal slices for both spherical and Iwahori orbits inside LX and establish several foundational results about them. In Section 7, we prove a codimension formula for spherical orbits and deduce from it the Lagrangian property of spherical slices. In Section 8, we construct the affine Lusztig-Vogan modules and introduce and study the affine Kazhdan-Lusztig-Vogan polynomials. We explain an algorithm to compute them. In Section 9, we prove the pointwise purity and parity vanishing results for IC-complexes for orbit closures and deduce from it the positivity of affine Kazhdan-Lusztig-Vogan polynomials. In Section 10, we discuss applications to the relative Langlands duality. In Section 11, we discuss the case when K is disconnected (we assume K is connected from Section 4-Section 10). In Appendix A, we recall and prove some basic results about the geometry of placid ind-schemes and sheaf theory, including the discussion of dimension theory, t-structures and perverse sheaves, Verdier duality, etc.

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2. Group data

2.1. **Symmetric varieties.** We recall some basic notations about symmetric varieties following [Spr85, Spr87] and [RS93]. Let G be a connected reductive group over $k = \overline{\mathbb{F}}_p$ of characteristic $p \neq 2$, and $\theta : G \to G$ an involution. We fix a θ-stable Borel pair (B_0, T_0).

Let $K = G^\theta$ be the associated symmetric subgroup. The quotient scheme $X = G/K$ is a symmetric variety. Let $\tau : G \to G, \tau(g) = g\theta(g)^{-1}$. It induces a G-equivariant isomorphism $\tau : X \cong G^{\mathrm{inv}\circ\theta,\circ}$ (denoted again by τ) between X and the identity component of the subvariety of G on which θ acts as inverse. Here G acts on $G^{\mathrm{inv}\circ\theta,\circ}$ by the twisted conjugation action $g \cdot x = gx\theta(g)^{-1}$.

A torus $S \subset G$ is called θ-split if $\theta(s) = s^{-1}$ on S. A parabolic P is called θ-split if $P \cap \theta(P)$ is a Levi subgroup of P. We fix a maximal θ-split torus S and a minimal θ-stable parabolic P such that $M = Z_G(S) = P \cap \theta(P)$. We pick a Borel pair (B, T) such that $S \subset T \subset B \subset P$. Note that T is automatically θ-stable. We call such a pair a θ-split Borel pair.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra. The tangent map $d\theta : \mathfrak{g} \to \mathfrak{g}$ has eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{p}$ is the (-1)-eigenspace.

2.2. **Loop and arc spaces.** Let $F = k(\!(t)\!)$ and $\mathcal{O} = k[\![t]\!]$ be the Laurent and Taylor series ring respectively. For a scheme Y over k, we will denote its loop space and arc space by LY and L^+Y such that $LY(k) = Y(F)$ and $L^+Y(k) = Y(\mathcal{O})$. For any $j \in \mathbb{Z}_{\geq 0}$, we have j-th arc space L_j^+Y such that $L_j^+Y(k) = Y(\mathcal{O}/t^j)$. When $Y = G$, the corresponding loop and arc spaces LG and L^+G are called the loop group and arc group of G. We denote by $I_0 \subset L^+G$ the Iwahori subgroup associated to B_0. We let $\mathrm{Gr} = LG/L^+G$ be the affine Grassmannian and $\mathcal{F}\ell = LG/I_0$ the affine flag variety.

3. Spherical orbits

3.1. **Spherical orbits parametrizations.** We have the following parametrization of L^+G-orbits on LX. Recall the embedding $\tau : X \cong G^{\mathrm{inv}\circ\theta,\circ} \subset G$ in Section 2.1 induces a map $\tau : LX \to LG$ on the corresponding loop spaces.

We denote by $\Phi, \Phi^+, \check{\Phi}, \check{\Phi}^+$ the set of roots, positive roots, coroots, positive coroots with respect to the θ-split Borel pair (B, T). We denote by $\Delta \subset \Phi$ and $\check{\Delta} \subset \check{\Phi}$ the set of simple roots and simple coroots. We denote by Λ_T and Λ_T^+ the sets of coweights and dominant coweights. We denote by $\Lambda_S \subset \Lambda_T$ the coweights of S and $\Lambda_S^+ = \Lambda_S \cap \Lambda_T^+$ the set of dominant coweights of S. Λ_S is naturally ordered by the restriction of the ordering from the coweights Λ_T: we have $\mu \leq \lambda \in \Lambda_S$ if and only if $\lambda - \mu$ is a non-negative sum of positive coroots. We write 2ρ the sum of positive roots.

Proposition 17. (i) *The L^+G-orbits on LX are represented by Λ_S^+: for each $\mu \in \Lambda_S^+$, there exists $x_\mu \in LX(k) = X(F)$ such that $\tau(x_\mu) = t^\mu \in S(F)$ and $LX_\mu = LG^+ \cdot x_\mu$ is the corresponding orbit.*

(ii) *Assume K is connected. The natural map $LG \to LX$ induces a bijection $G(F)/K(F) \cong X(F)$*

Proof. Part (i) is proved in [GN10, Theorem 8.2.9] and [Nad04, Theorem 4.2.]. Part (ii) follows from [Nad04, Theorem 4.3.] (Since $\mathcal{L} = \Lambda_S^+$ in *loc. cit.* as K is connected). □

3.2. **Closure relations.** Let $\overline{LX}_\mu$ be the closure of LX_μ in LX (with the reduced scheme structure). The goal of this section is to prove the following closure relation.

Theorem 18. *For $\forall \mu \in \Lambda_S^+$, $\overline{LX}_\mu = \bigsqcup_{\lambda \leq \mu} LX_\lambda$.*

Remark 19. If $k = \mathbb{C}$, the theorem is proved in [Nad04, Theorem 10.1]. It is interesting to note that the proof of in *loc. cit.* makes use of real affine Grassmannians. Our argument is algebraic and works over an arbitrary algebraically closed field whose characteristic is not two.

3.2.1. *The case* $\mathrm{GL}_2/\mathrm{O}_2$. We begin with the fundamental case of $G = \mathrm{GL}_2$, $\theta(g) = (g^t)^{-1}$, $K = \mathrm{O}_2$, $X = \mathrm{GL}_2/\mathrm{O}_2$. We will see later that the general orbit closures can be reduced to this situation.

In this case $S = T$. Note that in [Nad04], Proposition 17.(i) is proved without assuming K to be connected. Thus L^+G-orbits on LX are parameterized by $\Lambda_T^+ = \{\lambda = (p, q) \in \mathbb{Z}^2 \mid p \geq q\}$. We will work with the image of LX under embedding $\tau : LX \to LG$, $g \mapsto g\theta(g)^{-1} = gg^t$. The orbit representatives are $\tau(x_\lambda) = t^\lambda$, and the L^+G-action is $g \cdot t^\lambda = gt^\lambda g^t$. The orbits closure relation follows from the following statement:

Proposition 20. *For any integer $m \geq 0$, let $\mu_0 = \sqrt{(-1)^{m+1}}$. Then for any $s \in k^\times$, there exists $g = g(s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ such that*

(3)
$$g \begin{pmatrix} t^{m+1} & \\ & t^{-1} \end{pmatrix} g^t = \begin{pmatrix} a^2 t^{m+1} + b^2 t^{-1} & act^{m+1} + bdt^{-1} \\ act^{m+1} + bdt^{-1} & c^2 t^{m+1} + d^2 t^{-1} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{m+1} (-s^2)^{m+1-i} t^{i-1} & \mu_0 s^{m+2} t^{-1} \\ \mu_0 s^{m+2} t^{-1} & s^2 t^{-1} + 1 \end{pmatrix}.$$

Corollary 21. *For $\forall \mu \in \Lambda_S^+$, $\overline{LX}_\mu = \bigsqcup_{\lambda \leq \mu} LX_\lambda$.*

Proof. Since $L^+G \cdot t^\mu \subset L^+Gt^\mu L^+G$, by the orbits closure relation in affine Grassmannian we know $\overline{LX}_\mu \subset \bigsqcup_{\lambda \leq \mu} LX_\lambda$.

For the other direction, it suffices to show $t^{\mu-\alpha} \in \overline{LX}_\mu$ if $\mu - \alpha \in \Lambda_S^+ = \Lambda_T^+$, $\alpha = (1, -1)$. Acting by center, we can assume $\mu = (m+1, -1)$, $\mu - \alpha = (m, 0)$. By Proposition 20, we get $t^{(m,0)} = \lim_{s \to 0} g(s) \cdot t^{(m+1,-1)} \in \overline{LX}_\mu$. This proves the statement. □

In the rest of the subsection, we prove Proposition 20 in a few steps.

3.2.2. Write $a = \sum_{i \geq 0} a_i t^i, b = \sum_{i \geq 0} b_i t^i, c = \sum_{i \geq 0} c_i t^i, d = \sum_{i \geq 0} d_i t^i$. We first show that we can find a, b, c, d such that 3 holds modulo t^{m+1}. This only involves $b_0, ..., b_{m+1}, d_0, ..., d_{m+1}$. Explicitly, we need to find $b_0, ..., b_{m+1}, d_0, ..., d_{m+1} \in k[s, s^{-1}]$ depending only on m such that

$$b^2 \equiv \sum_{i=0}^{m+1} (-s^2)^{m+1-i} t^i, \quad d^2 \equiv s^2 + t, \quad bd \equiv \mu_0 s^{m+2} \mod t^{m+2}. \tag{4}$$

We first solve d_i. Let $d_i = \lambda_i s^{-(2i-1)}$, $0 \leq i \leq m+1$. Then $d \equiv \sum_{i=0}^{m+1} d_i t^i \mod t^{m+2}$. We get equation

$$d^2 \equiv \lambda_0^2 s^2 + 2\lambda_0\lambda_1 s^0 t + (2\lambda_0\lambda_2 + \cdots) s^{-2} t^2 + \cdots + (2\lambda_0\lambda_{m+1} + \cdots) s^{-2m} t^{m+1} \equiv s^2 + t \mod t^{m+2}.$$

Let $\lambda_0 = 1$, $d_0 = s$. From the above we can uniquely solve $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2^3}, ...,$ until λ_{m+1}, where the denominator of λ_i is a power of 2.

Next let $b_i = \mu_i s^{m+1-2i}$, $0 \leq i \leq m+1$, where $\mu_0 = \sqrt{(-1)^{m+1}} \neq 0$. From equation

$$\begin{aligned} bd \equiv & \mu_0 s^{m+2} + (\sum_{i=0}^{1} \mu_i \lambda_{1-i}) s^m t + (\sum_{i=0}^{2} \mu_i \lambda_{2-i}) s^{m-2} t^2 + \cdots + (\sum_{i=0}^{m+1} \mu_i \lambda_{m+1-i}) s^{-m} t^{m+1} \\ \equiv & \mu_0 s^{m+2} \mod t^{m+2}, \end{aligned}$$

$\mu_1, \mu_2, ..., \mu_{m+1}$ are uniquely determined.

We still need to check that the above $b_0, ..., b_{m+1}, d_0, ..., d_{m+1}$ satisfy the first equation in (4):

$$b^2 \equiv \sum_{i=0}^{m+1} (\sum_{k=0}^{i} \mu_k \mu_{i-k}) (s^2)^{m+1-i} t^i \equiv \sum_{i=0}^{m+1} (-s^2)^{m+1-i} t^i \mod t^{m+2}.$$

It amounts to show $\sum_{k=0}^{i} \mu_k \mu_{i-k} = (-1)^{m+1-i}$, $\forall\ 0 \leq i \leq m+1$. For this, first it is easy to prove by induction on i using the construction of μ_i, λ_i that $\mu_i = (\sum_{k=0}^{i} (-1)^k \lambda_{i-k}) \mu_0$, $\forall\ 0 \leq i \leq m+1$. Next we prove $N_i := \sum_{k=0}^{i} \mu_k \mu_{i-k} = (-1)^{m+1-i}$ by induction. Assume this

is true for $0 \le i \le r < m+1$, where $N_0 = \mu_0^2 = (-1)^{m+1}$. In $bd \equiv \mu_0 s^{m+2} \mod t^{m+2}$, we time both sides by b and evaluate at $s = 1$, we get

$$(b|_{s=1})^2 d|_{s=1} \equiv \sum_{i=0}^{m+1} N_i t^i \sum_{k=0}^{m+1} \lambda_i t^i \equiv \mu_0 \sum_{i=0}^{m+1} \mu_i t^i \mod t^{m+2}.$$

Compare the coefficient of t^{r+1} and using the inductive hypothesis, we get

$$\mu_0^2(\lambda_{r+1} - \lambda_r + \cdots + (-1)^r \lambda_1) + N_{r+1}\lambda_0 = \mu_0 \mu_{r+1}.$$

By the previous identity $\mu_i = (\sum_{k=0}^{i}(-1)^k \lambda_{i-k})\mu_0$, we obtain

$$N_{r+1} = \mu_0(\mu_{r+1} - \mu_0(\lambda_{r+1} - \cdots + (-1)^r \lambda_1)) = (-1)^{r+1}\mu_0^2 = (-1)^{m+1-r-1}.$$

Thus we have verified that the above $b_0, ..., b_{m+1}, d_0, ..., d_{m+1}$ satisfy (4).

3.2.3. Next we show that we can further find $a, c, b_i, d_i, i \ge m+2$ such that (3) holds. Denote $b_{\le m+1} = \sum_{i=0}^{m+1} b_i t^i$, $\tilde{b}_i = b_{m+1+i}$, $\tilde{b} = \sum_{i \ge 0} \tilde{b}_i t^i$, then $b = b_{\le m+1} + \tilde{b} t^{m+2}$. Similarly denote $d_{\le m+1}, \tilde{d}, \tilde{d}_i, d = d_{\le m+1} + \tilde{d} t^{m+2}$. Then

$$\begin{aligned}
b^2 &= (\sum_{i=0}^{m+1}(-s^2)^{m+1-i} t^i) + (\tilde{b}^2 t^{m+2} + 2b_{\le m+1}\tilde{b} + \widetilde{b^2_{\le m+1}}) t^{m+2}, \\
d^2 &= (s^2 + t) + (\tilde{d}^2 t^{m+2} + 2d_{\le m+1}\tilde{d} + \widetilde{d^2_{\le m+1}}) t^{m+2}, \\
bd &= (\mu_0 s^{m+2}) + (\tilde{b}\tilde{d} t^{m+2} + b_{\le m+1}\tilde{d} + d_{\le m+1}\tilde{b} + \widetilde{b_{\le m+1} d_{\le m+1}}) t^{m+2},
\end{aligned}$$

where $\widetilde{b^2_{\le m+1}}, \widetilde{d^2_{\le m+1}}, \widetilde{b_{\le m+1} d_{\le m+1}}$ are polynomials in terms of $t, b_1, ..., b_{m+1}, d_1, ..., d_{m+1}$ that fit into the above identities. The problem is reduced to finding $a, c, \tilde{b}, \tilde{d} \in \mathcal{O}$ satisfying the following equations:

$$\begin{aligned}
&a) \quad a^2 + \tilde{b}^2 t^{m+2} + \widetilde{b^2_{\le m+1}} + 2b_{\le m+1}\tilde{b} = 0, \\
&b) \quad c^2 + \tilde{d}^2 t^{m+2} + \widetilde{d^2_{\le m+1}} + 2d_{\le m+1}\tilde{d} = 0, \\
&c) \quad ac + \tilde{b}\tilde{d} t^{m+2} + \widetilde{b_{\le m+1} d_{\le m+1}} + b_{\le m+1}\tilde{d} + d_{\le m+1}\tilde{b} = 0.
\end{aligned}$$

First we let $a = s^{-1}$. Since the constant term of $b_{\le m+1}$ is $b_0 = \mu_0 s^{m+1} \ne 0$, we can solve $\tilde{b}$ uniquely from equation a). From equation c), we can solve c in terms of $\tilde{d}$. Now plug the c into equation b), we obtain an equation of $\tilde{d}$:

$$\begin{aligned}
&(t^{m+2} + s^2(\tilde{b} t^{m+2} + b_{\le m+1})^2)\tilde{d}^2 \\
&+(2s^2(\widetilde{b_{\le m+1} d_{\le m+1}} + \tilde{b} d_{\le m+1})(\tilde{b} t^{m+2} + b_{\le m+1}) + 2d_{\le m+1})\tilde{d} \\
&+(s^2(\widetilde{b_{\le m+1} d_{\le m+1}} + \tilde{b} d_{\le m+1})^2 + \widetilde{d^2_{\le m+1}}) = 0.
\end{aligned}$$

To guarantee a solution of $\tilde{d}$, it suffices to show that the determinant Δ of the above quadratic equation in $\tilde{d}$ has a square root in $\mathcal{O}^\times$, for which it suffices to show Δ has nonzero

constant term. Denote the constant term of $\widetilde{X}$ by $\widetilde{X}^0$, $X = b^2_{\leq m+1}, d^2_{\leq m+1}, b_{\leq m+1}d_{\leq m+1}$. Note that $\tilde{b}_0 = -\frac{s^{-2}+\widetilde{b^2_{\leq m+1}}^0}{2b_0}$. Then the constant term of the determinant is

$$\begin{aligned}
\Delta_0 &= (2s^2(\widetilde{b_{\leq m+1}d_{\leq m+1}}^0 + \tilde{b}_0 d_0)b_0 + 2d_0)^2 - 4s^2 b_0^2(s^2(\widetilde{b_{\leq m+1}d_{\leq m+1}}^0 + \tilde{b}_0 d_0)^2 + \widetilde{d^2_{\leq m+1}}^0) \\
&= 4s^2(2b_0 d_0 \widetilde{b_{\leq m+1}d_{\leq m+1}}^0 - \widetilde{b^2_{\leq m+1}}^0 d_0^2 - b_0^2 \widetilde{d^2_{\leq m+1}}^0) \\
&= 4s^2(2\mu_0\lambda_0 \sum_{i=1}^{m+1} \mu_i\lambda_{m+2-i} - \sum_{i=1}^{m+1} \mu_i\mu_{m+2-i}\lambda_0^2 - \mu_0^2 \sum_{i=1}^{m+1} \lambda_i\lambda_{m+2-i}) \\
&= -4s^2\mu_0^2 \sum_{i=1}^{m+1} (\mu_i\mu_0^{-1} - \lambda_i)(\mu_{m+2-i}\mu_0^{-1} - \lambda_{m+2-i}) \\
&= -4s^2 \sum_{i=0}^{m} \mu_i\mu_{m-i}.
\end{aligned}$$

The last identity above used $\mu_i = (\sum_{k=0}^{i}(-1)^k\lambda_{i-k})\mu_0$. By induction on m, it is easy to show $\sum_{i=0}^{m} \mu_i\mu_{m-i} = (-1)^m\mu_0^2 = -1$. Thus $\Delta_0 = 4s^2 \neq 0$ for $s \neq 0$. Therefore, we can solve $\tilde{d}$ out of the above equation unique up to a choice of the constant term $\tilde{d}_0$ satisfying a quadratic equation.

3.2.4. Now we have found $a, b, c, d \in \mathcal{O}$ satisfying (3). It remains to show $g \in \mathrm{SL}_2(\mathcal{O})$, i.e. $\det(g) = ad - bc = 1$. In fact, taking determinant on both sides of (3), we obtain

$$\det(g)^2 t^m = (\sum_{i=0}^{m+1}(-s^2)^{m+1-i}t^{i-1})(s^2t^{-1} + 1) - \mu_0 s^{m+2}t^{-1}\mu_0 s^{m+2}t^{-1} = t^m.$$

Thus $\det(g) = \pm 1$. We can choose the constant term of $\tilde{d}$ such that the constant term of $ad - bc$ is 1, then $ad - bc = 1$. This completes the proof of Proposition 20.

3.2.5. *Proof of Theorem 18.* As in the proof of Proposition 21, we know $\overline{LX}_\mu \subset \bigsqcup_{\lambda \leq \mu} LX_\lambda$. It remains to show that for $\lambda < \mu$, $\lambda \in \Lambda_S^+$, we have $x_\lambda \in \overline{LX}_\mu$. By [Nad05, Lemma 2.1.1], it suffices to prove for two cases: $\mu - \lambda = \alpha$ is a positive coroot $\alpha \in \Lambda_S^+ \cap \check{\Phi}^+$, or $\mu - \lambda = \alpha - \theta(\alpha)$ is not a multiple of coroots where $\alpha \in \check{\Phi}^+$.

Case 1: $\mu - \lambda = \alpha \in \Lambda_S^+ \cap \check{\Phi}^+$.

Since θ acts on S as inverse, it acts as -1 on Λ_S. As $\alpha = \mu - \lambda \in \Lambda_S$, we have $\theta(\alpha) = -\alpha$. The Levi subgroup G_α corresponding to α is preserved by θ. Under the isomorphism $G_\alpha \simeq \mathrm{SL}_2 \times \mathbb{G}_m^r, \mathrm{GL}_2 \times \mathbb{G}_m^r$, or $\mathrm{PGL}_2 \times \mathbb{G}_m^r$, consider the action of θ on the first factor, we can see it has the same action on roots as transpose inverse $\theta_0(g) = (g^t)^{-1}$. Thus θ and θ_0 different by an inner automorphism by some $t \in T$. Let $r \in T$ be such that $r\theta_0(r)^{-1} = r^2 = t$, then $\theta = \mathrm{Ad}_t \circ \theta_0 = \mathrm{Ad}_r \circ \theta_0 \circ \mathrm{Ad}_{r^{-1}}$. Since $x_\lambda, x_\mu \in L(G_\alpha/(G_\alpha)^\theta) \hookrightarrow LX$ and Ad_r fixes t^λ, t^μ, it suffices to prove for $G = \mathrm{SL}_2, \mathrm{GL}_2$, or PGL_2 and $\theta = \theta_0 : g \mapsto (g^t)^{-1}$. Thus the closure relation $x_\lambda \in \overline{LX}_\mu$ follows from Corollary 21. Note that the corollary applies to SL_2 because in Proposition 20 we only used action of $\mathrm{SL}_2(\mathcal{O})$.

Case 2: $\mu - \lambda = \alpha - \theta(\alpha)$ is not a multiple of coroots, and $\alpha + \theta(\alpha)$ is also not a coroot.

In this case $\alpha \neq \pm\theta(\alpha)$. Let H be the subgroup generated by Levi subgroups G_α and $G_{\theta(\alpha)}$. The root system of H is of type $A_1 \times A_1$. Fix isomorphisms of root subgroups $U_{\pm\alpha} \simeq \mathbb{G}_a, U_{\pm\theta(\alpha)} \simeq \mathbb{G}_a$. Consider the following subgroups of H:

$$U_+ = \mathrm{Im}(\mathbb{G}_a \xrightarrow{\Delta} \mathbb{G}_a \times \mathbb{G}_a \simeq U_\alpha \times U_{-\theta(\alpha)}), \quad U_- = \mathrm{Im}(\mathbb{G}_a \xrightarrow{\Delta} \mathbb{G}_a \times \mathbb{G}_a \simeq U_{-\alpha} \times U_{\theta(\alpha)}).$$

Then $\theta(U_+) = U_-$. Let A be the subgroup generated by U_+, U_-, then $A \simeq \mathrm{SL}_2$ or PSL_2 with unique coroots $\pm(\alpha - \theta(\alpha))$. Similar as in Case 1, we can assume θ acts as transpose inverse on A. Consider the action of $A(\mathcal{O})$ on $L(H/H^\theta) \hookrightarrow LX$, then again by Proposition 20 we conclude $x_\lambda \in \overline{LX}_\mu$.

Case 3: $\mu - \lambda = \alpha - \theta(\alpha)$ is not a multiple of coroots, but $\alpha + \theta(\alpha)$ is a coroot.

Let H be the subgroup generated by Levi subgroups $G_\alpha, G_{\theta(\alpha)}$. By [Spr85, Lemma 2.5], the root system of H is of type A_2, with coroots $\pm\alpha, \pm\theta(\alpha), \pm(\alpha + \theta(\alpha))$. Observe that θ exchanges $\alpha, \theta(\alpha)$, and fixes $\alpha + \theta(\alpha)$. Therefore the statement is reduced to the case of $G = \mathrm{GL}_3, \mathrm{SL}_3$, or other isogeny types, T is the diagonal, $J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, and θ has the same action on roots as involution $\theta_0(g) = \mathrm{Ad}_J(g^t)^{-1}$. Similar as in Case 1, we may assume $\theta = \theta_0$. Note

$$S = T^{\mathrm{inv}\circ\theta} = \{ \begin{pmatrix} a & & \\ & b & \\ & & a \end{pmatrix} \mid a, b \in k^\times \}.$$

It suffices to prove for $\mu = (p, q, p), \lambda = (p-1, q+2, p-1)$, where $p > q+2$.

Let $\theta_0(g) = (g^t)^{-1}$, then $\theta = \mathrm{Ad}_J \circ \theta_0 = \theta_0 \circ \mathrm{Ad}_J$. Let $i = \sqrt{-1} \in k$. Consider matrix

$$h = \begin{pmatrix} -i & 0 & 1 \\ 0 & 1 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Then $J = hh^t = h\theta_0(h)^{-1}$, $\theta = \mathrm{Ad}_{\theta_0(h)} \circ \theta_0 \circ \mathrm{Ad}_{\theta_0(h)^{-1}}$. Observe that $\mathrm{Ad}_{\theta_0(h)}$ acts trivially on S. Thus in order to show that the t^λ lies in the orbit closure of t^μ with respect to the θ-twisted conjugation of $\mathrm{SL}_3(\mathcal{O})$, it is equivalent to show that $\mathrm{Ad}_{\theta_0(h)^{-1}} t^\lambda = t^\lambda$ is in the orbit closure of $\mathrm{Ad}_{\theta_0(h)^{-1}} t^\mu = t^\mu$ with respect to the θ_0-twisted conjugation of $\mathrm{SL}_3(\mathcal{O})$. The later follows from applying Proposition 20 twice. This completes the proof.

4. Iwahori orbits

We will assume K is connected from Section 4-Section 10. In the last Section 11, we discuss the case when K is disconnected.

4.1. **Iwahori orbits parametrization.** Let $I_0 \subset LG$ be an Iwahori subgroup assoicated to the θ-stable Borel B_0. We consider I_0-orbits on LX. Many properties of B_0-orbits on X have been studied in [Spr85, RS90, MS98]. Our goal is to generalize these finite dimensional results to the affine setting.

In the following, we denote the pro-unipotent radical of an Iwahori subgroup I by I^+. For a split group H, we denote the pro-unipotent radical of $L^+H = H(\mathcal{O})$ by $L^{++}H$. Recall $T_0 \subset B_0$ is a θ-stable maximal torus contained in a θ-stable Borel subgroup. Then L^+T_0, LT_0

are θ-stable. The preimage of B_0 under the evaluation map $L^+G \to G$ defines a θ-stable Iwahori subgroup I_0 that contains L^+T_0.

For every split maximal torus $T \subset LG$, via the canonical isomorphism $X_*(T) \otimes_{\mathbb{Z}} F^\times \simeq T$, we can define a canonical integral model $L^+T := X_*(T) \otimes_{\mathbb{Z}} \mathcal{O}^\times \hookrightarrow T$. Denote $N_0 = N_{LG}(I_0)$. Note that $N_0\backslash LG$ parametrizes Iwahori subgroups via $g \mapsto \mathrm{Ad}_{g^{-1}} I_0$. Consider sets

$$\mathcal{C} = \{(T, I) \mid I \in N_0\backslash LG,\ L^+T \subset I\}, \quad \mathcal{C}^\theta = \{(T, I) \in \mathcal{C} \mid \theta(T) = T\},$$

where T is any split maximal torus of LG. Note that K acts on $\mathcal{C}^\theta$ via adjoint action. We can define a map

$$\phi : LG \to \mathcal{C}, \quad g \mapsto (\mathrm{Ad}_{g^{-1}} LT_0, \mathrm{Ad}_{g^{-1}} I_0).$$

Note that

$$\mathcal{V} := \phi^{-1}(\mathcal{C}^\theta) = \{g \in LG \mid \tau(g) = g\theta(g)^{-1} \in N_{LG}(LT_0)\},$$

and $\phi|_{\mathcal{V}}$ factors through the quotient $L^+T_0\backslash\mathcal{V}$ by left multiplication of L^+T_0.

We will need the following presumably well-known fact:

Lemma 22. *For any Iwahori subgroup I, all the maximal tori in I can be conjugated to each other by an element of pro-unipotent radical I^+.*

Proof. By conjugation, we can assume $I = I_0$ is the standard Iwahori, where $I_0 = T_0 I_0^+$. It suffices to show that any maximal torus $H \subset I_0$ can be conjugated to T_0 by I_0^+.

First for any $m > 0$, images of H, T_0 in I_0/L^mG are conjugated by some $u_m \in I_0^+$, i.e. $\mathrm{Ad}_{u_m} H \equiv T_0 \mod L^mG$. Next the images of $\mathrm{Ad}_{u_m} H, T_0$ in $I_0/L^{m+1}G$ are conjugated by some $g_m \in I_0^+$, i.e. $\mathrm{Ad}_{g_m}\mathrm{Ad}_{u_m} H \equiv T_0 \mod L^{m+1}G$. Priori, $\mathrm{Ad}_{g_m}\mathrm{Ad}_{u_m} H \equiv \mathrm{Ad}_{g_m} T_0 \equiv T_0 \mod L^mG$. Let $\bar{g}_m$ be the image of g_m in I_0/L^mG, then $\bar{g}_m \in N_{I_0/L^mG}(T_0)$. As T_0 is the maximal torus of solvable group I_0/L^mG, we have $\bar{g}_m \in N_{I_0/L^mG}(T_0) = C_{I_0/L^mG}(T_0)$. Observe that $C_{I_0/L^{m+1}G}(T_0) = T_0(\mathcal{O}/t^{m+1}) \twoheadrightarrow C_{I_0/L^mG}(T_0) = T_0(\mathcal{O}/t^m)$. Thus we can write $g_m = a_m b_m$, $a_m \in I_0^+$ has image in $C_{I_0/L^{m+1}G}(T_0)$, $b_m \in L^{m+1}G$. Let $u_{m+1} = b_m u_m \in I_0^+$, then $\mathrm{Ad}_{u_{m+1}} H \equiv T_0 \mod L^{m+1}G$. Repeat the above, we get a convergent sequence $u_m \in I_0^+$, $m > 0$, with limit $u \in I_0^+$, such that $\mathrm{Ad}_u H \equiv \mathrm{Ad}_{u_m} H \equiv T_0 \mod L^mG$, $\forall m$. Thus $\mathrm{Ad}_u H = T_0$. □

Now we give the parametrization of orbits.

Lemma 23. (i) *We have bijection $\bar{\phi} : (LT_0 \cap N_0)\backslash\mathcal{V} \simeq \mathcal{C}^\theta$.*

(ii) *The projection $p : \mathcal{C}^\theta \to N_0\backslash LG, (T, I) \mapsto I$ induces a bijection $\mathcal{C}^\theta/LK \simeq N_0\backslash LG/LK$ on LK-orbits.*

(iii) *The inclusion $\mathcal{V} \to LG$ induces bijections $\mathscr{V} := L^+T_0\backslash\mathcal{V}/LK \simeq I_0\backslash LG/LK \simeq I_0\backslash LX$.*

Proof. (i): Clearly ϕ is surjective, so is its restriction $\bar{\phi}$. To see it is injective, suppose $\phi(g_1) = \phi(g_2)$. Then $g_2 g_1^{-1}$ normalizes LT_0 and I_0, thus belongs to $LT_0 \cap N_0$.

(ii): First we show the surjectivity. For any Iwahori I, we need to find a θ-stable maximal torus $T \subset LG$ with $L^+T \subset I$. Note that θ acts on $I \cap \theta(I)$ and that the intersection of two Iwahori subgroups $I, \theta(I)$ contains a torus A of maximal rank in I. Denote by I_m the mth Moy-Prasad subgroup of I_m. Observe that $I_m \cap \theta(I_m)$ is a θ-stable normal subgroup of $I \cap \theta(I)$ of finite codimension. Thus θ acts on the finite dimensional solvable group $I \cap \theta(I)/I_m \cap \theta(I_m)$. Let m be large enough such that A is isomorphic to its image in $I \cap \theta(I)/I_m \cap \theta(I_m)$, so that the quotient group is of the same rank as I. By [Ste68, Theorem 7.5], there exists a θ-stable

maximal split torus $T_m \subset I \cap \theta(I)/I_m \cap \theta(I_m)$. We can lift T_m to LG as follows. Let H_m be the preimage of T_m in $I \cap \theta(I)$. In $H_m/I_{m+1} \cap \theta(I_{m+1})$, we can similarly find maximal torus T_{m+1}, and take its preimage H_{m+1} in H_m. Repeat this and let $T' = \cap_m H_m$, then $T' \simeq T_m$ is a θ-stable torus in $I \cap \theta(I)$. Moreover, $T = C_{LG}(T')$ is a split θ-stable maximal torus of LG. In fact, note that this is true if $T' \subset G \subset LG$. Since T' can be conjugated into the standard Iwahori I_0, and by Lemma 22 we know all the tori in I_0 can be conjugated into G, we obtain that T is split. Thus T satisfies the requirements.

For the injectivity, suppose $(I, T), (I, T') \in \mathcal{C}^\theta$. We need to find $g \in LK$ such that $\mathrm{Ad}_g I = I$, $\mathrm{Ad}_g T = T'$. Here $L^+T, L^+T' \subset I \cap \theta(I)$. For the split torus $T \simeq X_*(T) \otimes_{\mathbb{Z}} F^\times$, consider the constant subtorus $\mathring{T} := X_*(T) \otimes_{\mathbb{Z}} k^\times \subset T$. Then $T \simeq L\mathring{T}$, $L^+T \simeq L^+\mathring{T}$. Similarly define $\mathring{T}' \subset T'$. By the same proof of Lemma 22, $\mathring{T}, \mathring{T}'$ can be conjugated by an element of the pro-unipotent radical $(I \cap \theta(I))_u$ of $I \cap \theta(I)$. Let $g_1 \in (I \cap \theta(I))_u$ be such that $\mathrm{Ad}_{g_1}\mathring{T} = \mathring{T}'$. Then $\mathrm{Ad}_{g_1} L^+T = L^+T'$, $\mathrm{Ad}_{g_1} T = \mathrm{Ad}_{g_1} T'$. Since T, T' are θ-stable, we get $g_1^{-1}\theta(g_1) \in (N(T) \cap I^+)^{\mathrm{inv}\circ\theta} = (L^{++}T)^{\mathrm{inv}\circ\theta}$. Since $(L^{++}T)^{\mathrm{inv}\circ\theta}$ is pro-unipotent, we can find $t_1 \in (L^{++}T)^{\mathrm{inv}\circ\theta}$ such that $t_1\theta(t_1^{-1}) = t_1^2 = g_1^{-1}\theta(g_1)$. Thus $g := g_1 t_1 \in LK \cap I$ maps (I, T) to (I, T').

(iii): From (i), (ii) and Proposition 17 (ii), we obtain bijection

$$(LT_0 \cap N_0)\backslash\mathcal{V}/LK \simeq N_0\backslash LG/LK.$$

We have commutative diagram

$$\begin{array}{ccc} L^+T_0\backslash\mathcal{V}/LK & \xrightarrow{\iota} & I_0\backslash LG/LK \\ \downarrow & & \downarrow \\ (LT_0 \cap N_0)\backslash\mathcal{V}/LK & \xrightarrow{\iota'} & N_0\backslash LG/LK. \end{array}$$

Let $Z_G = Z(G)$. Observe that $LT_0 \cap N_0 = L^+T_0 LZ_G$. In fact, we can write $LT_0 = t^{X_*(T_0)}L^+T_0$. Then $t^{X_*(T_0)} \cap N_0$ consists of t^λ for $\langle \pm\lambda, \alpha\rangle \geq 0$, α any simple root. Thus $\langle \lambda, \alpha\rangle = 0$, $t^\lambda \in LZ_G$.

We first show that ι is injective. Given $u, v \in \mathcal{V}$ such that $\iota(u) = \iota(v)$. By the injectivity of ι', we have $u = tzvk$ for some $t \in L^+T_0$, $z \in LZ_G$, $k \in LK$. Then $\iota(u) = I_0 zvLK = I_0 vLK = \iota(v)$. Taking τ, we obtain that $\tau(u)\tau(z) \equiv \tau(u) \in \widetilde{\mathrm{W}} = N_{LG}(LT_0)/L^+T_0$. We deduce that $\tau(z) \in L^+T_0 \cap LZ_G = L^+Z_G$, $z \in L^+Z_G L(Z_G \cap K)$. We obtain $L^+T_0 uLK = L^+T_0 vLK$.

Next we show ι is surjective. From the surjectivity of ι', any I_0-orbit $I_0 gLK$ satisfies $g = nvk$ for some $v \in \mathcal{V}$, $n \in N_0$, $k \in LK$. Since $N_0/I_0 \simeq (N_{LG}(LT_0) \cap N_{LG}(I_0))/L^+T_0$, we can write $n = he$, $h \in I_0$, $e \in N_{LG}(I_0) \cap N_{LG}(LT_0)$. Note $\tau(ev) \in N_{LG}(LT_0)$. Thus $I_0 gLK = \iota(ev)$ for $ev \in \mathcal{V}$. □

Definition 24. *For any $v \in \mathscr{V} := L^+T_0\backslash\mathcal{V}/LK$, we let $\mathcal{O}_v \subset LX$ be the corresponding I_0-orbit on LX under the bijection in Lemma 23 (iii).*

4.2. **Twisted involutions in extended affine Weyl groups.** We denote by Δ_{aff}, Φ_{aff}, Φ^+_{aff}, Φ^-_{aff} the set of simple affine roots, affine roots, positive affine roots, and negative affine roots with respect to T_0 and I_0. Since T_0, I_0 are θ-stable, θ acts on Φ_{aff} and Φ^+_{aff}.

Let $\widetilde{\mathrm{W}} = N_{LG}(LT_0)/L^+T_0$ be the extened affine Weyl group. We denote by $\mathrm{W}_{\mathrm{aff}} \subset \widetilde{\mathrm{W}}$ the affine Weyl group generated by the simple reflections $\{s_\alpha\}_{\alpha\in\Delta_{\mathrm{aff}}}$ and $\Omega = \widetilde{\mathrm{W}}/\mathrm{W}_{\mathrm{aff}}$ the quotient. We denote by $l : \widetilde{\mathrm{W}} \to \mathbb{Z}_{\geq 0}$ the length function.

The natural map $N_{LG}(LT_0) \to LG$ induces a bijection $\widetilde{\mathrm{W}} = N_{LG}(LT_0)/L^+T_0 \cong I_0\backslash LG/I_0$. The composition $N_{LG}(I_0)/I_0 \subset I_0\backslash LG/I_0 \cong \widetilde{\mathrm{W}} \to \Omega$ induces an isomorphism of groups $N_{LG}(I_0)/I_0 \cong \Omega$, which gives rise to an isomorphism $\widetilde{\mathrm{W}} \cong \mathrm{W}_{\mathrm{aff}} \rtimes \Omega$ where $\Omega = \{w \in \widetilde{\mathrm{W}} | l(w) = 0\}$ consists of elements of length zero.

Since L^+T_0 and $N_{LG}(LT_0)$ are θ-stable, we get an induced action of θ on the extended affine Weyl group $\widetilde{\mathrm{W}}$. Let $\mathscr{I} = \{w \in \widetilde{\mathrm{W}} | \theta(w) = w^{-1}\}$. We say that the elements in $\mathscr{I}$ are twisted involutions.

Note that the map $\tau : G \to G$, $\tau(g) = g\theta(g)^{-1}$ induces a map $p : \mathscr{V} = L^+T_0\backslash\mathcal{V}/LK \to \mathscr{I}$ from the set of I_0-orbits on LX to the set of twisted involutions in the extended affine Weyl group. We study some basic properties of it.

We first need a lemma.

Lemma 25. *Any coset $v \in \mathscr{V}$ contains a representative $\dot{v} \in \mathcal{V}$ such that $\tau(\dot{v}) = n_v = \bar{n}_v t^\lambda$, where $\bar{n}_v \in N_G(T_0)$ and $\lambda \in X_*(T_0)$ satisfy $\theta(\bar{n}_v) = \bar{n}_v^{-1}$, $\mathrm{Ad}_{\bar{n}_v} t^{-\lambda} = t^{\theta(\lambda)}$.*

Proof. We can uniquely write $\tau(\dot{v}) = \dot{v}\theta(\dot{v})^{-1} \in N_{LG}(LT_0)$ as $\tau(\dot{v}) = \bar{n}_v t^\lambda \gamma$, where $\bar{n}_v \in N_G(T_0), \lambda \in X_*(T_0), \gamma \in L^{++}T_0$. Since $\tau(\dot{v}) \in (LG)^{\mathrm{inv}\circ\theta}$, we have

$$\theta(\tau(\dot{v})) = \theta(\bar{n}_v)t^{\theta(\lambda)}\theta(\gamma) = (\tau(\dot{v}))^{-1} = \gamma^{-1}t^{-\lambda}(\bar{n}_v)^{-1} = (\bar{n}_v)^{-1}t^{-\mathrm{Ad}_{\bar{n}_v}\lambda}\mathrm{Ad}_{\bar{n}_v}\gamma^{-1}.$$

By uniqueness, we get $\theta(\bar{n}_v) = \bar{n}_v^{-1}$, $t^{\theta(\lambda)} = t^{-\mathrm{Ad}_{\bar{n}_v}\lambda}$, $\theta(\gamma) = \mathrm{Ad}_{\bar{n}_v}\gamma^{-1}$. Denote $\theta_v := \mathrm{Ad}_{\bar{n}_v^{-1}} \circ \theta$, then $\theta_v(\gamma) = \gamma^{-1}$. Since θ_v acts on the pro-unipotent group $L^{++}T_0$ as an involution, there exists $\beta \in (L^{++}T_0)^{\mathrm{inv}\circ\theta_{\bar{v}}}$ such that $\beta^2 = \gamma$, i.e. $\beta\theta_{\bar{v}}(\beta)^{-1} = \gamma$. Then $\dot{v}' = \theta(\beta)\dot{v}$ satisfies $\tau(\dot{v}') = \bar{n}_v t^\lambda$, giving the desired representative. □

Lemma 26. *The map $p : \mathscr{V} \to \mathscr{I}$ has finite fibers.*

Proof. (i): For $u, v \in \mathscr{V} = L^+T_0\backslash\mathcal{V}/LK$ with representatives $\dot{u}, \dot{v}$ as in Lemma 25, suppose $\tau(\dot{u}) = \bar{n}_u t^\lambda, \tau(\dot{v}) = \bar{n}_v t^\mu$ have the same images in $\widetilde{\mathrm{W}}$. Then $\lambda = \mu$ and $\bar{n}_u = c\bar{n}_v$ for some $c \in T_0$. Since $\tau(\dot{u}), \tau(\dot{v}) \in (LG)^{\mathrm{inv}\circ\theta}$, we have

$$\theta(c\tau(\dot{v})) = \theta(c)\tau(\dot{v})^{-1} = \theta(c)t^{-\mu}\bar{n}_v^{-1} = \tau(\dot{v})^{-1}c^{-1} = t^{-\mu}\bar{n}_v^{-1}c^{-1}.$$

Thus $\psi_{\bar{v}}(c) = c^{-1}$ where $\psi_{\bar{v}} := \mathrm{Ad}_{\bar{n}_v} \circ \theta$ is an involution. Denote $\tau_{\bar{v}}(g) = g\psi_{\bar{v}}(g)^{-1}$. Observe that T_0 is $\psi_{\bar{v}}$-stable. Write $T_0^{\mathrm{inv}\circ\psi_{\bar{v}}} = \bigsqcup_i t_i T_0^{\mathrm{inv}\circ\psi_{\bar{v}},\circ}$, where $T_0^{\mathrm{inv}\circ\psi_{\bar{v}},\circ} = \tau_{\bar{v}}(T_0)$. There exists $a \in T_0$ such that $c = \tau_{\bar{v}}(a)t_i$ for some t_i. Thus $\tau(a\dot{u}) = t_i\tau(\dot{v})$. The preimage of p over the image of $\tau(\dot{v})$ is at most the number of connected components of $T_0^{\mathrm{inv}\circ\psi_{\bar{v}}}$, which is finite. □

Corollary 27. (i) *The closure of each I_0-orbit contains only finitely many I_0-orbits.*
(ii) *Each L^+G-orbit on LX is a union of finitely many I_0-orbits.*

Proof. (i): For an I_0-orbit $\mathcal{O}_v$, $\tau(\mathcal{O}_v) \subset I_0 n_v I_0$, $\tau(\overline{\mathcal{O}}_v) \subset \overline{I_0 n_v I_0}$. We know $\overline{I_0 n_v I_0}$ is a finite union of double cosets. By Lemma 26, the preimage under τ of each double coset consists of finitely many I_0-orbit. Part (i) follows.

(ii): Note that $\tau(L^+Gx_v) \subset L^+Gn_vL^+G$ where $L^+Gn_vL^+G$ is a finite union of I_0-double cosets. Then part (ii) follows from Lemma 26. □

4.3. **Simple parahoric orbits.** In this section we collect some basic facts on Iwahori orbits.

For a simple affine root $\alpha \in \Delta_{\mathrm{aff}}$, let P_α be the standard parahoric subgroup associated to α. Let P_α^+ and L_α be the pro-unipotent radical and Levi subgroup of P_α. Denote by

$s_\alpha \in W_{\mathrm{aff}}$ the simple reflection with respect to the hyperplane defined by α, then

$$P_\alpha = I_0 \sqcup I_0 s_\alpha I_0.$$

Let $\mathcal{O}_v = I_0 \dot{v} LK/LK = I_0 x_v$ be an I_0-orbit on LX, where $\dot{v} \in \mathcal{V}$ is as in Lemma 25. Denote by $w_v = p(v) \in \mathscr{I}$ the associated twisted involution in the extended affine Weyl group. Recall we associated to v an involution $\psi_v = \mathrm{Ad}_{n_v} \circ \theta$ so that $\mathrm{Stab}_{LG}(x_v) = (LG)^{\psi_v}$. The stabilizer of P_α on $\dot{v}$ is $P_{\alpha,v} = P_\alpha^{\psi_v}$. For $\alpha \in \Delta_{\mathrm{aff}}$, observe

$$I_0 \backslash P_\alpha \mathcal{O}_v \simeq I_0 \backslash P_\alpha / P_{\alpha,v} \simeq \mathbb{P}^1 / H_{\alpha,v}$$

where $H_{\alpha,v} \subset \mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}_2$ is the image of the right action of $P_{\alpha,v}$ on $I_0 \backslash P_\alpha \simeq \mathbb{P}^1$.

Denote by $\phi : P_\alpha \to \mathrm{Aut}(\mathbb{P}^1)$ the action. Let $U_{\pm\alpha} \subset P_\alpha$ be the affine root subgroups for $\pm\alpha$. Choose isomorphisms $u_{\pm\alpha} : k \xrightarrow{\sim} U_{\pm\alpha}$ as in [MS98, §4.1.1], so that $n_\alpha = u_\alpha(1) u_{-\alpha}(-1) u_\alpha(1)$ is a representative of s_α. Recall from [MS98, §4.1.3, §4.1.4] the following classification of infinite $H_{\alpha,v}$:

Lemma 28. *Up to conjugation by $\phi(I_0)$, any infinite subgroup $H = H_{\alpha,v} = \phi(P_\alpha) \subset \mathrm{PGL}_2$ is one of the following classes:*

(I)	$H = \mathrm{PGL}_2$,	$P_\alpha \mathcal{O}_v = \mathcal{O}_v$.
(IIa)	$\phi(U_\alpha) \subset H \subset \phi(I_0)$,	$P_\alpha \mathcal{O}_v = \mathcal{O}_v \sqcup \mathcal{O}_{v'}$ *in which* $\mathcal{O}_{v'}$ *is open.*
(IIb)	$\phi(U_{-\alpha}) \subset H \subset \phi(\mathrm{Ad}_{s_\alpha} I_0)$,	$P_\alpha \mathcal{O}_v = \mathcal{O}_v \sqcup \mathcal{O}_{v'}$ *in which* $\mathcal{O}_v$ *is open.*
(IIIa)	$H = \phi(T_0)$,	$P_\alpha \mathcal{O}_v = \mathcal{O}_v \sqcup \mathcal{O}_{v'} \sqcup \mathcal{O}_{v''}$ *in which* $\mathcal{O}_{v''}$ *is open.*
(IIIb)	$H = \phi(\mathrm{Ad}_{n_\alpha u_{\alpha(-1)}} T_0)$,	$P_\alpha \mathcal{O}_v = \mathcal{O}_v \sqcup \mathcal{O}_{v'} \sqcup \mathcal{O}_{v''}$ *in which* $\mathcal{O}_v$ *is open.*
(IVa)	H *is the normalizer of* $\phi(T)$,	$P_\alpha \mathcal{O}_v = \mathcal{O}_v \sqcup \mathcal{O}_{v'}$ *in which* $\mathcal{O}_{v'}$ *is open.*
(IVb)	H *is the normalizer of* $\phi(\mathrm{Ad}_{n_\alpha u_{\alpha(-1)}} T_0)$,	$P_\alpha \mathcal{O}_v = \mathcal{O}_v \sqcup \mathcal{O}_{v'}$ *in which* $\mathcal{O}_v$ *is open.*

We say an orbit $\mathcal{O}_v$ is of type I, IIa,... for s_α if $H_{\alpha,v}$ is of type I, IIa,... as in the above. Observe that $\mathcal{O}_v$ is open in $P_\alpha \mathcal{O}_v$ if and only if $\mathcal{O}_v$ is of type I, IIb, IIIb, or IVb for s_α.

Note that the involution $\psi_v = \mathrm{Ad}_{n_v} \circ \theta$ preserves T_0, thus acts on the set of affine roots. Since $\theta(n_v) = n_v^{-1}$, the action of ψ_v on affine roots is given by $w_v \circ \theta = \theta \circ w_v^{-1}$. By the same discussion as in [MS98, §6.5], the type of $\mathcal{O}_v$ for s_α can be read off from this action as follows:

Lemma 29.

(i) $\mathcal{O}_v$ *is of type I for s_α if and only if $\psi_v \alpha = \alpha$ and $\psi_v|_{U_{\pm\alpha}} = \mathrm{Id}$.*
(ii) $\mathcal{O}_v$ *is of type IIa or IIb for s_α if and only if $\psi_v \alpha \neq \pm\alpha$.*
(iii) $\mathcal{O}_v$ *is of type IIIa or IVa for s_α if and only if $\psi_v \alpha = \alpha$ and $\psi_v(g) = g^{-1}$ on $U_{\pm\alpha}$.*
(iv) $\mathcal{O}_v$ *is of type IIIb or IVb for s_α if and only if $\psi_v \alpha = -\alpha$.*
(v) *If $\mathcal{O}_v$ is of type I, IIa, IIIa, or IVa for s_α, then $\psi_v \alpha > 0$, i.e. $w_v^{-1}\alpha > 0$. If $\mathcal{O}_v$ is of type b, i.e. IIb, IIIb, or IVb for s_α, then $\psi_v \alpha < 0$, i.e. $w_v^{-1}\alpha < 0$.*

In particular, the stabilizer $H_{\alpha,v}$ is always infinite.

4.4. **Characterization of closed orbits.** We give a characterization of closed orbits. To this end, we need the following useful lemma:

Lemma 30. *Let ψ be an involution on a parahoric subgroup P that preserves its unipotent radical P^+ and Levi subgroup L_P. Then the embedding $\tau(g) = g\psi(g)^{-1}$ has image $\tau(P) = P^{\mathrm{inv} \circ \psi, \circ}$. In particular, the image is closed in P and in LG.*

Proof. It suffices to show the surjectivity. For any $\ell u \in P^{\mathrm{inv}\circ\psi,\circ}$,

$$\psi(\ell u) = \psi(\ell)\psi(u) = (\ell u)^{-1} = \ell^{-1}\mathrm{Ad}_{\ell} u^{-1}.$$

Denote $\psi_{\ell^{-1}} = \mathrm{Ad}_{\ell^{-1}} \circ \psi$. From the decomposition $P = L_P P^+$, we obtain $\psi(\ell) = \ell^{-1}$ and $\psi_{\ell^{-1}}(u) = u^{-1}$. This shows that the composition $P^{\mathrm{inv}\circ\psi} \hookrightarrow P \twoheadrightarrow L_P$ lands inside $L_P^{\mathrm{inv}\circ\psi}$. In particular, the neutral component $P^{\mathrm{inv}\circ\psi,\circ}$ maps into $L_P^{\mathrm{inv}\circ\psi,\circ}$, so that $\ell \in L_P^{\mathrm{inv}\circ\psi,\circ}$.

Since (L_P, ψ) is a symmetric pair of finite type, we know $\tau(L_P) = L_P^{\mathrm{inv}\circ\psi,\circ}$, so that $\ell = a\psi(a)^{-1}$ for some $a \in L_P$. Also, since P^+ is a pro-unipotent subgroup with a sequence of Moy-Prasad subgroups stable under $\psi_{\ell^{-1}}$ by the assumptions on ψ, we know the square map on $P^{+,\mathrm{inv}\circ\psi_{\ell^{-1}}}$ is onto. Thus there exists $b \in P^{+,\mathrm{inv}\circ\psi_{\ell^{-1}}}$ with $u = b^2$. We have

$$\tau(a\mathrm{Ad}_{\psi(a)^{-1}}(b)) = a\psi(a)^{-1}b\psi(a)a^{-1}\psi(b)^{-1}a\psi(a)^{-1} = \ell b\psi_{\ell^{-1}}(b)^{-1} = \ell b^2 = \ell u.$$

□

As in the finite dimensional situation [MS98, §6.6], we have the following important characterization of closed orbits:

Lemma 31. *For an I_0-orbit $\mathcal{O}_v$, the following are equivalent:*

(i) *$\mathcal{O}_v$ is a closed orbit,*
(ii) *n_v normalizes I_0,*
(iii) *$w_v = p(v) \in \Omega$, that is, the twisted involution $w_v \in \mathscr{I}$ has length zero,*
(iv) *$\mathcal{O}_v$ is not of type IIb, IIIb, or IVb for any simple reflection s_α.*

Proof. *(ii)* and *(iii)* are equivalent since $\Omega \cong N_{LG}(I_0)/I_0$. We show implications *(i)* $\Rightarrow$ *(iv)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(i)*.

Let $\mathcal{O}_v$ be a closed orbit. If it is of type b for a simple reflection s_α, then from the list in Lemma 28, the closure of $\mathcal{O}_v$ contains at least one distinct orbit, so that it cannot be closed. Thus (iv) must be satisfied.

If $\mathcal{O}_v$ satisfies (iv), then by Lemma 29.(5), $w_v^{-1}\alpha > 0$ for any simple affine root α. Thus w_v^{-1} and w_v fix the fundamental alcove and it follows that n_v normalizes I_0.

If n_v normalizes I_0, we show the orbit $\mathcal{O}_v$ is closed. The involution $\psi_v = \mathrm{Ad}_{n_v} \circ \theta$ acts on I_0 and preserves $T_0 \subset I_0$. Let $\tau_v(g) = g\psi_v(g)^{-1}$. Observe that $\tau(I_0 x_v) = \tau_v(I_0)n_v$. By Lemma 30, $\tau_v(I_0) = I_0^{\mathrm{inv}\circ\psi_v,\circ}$, which is closed in $I_0^{\mathrm{inv}\circ\psi_v}$ and further closed in I_0. Thus $\tau(I_0 x_v) = \tau_v(I_0)n_v$ is closed in $I_0 n_v$, thus closed in LG. We obtain that $\tau(I_0 x_v)$ is closed in $\tau(LX)$, so that $I_0 x_v$ is a closed orbit in LX.

□

Remark 32. The equivalence of (i) and (ii) in the above is an affine generalization of [RS93, Proposition 1.4.2]. Moreover, when G is simply connected (hence $\Omega = 0$), for a closed $\mathcal{O}_v$ we have $\tau(x_v) \in I_0 \subset L^+G$. Thus $\mathcal{O}_v \subset L^+X$ is the base change of a B_0-orbit $B_0\bar{x}_v$ on X along the evaluation map $L^+X \to X$ such that $\tau(\bar{x}_v) \in N_G(T_0) \cap B_0 = T_0$. By [RS93, Proposition 1.4.2], $B_0\bar{x}_v$ is a closed orbit on X. This gives a bijection between closed I_0-orbits on LX and closed B_0-orbits on X.

Remark 33. As mentioned in [MS98, §6.6], although part (i) and (iv) of the above lemma make sense for general spherical varieties, they are not equivalent in general.

5. Placidness of orbit closures

In this section, we show that spherical and Iwahori obits on the loop space LX are placid schemes. In particular, although those orbit closures are infinite dimensional, their singularities are in fact finite dimensional. We will use some basic definitions and constructions of placid schemes collected in Appendix A.4.

5.1. **Affine degenerations.** We recall the construction of affine degenerations of symmetric varieties and its relationship with spherical orbits in LX. The main reference is [SW22, Section 2.2].[6]

Let $X^\circ = BK/K \subset X$ be the open B-orbit. The action of B/U on $X^\circ//U = \mathrm{Spec}(k[X^\circ]^U)$ factors through a quotient torus $T \to T/T\cap K \overset{\tau}{\cong} S$. Let $\check{\Lambda}_S \subset X^*(T)$ be the characters that factor through $T \overset{\tau}{\to} S$. Consider the filtration $k[X]_\chi, \chi \in \check{\Lambda}_S$ of $k[X]$ where $k[X]_\chi$ consists of $f \in k[X]$ such that the representation of G generated by f has highest weights $\chi' \in \check{\Lambda}_S$ satisfying $\langle \chi - \chi', \Lambda_S^+\rangle \leq 0$.[7] The *affine degeneration* $\mathcal{X}$ of X is the affine scheme defined by

$$k[\mathcal{X}] = \bigoplus_{\chi\in\check{\Lambda}_S} k[X]_\chi \otimes e^\chi \subset k[X\times S],$$

where $e^\chi \subset k[S]$ is the function corresponding to the character χ. Consider the following affine toric scheme $A = \mathrm{Spec}(\bigoplus_{\chi\in\check{\Lambda}_S,\langle\chi,\Lambda_S^+\rangle\leq 0} ke^\chi)$. The natural inclusion $\bigoplus_{\chi\in\check{\Lambda}_S,\langle\chi,\Lambda_S^+\rangle\leq 0} ke^\chi \to k[\mathcal{X}]$ gives rise to $G\times S$-equivariant morphism

$$\pi : \mathcal{X} \to A.$$

For any $a \in A(k)$, the fiber $\mathcal{X}_a = \pi^{-1}(a)$ is an affine spherical variety for G (not necessary smooth and homogeneous). We let $\mathcal{X}^\bullet \subset \mathcal{X}$ be the open subscheme such that the fiber $\mathcal{X}_a^\bullet \subset \mathcal{X}_a$ over a is the open G-orbit.

Let $\lambda \in \Lambda_S^+$. Then the composition $\mathbb{G}_m \xrightarrow{\lambda^{-1}} S \to A$ extends to a map $\iota_\lambda : \mathbb{A}^1 \to A$. We let $\mathcal{X}_\lambda = \mathcal{X}\times_{A,\iota_\lambda}\mathbb{A}^1$ and $\mathcal{X}_\lambda^\bullet = \mathcal{X}^\bullet\times_{A,\iota_\lambda}\mathbb{A}^1$. Then $\pi_\lambda : \mathcal{X}_\lambda \to \mathbb{A}^1$ is a $G\times\mathbb{G}_m$-equivariant affine family and $\pi_\lambda^\bullet : \mathcal{X}_\lambda^\bullet \to \mathbb{A}^1$ is an open subfamily. The preimages of both π_λ and $\pi_\lambda^\bullet$ over $\mathbb{G}_m$ are isomorphic to $X\times\mathbb{G}_m$.

The open inclusion $X\times\mathbb{G}_m \cong \pi_\lambda^{-1}(\mathbb{G}_m) \subset \mathcal{X}_\lambda$ defines an inclusion $LX\times L\mathbb{G}_m \subset L\mathcal{X}_\lambda$. Consider the embedding

$$f_\lambda : LX \to LX\times L\mathbb{G}_m \subset L\mathcal{X}_\lambda, \qquad \gamma \to (\gamma, t). \tag{5}$$

Here $t \in k((t)) = L\mathbb{G}_m(k)$. We have an fp-open embedding $L^+\mathcal{X}_\lambda^\bullet \subset L^+\mathcal{X}_\lambda$ and a closed embedding $L^+\mathcal{X}_\lambda \to L\mathcal{X}_\lambda$. We have the following description of LX_λ in terms of the arc space $L^+\mathcal{X}_\lambda^\bullet$:

Lemma 34. *The map f_λ induces an isomorphism*

$$LX_\lambda \cong LX\times_{f_\lambda, L\mathcal{X}_\lambda} L^+\mathcal{X}_\lambda^\bullet.$$

Proof. According to [MS98, Section 6.3], the stabilizer of B on X° is a smooth subgroup and the lemma follows from [SW22, Lemma 2.3.10]. □

[6]Note that in *loc. cit.*, the authors made some assumptions on the characteristic of k. Thanks to the recent work [BS25], those assumptions can be removed.

[7]Since $k[X]^U \subset k[X^\circ]^U$, the action of T on highest weight vectors factors through the quotient torus S and hence the highest weights are contained in Λ_S

5.2. **Placidness of spherical and Iwahori orbit closures.** We have the following basic placidness results for loop spaces due to Drinfeld:

Theorem 35. *Let Y be a smooth affine scheme of finite type over k. The loop space LY of Y is a placid ind-scheme. More precisely, for any affine embedding $Y \subset \mathbb{A}^n$ let $L^iY = LY \cap t^{-i}L^+\mathbb{A}^n$. Then each LY^i is a placid scheme and $LY \cong \operatorname{colim}_i LY^i$ is an ind-placid presentation of LY.*

Proof. This is [Dri06, Theorem 6.3]. □

We shall deduce the placidness of orbit closures from Lemma 34 and Theorem 35.

Theorem 36. (i) *The orbit closures $\overline{\mathcal{O}}_v$ (resp. $\overline{LX}_\lambda$) are irreducible placid schemes and the closed embeddings $\overline{\mathcal{O}}_v \to LX$ (resp. $\overline{LX}_\lambda \to LX$) are finitely presented.*

(ii) *The open embedding $\mathcal{O}_v \to \overline{\mathcal{O}}_v$ (resp. $LX_\lambda \to \overline{LX}_\lambda$) and the closed embedding $\overline{\mathcal{O}}_v \to \overline{\mathcal{O}}_{v'}$, $v \le v'$ (resp. $\overline{LX}_\lambda \to \overline{LX}_{\lambda'}$, $\lambda \le \lambda'$) are finitely presented.*

Proof. We first deal with L^+G-orbit closures. Recall the $G \times \mathbb{G}_m$-equivariant affine family $\pi_\lambda : \mathcal{X}_\lambda \to \mathbb{A}^1$ and the subschemes $X \cong \pi_\lambda^{-1}(1) \subset X \times \mathbb{G}_m \cong \pi_\lambda^{-1}(\mathbb{G}_m) \subset \mathcal{X}_\lambda^\bullet \subset \mathcal{X}_\lambda$ in Section 5.1. Since $\mathcal{X}_\lambda$ is an affine $G \times \mathbb{G}_m$-scheme of finite type, we can find a $G \times \mathbb{G}_m$-equivariant closed embedding $\mathcal{X}_\lambda \subset V$ into a finite dimensional representation V of $G \times \mathbb{G}_m$.

We have a $LG \times L\mathbb{G}_m$-equivariant closed embedding $L\mathcal{X}_\lambda \subset LV$ and for any $i \in \mathbb{Z}_{\le 0}$ we can form the $L^+G \times L^+\mathbb{G}_m$-invariant closed subschemes $L^i\mathcal{X}_\lambda = L\mathcal{X}_\lambda \cap t^iL^+V \subset L\mathcal{X}_\lambda$ and L^+G-invariant closed subscheme $L^iX = LX \cap t^iL^+V \subset LX$. Since X is smooth, Theorem 35 implies L^iX is a placid scheme and $LX \cong \operatorname{colim}_i L^iX$ is a placid presentation.

We claim that for any $i \in \mathbb{Z}_{\le 0}$, there exists $i' \in \mathbb{Z}_{\le 0}$ such that the embedding $f_\lambda : LX \to LX \times L\mathbb{G}_m \subset L\mathcal{X}_\lambda$, $f_\lambda(\gamma) = (\gamma, t)$ in (5) satisfies

$$f_\lambda(L^iX) \subset L^{i'}\mathcal{X}_\lambda. \tag{6}$$

Indeed, let $V = \bigoplus_{j\in\mathbb{Z}} V_j$ be the weight spaces decomposition with respect to the $\mathbb{G}_m$-action, that is, $s \cdot v_j = s^j v_j$ for $v_j \in V_j$. Then for any $w_j \in LV_j$ we have $t \cdot w_j = t^j w_j$ and it follows that $f_\lambda(L^iX) \subset t \cdot (LX \cap t^iL^+V) \subset t^{i'}L^+V$ for $i' := \min\{i + j | j \in \mathbb{Z}, V_j \ne 0\}$. The claim follows.

For any $\lambda \in \Lambda_S^+$, choose $i, i' \in \mathbb{Z}_{\le 0}$ satisfying (6) and $LX_\lambda \subset L^iX$. Then Lemma 34 implies that there is a Cartesian diagram

$$\begin{array}{ccc} LX_\lambda \cong L^iX \times_{L^{i'}\mathcal{X}_\lambda} L^+\mathcal{X}_\lambda^\bullet & \longrightarrow & L^+\mathcal{X}_\lambda^\bullet \\ \downarrow{\scriptstyle \eta_\lambda} & & \downarrow{\scriptstyle \eta} \\ L^iX & \xrightarrow{f_\lambda} & L^{i'}\mathcal{X}_\lambda \end{array}.$$

Note that $\eta : L^+\mathcal{X}_\lambda^\bullet \to L^+\mathcal{X}_\lambda \to L^i\mathcal{X}_\lambda$ is a fp-locally closed embedding and hence η_λ is also a fp-locally closed embedding. Since L^iX is placid, the Noetherian descent for fp-morphisms implies that there is a unique decomposition

$$\eta_\lambda : LX_\lambda \xrightarrow{j_\lambda} \overline{LX_\lambda} \xrightarrow{i_\lambda} L^iX$$

where j_λ is a placid fp-open embedding and i_λ is a placid fp-closed emebdding (see, e.g., [BKV11, Lemma 1.3.6 and Lemma 1.3.11]). Now the placidness of $\overline{LX_\lambda}$ follows from A.1 (2).

For the case of Iwahori orbits, it suffices to show that the embedding $\mathcal{O}_v \to L^+Gx_v$ into the spherical orbit through $x_v \in \mathcal{O}_v$ is finitely presented. Let $H \subset L^+G$ and $H' \subset I_0$ be the stabilizers of x_v in L^+G and I_0 respectively. For any $j \in \mathbb{Z}_{\geq 0}$, let $H_j \subset L_j^+G$ and $H'_j \subset (I_0)_j)$ be the images of H and H' along the map $L^+G \to L_j^+G$ and $I_0 \to (I_0)_j$. Here L_j^+G is the j-th arc group of G and $(I_0)_j = \mathrm{Im}(I_0 \to L^+G \to L_j^+G)$. We have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{O}_v \cong \lim_j (I_0)_j/H'_j & \longrightarrow & L^+Gx_v \cong \lim_j L_j^+G/H_j \\ \downarrow & & \downarrow \\ (I_0)_j/H'_j & \xrightarrow{f} & L_j^+G/H_j \end{array}$$

where the bottom horizontal arrow is induced by the natural inclusion

$$f : (I_0)_j/H'_j \cong (I_0)_j/H_j \cap (I_0)_j \to L_j^+G/H^j.$$

Since f is finitely presented it follows that the top inclusion $\mathcal{O}_v \to L^+Gx_v$ is finitely presented. The desired claim follows. □

Remark 37. Since [SW22, Lemma 2.3.10] and Theorem 35 are valid for general spherical varieites, Theorem 36 remains true for general spherical varieties.

5.3. **Filtered structures on orbits.** We shall show that, on each connected component of LX, the orbits closure relation defines filtered partial orders on the sets of I_0-orbits and L^+G-orbits.

We have isomorphisms of exact sequences

$$\begin{array}{ccccc} \pi_0(LG) & \longrightarrow & \pi_0(LX) & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \pi_1(G) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_0(K) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \\ \Lambda_T/Q & \xrightarrow{\tau} & \Lambda_S/\Lambda_S \cap Q & \longrightarrow & 0 \end{array} \tag{7}$$

induced from the fibration $K \to G \to G/K = X$ (note that we assume K is connected), here $Q \subset \Lambda_T$ is the coroot lattice and the map τ is given by $\tau(\lambda) = -\theta(\lambda) + \lambda$ (see, e.g., [Nad05, Section 3.9]).

Let $LX = \bigsqcup_{\beta \in \pi_0(LX)} LX_\beta$ be the decomposition into connected components. We denote by $\mathscr{V}_\beta = \{v \in \mathscr{V} | \mathcal{O}_v \subset LX_\beta\}$ and $\Lambda_{S,\beta}^+ = \{\lambda \in \Lambda_S^+ | LX_\lambda \subset LX_\beta\}$ and consider the set $\underline{\mathscr{V}} = \{\underline{v} = \{v_\beta\}_{\beta \in \pi_0(LX)} | v_\beta \in \mathscr{V}_\beta\}$ and $\underline{\Lambda}_S^+ = \{\underline{\lambda} = \{\lambda_\beta\}_{\beta \in \pi_0(LX)} | \lambda_\beta \in \Lambda_{S,\beta}^+\}$.

Lemma 38. (i) *The set $\Lambda_{S,\beta}^+$ together with the partial order $\lambda \leq \lambda'$ if $LX_\lambda \subset \overline{LX}_{\lambda'}$ is a filtered set.*

(ii) *The set $\mathscr{V}_\beta$ together with the partial order $\bar{v} \leq \bar{v}'$ if $\mathcal{O}_v \subset \overline{\mathcal{O}}_{v'}$ is a filtered set*

Proof. Proof of (i). Let $\lambda, \lambda' \in \Lambda_{S,\beta}^+$. According to (7), we have $\lambda - \lambda' \in \Lambda_S \cap Q$. Let $M = Z_G(S) = P \cap \theta(P)$ be the Levi subgroup of a minimal θ-split parabolic P. Let Δ (resp. Δ_M) be the set of simple roots of G (resp. M) and let $\mu = 2\check{\rho} - 2\check{\rho}_M$ where $2\check{\rho}$ (resp. $2\check{\rho}_M$) is the sum of positive coroots of G (resp. M). We have the following properties of μ:

(1) Note that $\mu = 2\sum_{\alpha\in\Delta\setminus\Delta_M}\check{\omega}_\alpha$, where $\check{\omega}_\alpha$ is the fundamental coweight corresponding to the simple root α.

(2) By [Tak, Lemma 8.1], we have $\theta(\mu) = -\mu$, and hence $\mu \in \Lambda_S \cap Q$.

(3) For any $\alpha \in \Delta$, it follows from (1) that $\langle \mu, \alpha\rangle = 0$ if $\alpha \in \Delta_M$ and $\langle \mu, \alpha\rangle = 2$ if $\alpha \in \Delta \setminus \Delta_M$.

(4) μ is a sum of simple coroots that has positive integer coefficient in each simple coroot. This follows from the fact the coefficients of fundamental coweights $\check{\omega}_\alpha$ in simple roots are entries of the inverse of Cartan matrix, which are all positive (checked case-by-case).

Using the above properties of μ, it is straightforward to check that that for large enough integer n, we have $\gamma = \lambda + n\mu \in \Lambda_S^+$, $\gamma \geq \lambda$, and $\gamma \geq \lambda'$. Part (i) follows.

Proof of (ii). Let $v, v' \in \mathscr{V}_\beta$. Let $LX_\lambda \supset \mathcal{O}_v$ and $LX_{\lambda'} \supset \mathcal{O}_{v'}$ be the L^+G-orbits through x_v and $x_{v'}$ and let $\gamma \geq \lambda, \lambda'$ be as in (1). By Corolalry 27, there is a unique open I_0-orbit $\mathcal{O}_{v''} \subset LX_\gamma$, and it follows that $\overline{\mathcal{O}}_{v''} = \overline{LX_\gamma} \supset LX_\lambda \cup LX_{\lambda'} \supset \mathcal{O}_v \cup \mathcal{O}_{v'}$. Thus $v'' \in \mathscr{V}_\beta$ satisfies $v'' \geq v$ and $v'' \geq v'$. Part (ii) follows.

□

We denote $\mathscr{V}_\beta = \{v \in \mathcal{V} | \mathcal{O}_v \subset LX_\beta\}$ and $\Lambda^+_{S,\beta} = \{\lambda \in \Lambda_S^+ | LX_\lambda \subset LX_\beta\}$ and consider the set $\underline{\mathscr{V}} = \{\underline{v} = \{v_\beta\}_{\beta\in\pi_0(LX)} | v_\beta \in \mathscr{V}_\beta\}$ and $\underline{\Lambda}_S^+ = \{\underline{\lambda} = \{\lambda_\beta\}_{\beta\in\pi_0(LX)} | \lambda_\beta \in \Lambda^+_{S,\beta}\}$. It follows from Lemma 38 that the set $\underline{\Lambda}_S^+$ (resp. $\underline{\mathscr{V}}$) together with the partial order $\underline{\lambda} \leq \underline{\lambda}'$ if $\lambda_\beta \leq \lambda'_\beta$ (resp. $\underline{v} \leq \underline{v}'$ if $v_\beta \leq v'_\beta$) is a filtered set.

The following proposition following immediately from Theorem 36.

Proposition 39. (i) *The union $\overline{\mathcal{O}}_{\underline{v}} = \bigsqcup_\beta \overline{\mathcal{O}}_{\bar{v}_\beta}$ is a locally equidimensional placid scheme and $LX \cong \mathrm{colim}_{\underline{v}}\, \overline{\mathcal{O}}_{\underline{v}}$ is a locally equidimensional I_0-placid presentation.*

(ii) *The disjoint union $\overline{LX}_{\underline{\lambda}} = \bigsqcup_\beta \overline{LX}_{\lambda_\beta}$ is a locally equidimensional placid scheme and $LX \cong \mathrm{colim}_{\underline{\lambda}}\, \overline{LX}_{\underline{\lambda}}$ is a locally equidimensional L^+G-placid presentation.*

5.4. **Dimension theory for LX.** Following A.4 (6), a dimension theory for the locally equidimensional placid ind-scheme LX is an assignment to each locally equidimensional placid subscheme $S \subset LX$ a locally constant function $\delta_S : S \to \mathbb{Z}$ such that for any finitely presented locally closed embedding $S \subset S' \subset LX$ we have

$$\delta_S - \delta_{S'}|_S = \dim_{S/S'} : S \to \mathbb{Z}, \tag{8}$$

here $\dim_{S/S'} : S \to \mathbb{Z}$ is the dimension function associated to the inclusion $S \subset S'$ in Appendix A.4(4).

Lemma 40. *For any closed I_0-orbits $\mathcal{O}_u, \mathcal{O}_{u'}$ contained in $\overline{\mathcal{O}}_v$, both $\dim_{\mathcal{O}_u/\overline{\mathcal{O}}_v}$ and $\dim_{\mathcal{O}_{u'}/\overline{\mathcal{O}}_v}$ are constant functions with the same value.*

Proof. By the additive property of dimension functions in A.4(6) and the filtered property of I_0-orbits in Proposition 38, it suffices to verify the lemma for a particular $\overline{\mathcal{O}}_v$ containing $\mathcal{O}_u, \mathcal{O}_{u'}$.

Claim: $N_{LG}(I_0)$ has transitive action on connected components.

Since K is assumed to be connected, the natural map $LG \to LX$ induces a surjection $\pi_0(LG) \to \pi_0(LX)$ on connected components (see (7)). On the other hand, a well-know result of Kottwitz [Kot97] implies that the inclusion $N_{LG}(I_0) \subset LG$ induces a bijection $\Omega \cong N_{LG}(I_0)/I_0 \cong \pi_0(LG)$. The claim follows.

Since the action of Ω maps I_0-orbit to I_0-orbit, by the claim we can translate $\mathcal{O}_v$ to the neutral component. Then $\mathcal{O}_u, \mathcal{O}_{u'}$ are closed I_0-orbits in the neutral component, so that $\tau(\mathcal{O}_u) \subset I_0 n_u I_0 \subset N_{LG}(I_0)$ lands inside the neutral component of LG, same for u'. It follows that $\tau(\mathcal{O}_u), \tau(\mathcal{O}_{u'}) \subset I_0$ and hence $\mathcal{O}_u, \mathcal{O}_{u'} \subset L^+X$ (as $\tau^{-1}(L^+G) = L^+X$). Since the closed I_0-orbits on L^+X are base change from closed B_0-orbits on X via the pro-smooth morphism $L^+X \to X$, it suffices to show that closed orbits on X are equal-dimensional. As discussed in [RS93, §1.4], closed B_0-orbits on X all have the same dimension as the flag variety of K. This completes the proof. □

Proposition 41. *There is a dimension theory δ on LX.*

Proof. By A.4 (9) and Proposition 38, it suffices to assign $\delta_{\overline{\mathcal{O}}_v} : \overline{\mathcal{O}}_v \to \mathbb{Z}$ for each $v \in \mathscr{V}$ satisfying (8). To this end, we set $\delta_{\overline{\mathcal{O}}_v} = \dim_{\mathcal{O}_u/\overline{\mathcal{O}}_v} : \overline{\mathcal{O}}_v \to \mathbb{Z}$ where $\mathcal{O}_u \subset \overline{\mathcal{O}}_v$ is a closed I_0-orbit in $\overline{\mathcal{O}}_v$. Lemma 40 and additive property of dimension functions A.4(6) imply $\delta_{\overline{\mathcal{O}}_v}$ is well-defined and satisfies (8). The proposition follows. □

6. Transversal slices

We construct transversal slices of spherical and Iwahori obits on the loop space LX and establish some basic properties of them. The construction generalizes the one due to Mars-Springer [MS98, §6] in the case of B-orbits on X.

6.1. **Slices for spherical orbits.**

6.1.1. *Involution on Affine Grassmannian slices.* Let $L^-G = G[t^{-1}]$, $L^{<0}G = \ker(L^-G \to G)$, and $L^-\mathfrak{g}, L^{<0}\mathfrak{g}$ their Lie algebras. For any $\lambda \in \Lambda_T^+$, we write $LG_\lambda = L^+Gt^\lambda L^+G$ for the $L^+G \times L^+G$-orbit of t^λ in LG, and $\overline{LG_\lambda}$ its closure. The affine Grassmannian slice of LG_λ in LG is defined as $W^\lambda = (L^{<0}G \cap \mathrm{Ad}_{t^\lambda} L^{<0}G)t^\lambda$. For any pair $\lambda, \mu \in \Lambda_T^+$, we define $W^\lambda_\mu = W^\lambda \cap LG_\mu$ and $W^\lambda_{\leq\mu} = W^\lambda \cap \overline{LG_\mu}$. We have the following well-known facts, see for example [Zhu17]:

Lemma 42. (i) *W^λ is an ind-scheme of ind-finite type and formally smooth.*
(ii) *W^λ_μ is non-empty if and only if $\lambda \leq \mu$.*
(iii) *W^λ_μ is smooth and connected (if non-empty).*
(iv) *$W^\lambda_{\leq\mu}$ is of finite type, affine, and normal.*
(v) $\dim W^\lambda_\mu = W^\lambda_{\leq\mu} = \langle 2\rho, \mu - \lambda\rangle$.

Recall the involution $\mathrm{inv} \circ \theta : LG \to LG, g \to \theta(g)^{-1}$.

Lemma 43. *Let $\lambda \leq \mu \in \Lambda_S^+$. The map $\mathrm{inv} \circ \theta : LG \to LG$ restricts to involutions on W^λ, W^λ_μ, and $W^\lambda_{\leq\mu}$. Moreover, the identification $W^\lambda \cong L^{<0}G \cap \mathrm{Ad}_{t^\lambda} L^{<0}G$, sending $\gamma \cdot t^\lambda \to \gamma$, intertwines the involution $\mathrm{inv} \circ \theta|_{W^\lambda}$ with the involution $\mathrm{inv} \circ \psi_\lambda$ on $L^{<0}G \cap \mathrm{Ad}_{t^\lambda} L^{<0}G$ where $\psi_\lambda = \mathrm{Ad}_{t^\lambda} \circ \theta$.*

Proof. Note that LG_λ is stable under $\mathrm{inv} \circ \theta$ and $L^{<0}G \cap \mathrm{Ad}_{t^\lambda} L^{<0}G$ is stable under $\mathrm{inv} \circ \psi_\lambda$. Thus for any $\gamma t^\lambda \in W^\lambda = (L^{<0}G \cap \mathrm{Ad}_{t^\lambda} L^{<0}G)t^\lambda$ (resp. W^λ_μ, $W^\lambda_{\leq\mu}$), we have

$$\mathrm{inv} \circ \theta(\gamma t^\lambda) = \theta(t^\lambda)^{-1}\theta(\gamma)^{-1} = (t^\lambda \theta(\gamma)^{-1} t^{-\lambda})t^\lambda = \mathrm{inv} \circ \psi_\lambda(\gamma)t^\lambda \in W^\lambda \quad (\textit{resp. } W^\lambda_\mu, W^\lambda_{\leq\mu}).$$

The lemma follows.

□

For any $\lambda \in \Lambda_S^+$, consider the maps $\rho_\lambda(s) : LG \to LG$ sending $\gamma(t) \to \rho_\lambda(s)(\gamma(t)) = s^\lambda \gamma(s^{-2}t)s^\lambda$.

Lemma 44. *The two maps $\rho_\lambda(s)$ and $\mathrm{inv} \circ \theta$ commute with each other. Moreover, the map $\rho_\lambda(s)$ restricts to a $\mathbb{G}_m$-action ρ_s on $W^\lambda, W^\lambda_\mu, W^\lambda_{\leq\mu}$. The $\mathbb{G}_m$-action ρ_s on W^λ and $W^\lambda_{\leq\mu}$ is contrating with unique fixed point t^λ.*

Proof. The first claim is a direct computation. For any $\gamma(t)t^\lambda \in W^\lambda$, we have

$$\rho_\lambda(s)(\gamma(t)t^\lambda) = s^\lambda \gamma(s^{-2}t)(s^{-2}t)^\lambda s^\lambda = s^\lambda \gamma(s^{-2}t)s^{-\lambda}t^\lambda \in W^\lambda.$$

Thus $\rho_\lambda(s)$ preserves W^λ. Since $\gamma \in L^{<0}G \cap \mathrm{Ad}_{t^\lambda} L^{<0}G$, we have $s^\lambda \gamma(s^{-2}t)s^{-\lambda}$ goes to the unit $e \in LG$ as s goes to 0, and it follows that the $\mathbb{G}_m$-action on W^λ is contracting with unique fixed point t^λ. Since LG_μ is obviously stable under $\rho_\lambda(s)$, the $\mathbb{G}_m$-action preserves W^λ_μ and $W^\lambda_{\leq\mu}$. □

Consider the fixed points

$$L^\lambda = (W^\lambda)^{\mathrm{inv}\circ\theta}, \quad L^\lambda_\mu = (W^\lambda_\mu)^{\mathrm{inv}\circ\theta}, \quad L^\lambda_{\leq\mu} = (W^\lambda_{\leq\mu})^{\mathrm{inv}\circ\theta}. \tag{9}$$

Lemma 43 (2) implies there is an isomorphism

$$L^\lambda \cong (L^{<0}G \cap \mathrm{Ad}_{t^\lambda} L^{<0}G)^{\mathrm{inv}\circ\psi_\lambda}. \tag{10}$$

Lemma 45. (i) *L^λ is a connected ind-scheme of ind-finite type and formally smooth.*
(ii) *L^λ_μ is smooth.*
(iii) *$L^\lambda_{\leq\mu}$ is of finite type, affine, and connected.*
(iv) *The map $\rho_\lambda(s)$ restricts to a $\mathbb{G}_m$-action ρ_s on $L^\lambda, L^\lambda_\mu, L^\lambda_{\leq\mu}$. The $\mathbb{G}_m$-action ρ_s on L^λ and $L^\lambda_{\leq\mu}$ is contrating with unique fixed point t^λ.*

Proof. Part (i) and (ii) follow from Lemma 53 and Lemma 97. Since $W^\lambda_{\leq\mu}$ is of finite type and affine, the fixed points $L^\lambda_{\leq\mu}$ is a closed subscheme and hence of finite type and affine. Lemma 44 and 43 imply that those fixed points subschemes $L^\lambda, L^\lambda_\mu, L^\lambda_{\leq\mu}$ are stable under the $\mathbb{G}_m$-action ρ_s. Moreover since $t^\lambda \in L^\lambda$, the $\mathbb{G}_m$-action on L^λ and $L^\lambda_{\leq\mu}$ is contracting with unique fixed point t^λ. In particular, we see that $L^\lambda_{\leq\mu}$ is connected. Part (iii) and (iv) follow. □

6.1.2. *The contracting slices S^λ and $S^\lambda_{\leq\mu}$.* Recall the embedding $\tau : LX \to LG$ induced from $\tau(\gamma) = \gamma\theta(\gamma)^{-1} : X \to G$.

Definition 46. *(i) For any $\lambda \in \Lambda_S^+$, we define $S^\lambda \subset LX$ as the following fiber product*

$$\begin{array}{ccc} S^\lambda & \longrightarrow & LX \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ W^\lambda & \longrightarrow & LG. \end{array}$$

(ii) For any $\lambda, \mu \in \Lambda_S^+$, let $S^\lambda_\mu = S^\lambda \cap LX_\mu$ and $S^\lambda_{\leq\mu} = S^\lambda \cap \overline{LX}_\mu$.

The following proposition implies that S^λ defines a transversal contracting slice to the orbit LX_λ inside LX.

Proposition 47. (i) *The embedding τ induces an isomorphism $S^\lambda \cong L^\lambda$. In particular, S^λ is a connected ind-scheme of ind-finite type and formally smooth.*

(ii) *Let $r_s : t \mapsto s^{-2}t$ be rotation automorphism of LG, $s \in \mathbb{G}_m$. Then $\phi_s^\lambda = \phi_s : (x \mapsto s^\lambda r_s(x))$ defines a $\mathbb{G}_m$-action on S^λ, which fixes x_λ, and contracts S^λ to x_λ.*

(iii) *The multiplication map $m : L^+G \times S^\lambda \to LX$ is formally smooth.*

(iv) *We have $S^\lambda \cap LX_\lambda = \{x_\lambda\}$ and $S^\lambda \subset LX$ is transversal to $LX_\lambda = L^+G \cdot x_\lambda$ at x_λ.*

(v) *All the $\mathbb{G}_m$-actions ϕ_s^λ preserve all the L^+G-orbits LX_μ. Any L^+G-equivariant local system on LX_μ is $\mathbb{G}_m$-equivariant with respect to the action ϕ_s^λ for any $\mu \in \Lambda_S^+$.*

Proof. Part (i). Note that the image $\tau(LX) \cong L(G^{\mathrm{inv}\circ\theta,\circ}) \subset L(G^{\mathrm{inv}\circ\theta}) = (LG)^{\mathrm{inv}\circ\theta}$ is an union of components of the fixed point $(LG)^{\mathrm{inv}\circ\theta}$. Thus the image $\tau(S^\lambda) = W^\lambda \cap \tau(LX) \subset L^\lambda = (W^\lambda)^{\mathrm{inv}\circ\theta}$ is an union of component of L^λ. Now the desired claim follows from Lemma 45 (i).

Part (ii). A direct computation shows that the isomorphism $\tau : S^\lambda \cong L^\lambda$ is compatible the $\rho_s(\mathbb{G}_m)$-action. The claims follow from Lemma 45 (iv).

Part (iii). It is equivalent to show that the twisted conjugation map $f : L^+G \times L^\lambda \to LG^{\mathrm{inv}\circ\theta}, (g, ht^\lambda), \to ght^\lambda\theta(g)^{-1}$ is formally smooth. Since f is a map between placid ind-schemes, we can apply the formally smooth criterion in Lemma 96 and reduce to check that the differential

$$d_{(g,ht^\lambda)}f : T_gL^+G \times T_{ht^\lambda}L^\lambda \to T_aLX$$

is surjective for any R-point $(g, ht^\lambda) \in L^+G(R) \times L^\lambda(R)$ and $a = ght^\lambda\theta(g)^{-1} \in (LG)^{\mathrm{inv}\circ\theta}(R)$. Denote $\psi_a = \mathrm{Ad}_a \circ \theta : LG(R) \to LG(R)$ and $L\mathfrak{g}^{-\psi_a}(R)$ the -1 eigensubspace of ψ_a in $L\mathfrak{g}(R)$. Similarly define ψ_{ht^λ}. Note that

$$T_gL^+G \cong L^+\mathfrak{g}(R), \quad T_{ht^\lambda}L^\lambda \simeq (L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda}L^{<0}\mathfrak{g})^{-\psi_{ht^\lambda}}(R), \quad T_a(LG)^{\mathrm{inv}\circ\theta,\circ} \simeq (L\mathfrak{g})^{-\psi_a}(R).$$

We need to show that the tangent map

$$L^+\mathfrak{g}(R) \times (L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda}L^{<0}\mathfrak{g})^{-\psi_{ht^\lambda}}(R) \to (L\mathfrak{g})^{-\psi_a}(R), \quad (X, Y) \mapsto X - \mathrm{Ad}_a\theta(X) + \mathrm{Ad}_g(Y)$$

is surjective. For simplicity, in the proof below we will write $L^+\mathfrak{g} = L^+\mathfrak{g}(R)$, etc.

Applying $\mathrm{Ad}_{g^{-1}}$, we equivalently need to show that map

$$L^+\mathfrak{g} \times (L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda}L^{<0}\mathfrak{g})^{-\psi_{ht^\lambda}} \to (L\mathfrak{g})^{-\psi_{ht^\lambda}}, \quad (X, Y) \mapsto X - \mathrm{Ad}_{ht^\lambda}\theta(X) + Y$$

is surjective, i.e.

$$\tau_{ht^\lambda} : L^+\mathfrak{g} \to (L\mathfrak{g})^{-\psi_{ht^\lambda}}/(L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda}L^{<0}\mathfrak{g})^{-\psi_{ht^\lambda}}, \quad X \mapsto X - \mathrm{Ad}_{ht^\lambda}\theta(X) \tag{11}$$

is surjective. Consider subspaces

$$\begin{aligned}
A_1 &= L^+\mathfrak{g} \cap \mathrm{Ad}_{ht^\lambda}L^+\mathfrak{g},\\
A_2 &= L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{ht^\lambda}L^{<0}\mathfrak{g},\\
A_3 &= L^+\mathfrak{g} \cap \mathrm{Ad}_{ht^\lambda}L^{<0}\mathfrak{g} \oplus L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{ht^\lambda}L^+\mathfrak{g},\\
L\mathfrak{g} &= L^+\mathfrak{g} \oplus L^{<0}\mathfrak{g} = A_1 \oplus A_2 \oplus A_3.
\end{aligned}$$

Each of A_i is stable under ψ_{ht^λ}. Thus we have decompositions

$$A_i = A_i^{\psi_{ht^\lambda}} \oplus A_i^{-\psi_{ht^\lambda}},$$

which induces

$$L\mathfrak{g}^{-\psi_{ht^\lambda}} = A_1^{-\psi_{ht^\lambda}} \oplus A_2^{-\psi_{ht^\lambda}} \oplus A_3^{-\psi_{ht^\lambda}}.$$

The map (11) is equivalent to map

$$\tau' : L^+\mathfrak{g} \mapsto A_1^{-\psi_{ht^\lambda}} \oplus A_3^{-\psi_{ht^\lambda}}, \quad X \mapsto X - \psi_{ht^\lambda}(X). \tag{12}$$

Note that for any $X \in A_1^{-\psi_{ht^\lambda}} \subset L^+\mathfrak{g}$, $\tau'(\frac{1}{2}X) = X$. Thus τ' restricts to an isomorphism on $A_1^{-\psi_{ht^\lambda}}$. We are left to show that

$$\tau'' : L^+\mathfrak{g} \cap \mathrm{Ad}_{ht^\lambda} L^{<0}\mathfrak{g} \to A_3^{-\psi_{ht^\lambda}}, \quad X \mapsto X - \psi_{ht^\lambda}(X)$$

is surjective.

Denote $A = L^+\mathfrak{g} \cap \mathrm{Ad}_{ht^\lambda} L^{<0}\mathfrak{g} \subset L^+\mathfrak{g}$ and $\psi := \psi_{ht^\lambda}$. Note that $\psi(A) = L^{<0}\mathfrak{g} \cap \psi(L^+\mathfrak{g}) \subset L^{<0}\mathfrak{g}$, so that $A_3 = A \oplus \psi(A)$. For any $a + \psi(b) \in A_3$, $a, b \in A$,

$$a + \psi(b) \in A_3^{-\psi} \Leftrightarrow \psi(a + \psi(b)) = \psi(a) + b = -a - \psi(b) \Leftrightarrow b = -a.$$

Thus $a + \psi(b) = a - \psi(a) = \tau''(a)$, proving the surjectivity.

Part (iv). $S^\lambda \cap LX_\lambda = \{x_\lambda\}$ follows from $W^\lambda \cap LG_\lambda = \{t^\lambda\}$. To show L^λ is transversal to LX_λ at x_λ, we need to check

$$T_{t^\lambda}\tau(LX_\lambda) \oplus T_{t^\lambda} L_\lambda = T_{t^\lambda}(LG)^{-\psi_\lambda}.$$

Explicitly,

$$T_{t^\lambda}\tau(LX_\lambda) = \{X - \psi_\lambda(X) | X \in L^+\mathfrak{g}\}, \quad T_{t^\lambda} L^\lambda = (L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda} L^{<0}\mathfrak{g})^{-\psi_\lambda}.$$

It follows from part (iii) that the LHS spans RHS. It only remains to show the LHS is a direct sum, which follows from the decomposition

$$L\mathfrak{g} = L^+\mathfrak{g} \oplus (\mathrm{Ad}_{t^\lambda} L^+\mathfrak{g} \cap L^{<0}\mathfrak{g}) \oplus (L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda} L^{<0}\mathfrak{g}).$$

Part (v). For a L^+G-orbit $LX_\mu = L^+Gx_\mu$ and action ϕ_s^λ, for $g \in G(\mathcal{O})$, we have

$$\phi_s^\lambda(gx_\mu) = s^\lambda r_s(g) r_s(x_\mu) = (s^\lambda r_s(g) s^{-\mu}) \phi_s^\mu(x_\mu) = (s^\lambda r_s(g) s^{-\mu}) x_\mu.$$

As $s^\lambda r_s(g) s^{-\mu} \in G(\mathcal{O})$, ϕ_s^λ preserves LX_μ.

Now we show any local system $\mathcal{L}$ on LX_μ is ϕ_s^λ-equivariant. Consider $\mathbb{G}_m$-action on L^+G: $\gamma_s(g) = \mathrm{Ad}_{s^\lambda} r_s(g)$. Then the action of L^+G and $\mathbb{G}_m$ on LX_μ at x_μ can be integrated into an action of the semidirect product $L^+G \rtimes \mathbb{G}_m$ with respect to γ_s. Consider exact sequence

$$1 \to \mathrm{Stab}_{L^+G}(x_\mu) \to \mathrm{Stab}_{L^+G \rtimes \mathbb{G}_m}(x_\mu) \to \mathbb{G}_m, \quad g \mapsto (g, 1), \quad (g, s) \mapsto s.$$

Consider section $p : \mathbb{G}_m \to \mathrm{Stab}_{L^+G \rtimes \mathbb{G}_m}(x_\mu), p(s) = (s^{\mu - \lambda}, s)$. This is clearly a group homomorphism. Thus the above exact sequence is a split short exact sequence. We get semidirect product

$$\mathrm{Stab}_{L^+G \rtimes \mathbb{G}_m}(x_\mu) \simeq \mathrm{Stab}_{L^+G}(x_\mu) \rtimes \mathbb{G}_m.$$

Thus the component groups are

$$\pi_0(\mathrm{Stab}_{L^+G \rtimes \mathbb{G}_m}(x_\mu)) \simeq \pi_0(\mathrm{Stab}_{L^+G}(x_\mu) \rtimes \mathbb{G}_m) \simeq \pi_0(\mathrm{Stab}_{L^+G}(x_\mu)).$$

Therefore L^+G-equivariant local systems on LX_μ are ϕ_s^λ-equivariant for any μ. □

We also give an alternative proof of the above Proposition 47.(iii).

Lemma 48. *The multiplication map $m : L^+G \times S^\lambda \to LX$ is formally smooth.*

Proof. As before, it sufficies to check the surjectivity of the differential map. From standard results on affine Grassmanian slices, we know that the multiplication map

$$m : L^+G \times W^\lambda \times L^+G \to LG, \quad (g_1, x, g_2) \mapsto g_1 x g_2$$

is a submersion.

Assume $\lambda \in \Lambda_S^+$. Denote $\sigma = \mathrm{inv} \circ \theta$, which is an involution of space on LG. Also denote by $\gamma(g_1, x, g_2) = (\sigma(g_2), \sigma(x), \sigma(g_1)$, which is an involution on $L^+G \times W^\lambda \times L^+G$. Its fixed subspace is

$$\{(g_1, x, \sigma(g_1)) \mid g_1 \in L^+G, x \in (W^\lambda)^\sigma\} \simeq L^+G \times S^\lambda.$$

Observe that m intertwines γ and σ. The induced map on fixed subspaces is isomorphic to $m : L^+G \times S^\lambda \to LX$. Now for any $a = (g, x) \in L^+G \times S^\lambda \hookrightarrow L^+G \times W^\lambda \times L^+G$, the surjective differential map

$$\mathrm{d}m : T_a(L^+G \times W^\lambda \times L^+G) \to T_{m(a)}LG$$

intertwines γ and σ. Since both γ and σ are involutions, the induced map on fixed subspaces, which is $\mathrm{d}m : T_a(L^+G \times S^\lambda) \to T_{m(a)}LX$, is also surjective as desired. □

Corollary 49. *Let $\lambda, \mu \in \Lambda_S^+$.*

(i) *The multiplication map $m : L^+G \times S^\lambda_{\leq \mu} \to \overline{LX}_\mu$ is formally smooth.*
(ii) *There is a $\mathbb{G}_m$-action on $S^\lambda_{\leq \mu}$, which fixes x_λ, and contracts $S^\lambda_{\leq \mu}$ to x_λ.*
(iii) *$S^\lambda_\mu \neq \emptyset$ if and only if $\lambda \leq \mu$.*

Proof. The map in (i) is the base-change of the formally smooth map $L^+G \times S^\lambda \to LX$ in Proposition 47 (iii) along the embedding $\overline{LX}_\mu$ and hence is formally smooth. Part (ii) follows from Proposition 47 (ii).

Part (iii). Assume $S^\lambda_\mu = S^\lambda \cap LX_\mu \neq \varnothing$. Recall that we have $\mathbb{G}_m$-action ϕ_s(depended on λ) on S^λ. This action also acts on all the L^+G-orbits LX_μ. The contraction of $S^\lambda \cap LX_\mu$, which gives x_λ, must be contained in $\overline{LX}_\mu$. Thus $\lambda \leq \mu$.

Conversely, by Theorem 18 we have $x_\lambda \in S^\lambda_{\leq \mu} = S^\lambda \cap \overline{LX}_\mu \neq \emptyset$. Since $m : L^+G \times (S^\lambda_{\leq \mu} \cap \overline{LX}_\mu) \to \overline{LX}_\mu$ is formally smooth and $L^+G \times S^\lambda_{\leq \mu}$ admits a placid presentation $L^+G \times S^\lambda_{\leq \mu} \cong \lim_i((L^+G)_i \times S^\lambda_{\leq \mu})$ with surjctive transition maps, Lemma 100 and Lemma 101 imply that the image of m has nonempty intersection with the open dense subset LX_μ. It follows that $m^{-1}(LX_\mu) = L^+G \times S^\lambda_\mu \neq \emptyset$. Part (iii) follows. □

The corollary implies that $S^\lambda_{\leq \mu}$ defines a contracting slice for LX_λ inside $\overline{LX}_\mu$, see Definition 112).

Remark 50. Unlike the case of affine Grassmannian slices $W^\lambda_{\leq \mu}$ and W^λ_μ, the slices $S^\lambda_{\leq \mu}$ might be reducible and non-normal, and S^λ_μ might be disconnected (see Example 52).

6.2. **Lusztig embedding.** Let $G = \mathrm{GL}_n$. Consider the classical groups $K = \mathrm{O}_n$ and Sp_n(when n is even) of orthogonal group and symplectic group corresponding to the involution $\theta(g) = (g^t)^{-1}$ and $\theta(g) = J^{-1}(g^t)^{-1}J$ respectively. Here $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ with $n = 2m$. We have $\mathfrak{g} = \mathfrak{gl}_n$, $d\theta(g) = -g^t$ (resp. $d\theta(g) = -J^{-1}g^tJ$), $\mathfrak{k} = \mathfrak{g}^{d\theta} = \mathfrak{o}_n$ (resp. $\mathfrak{k} = \mathfrak{sp}_n$), $\mathfrak{p} = \mathfrak{g}^{-d\theta} = \{g \in \mathfrak{gl}_n \mid g^t = g\}$ (resp. $\mathfrak{p} = \{g \in \mathfrak{gl}_n \mid (gJ)^t = -gJ\}$).

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone and $\mathcal{N}_\mathfrak{p} = \mathcal{N} \cap \mathfrak{p}$ be the corresponding symmetric nilpotent cone. The G-orbits on $\mathcal{N}$, called the nilpotent orbits, are parametrized by partitions $\mathcal{P}_n = \{(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 | \sum \lambda_i = n\}$. Let $K^\circ \subset K$ be the identity component of K. The K°-orbits on $\mathcal{N}_\mathfrak{p}$, called the symmetric nilpotent orbits, are parametrized by $\mathcal{P}_n$ in the case of $K^\circ = \mathrm{SO}_n$ and $\mathcal{P}_m$ in the case of $K^\circ = K = \mathrm{Sp}_{2m}$. We will write $\mathcal{O}_\lambda$ and $\mathcal{O}_{\mathfrak{p},\lambda}$ for the corresponding nilpotent orbits and symmetric nilpotent orbits and $\overline{\mathcal{O}}_\lambda$ and $\overline{\mathcal{O}}_{\mathfrak{p},\lambda}$ the orbit closures.

We will identify the set Λ_T^+ with $\Lambda_T^+ = \{(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)\}$ so that

$$LG_\lambda = L^+G \,\mathrm{diag}(t^{\lambda_1 - 1}, ..., t^{\lambda_n - 1}) L^+G$$

for $\lambda \in \Lambda_T^+$. Recall the Lusztig embedding

$$\iota : \mathcal{N} \hookrightarrow W^0 = L^{<0}G, \quad x \mapsto (I_n - t^{-1}x). \tag{13}$$

It induces an isomorphism

$$\mathcal{N} \cong W^0_{\leq \omega_n} = L^{<0}G \cap \overline{LG}_{\omega_n}$$

where $\omega_n = (n \geq 0 \geq \cdots \geq 0)$. In addition, for $\lambda \in \mathcal{P}_n$, the above isomorphism restricts to isomorphisms

$$\mathcal{O}_\lambda \cong W^0_\lambda, \qquad \overline{\mathcal{O}}_\lambda \cong W^0_{\leq \lambda}.$$

Note that the Lusztig embedding intertwines the involution $-d\theta$ on $\mathcal{N}$ with $\mathrm{inv} \circ \theta$ on W^0 and hence induces isomorphisms

$$\mathcal{N}_\mathfrak{p} \cong (W^0_{\leq \omega_n})^{\mathrm{inv} \circ \theta} \cong S^0_{\leq \omega_n},$$
$$\mathcal{O}_{\mathfrak{p},\lambda} \cong (W^0_\lambda)^{\mathrm{inv} \circ \theta} \cong S^0_\lambda, \qquad \overline{\mathcal{O}}_{\mathfrak{p},\lambda} \cong (W^0_{\leq \lambda})^{\mathrm{inv} \circ \theta} \cong S^0_{\leq \lambda}$$

on the fixed points. In view of Proposition 47, we obtain

Proposition 51. *There is a $\mathbb{G}_m$-equivariant isomorphism $\overline{\mathcal{O}}_{\mathfrak{p},\lambda} \cong S^0_{\leq \lambda}$ between symmetric nilpotent orbit closures and transversal slices in loop spaces of classical symmetric varieties. For any $\lambda \leq \mu \leq \omega_n \in \mathcal{P}_n$, the variety $S^\lambda_{\leq \mu}$ defines a Slodowy slice to the symmetric nilpotent orbit $\mathcal{O}_{\mathfrak{p},\lambda}$ inside $\overline{\mathcal{O}}_{\mathfrak{p},\mu}$.*

Example 52. For $X = \mathrm{GL}_2/\mathrm{O}_2$, the slice $S^0_{\leq \omega_2}$ is isomorphic to $S^0_{\leq \omega_2} \cong \mathcal{N}_\mathfrak{p} \cong \{xy = 0\} \subset k^2$, which is reducible and non-normal.

6.3. **Slices for Iwahori orbits.**

6.3.1. *Involutions on affine flag slices.* Let $T_0 \subset B_0 \subset G$ and $I_0 = B_0(L^{++}G)$ be θ-stable maximal torus, Borel subgroup, and Iwahori of LG as before. Denote the opposite Iwahori by $I_0^- := B_0^- L^{<0}G$, where B_0^- is the Borel opposite to B_0. Let $I_0^{--} = U_0^- L^{<0}G$, U_0^- is the unipotent radical of B_0^-. For any $w \in \widetilde{\mathrm{W}} = N_{LG}(LT_0)/L^+T_0$ we denote by $\mathcal{F}\ell_w \subset \mathcal{F}\ell = LG/I_0$ the corresponding I_0-orbits on the affine flag variety and $\overline{\mathcal{F}\ell}_w$ its closure. We denote by $LG_w, \overline{LG}_w \subset LG$ the pre-images of $\mathcal{F}\ell, \overline{\mathcal{F}\ell}_w$ in LG.

Let $n \in N_{LG}(LT_0)$ be a lifting of w. The affine flag slice of LG_w in LG at n is defined as

$$W^n = (I_0^{--} \cap \mathrm{Ad}_n\, I_0^{--})n \subset LG.$$

For any pair $n, n' \in N_{LG}(LT_0)$ with images $w, w' \in \widetilde{W}$, we define $W^n_{n'} = W^n \cap LG_{w'}$ and $W^n_{\leq n'} = W^n \cap \overline{LG}_{w'}$. We have the following well-known facts:

Lemma 53. (i) *W^n is an ind-scheme of ind-finite type and formally smooth.*

(ii) $W^n_{n'}$ *is non-empty if and only if* $w \leq w'$.
(iii) $W^n_{n'}$ *is smooth and connected (if non-empty).*
(iv) $W^n_{\leq n'}$ *is of finite type, affine, and normal.*
(v) $\dim W^n_{n'} = \dim W^n_{\leq n'} = l(w') - l(w)$.

Proof. This follows from [Fal03, Theorem 7 and 8] and [KT95, Proposition 1.3.1 and 1.3.2]. □

Since $\theta(T_0) = T_0$ the normalizer $N_{LG}(LT_0)$ is stable under the involution $\mathrm{inv} \circ \theta$ of LG.

Lemma 54. *Let* $n, n' \in (N_{LG}(LT_0))^{\mathrm{inv}\circ\theta}$. *Then* W^n, $W^n_{n'}$, *and* $W^n_{\leq n'}$ *are stable under the involution* $\mathrm{inv} \circ \theta : LG \to LG$. *Moreover, the identification* $W^n \cong I_0^{--} \cap \mathrm{Ad}_n I_0^{--}$, *sending* $\gamma \cdot n \to \gamma$, *intertwines the involution* $\mathrm{inv} \circ \theta|_{W^\lambda}$ *with the involution* $\mathrm{inv} \circ \psi_n$ *on* $I_0^{--} \cap \mathrm{Ad}_n I_0^{--}$, *where* $\psi_n = \mathrm{Ad}_n \circ \theta : LG \to LG$.

Proof. A direct computation shows that that $LG_{w'}$ and $\overline{LG}_{w'}$ are stable under $\mathrm{inv} \circ \theta$ and $I_0^{--} \cap \mathrm{Ad}_n I_0^{--}$ is stable under $\mathrm{inv} \circ \psi_n$. Thus for any $\gamma n \in W^n = (I_0^{--} \cap \mathrm{Ad}_n I_0^{--})n$ (resp. $W^n_{n'}$, $W^n_{\leq n'}$), we have $\mathrm{inv} \circ \theta(\gamma n) = n\theta(\gamma)^{-1} = \mathrm{inv} \circ \psi_n(\gamma) n \in W^\lambda$ (resp. W^λ_μ, $W^\lambda_{\leq \mu}$). The lemme follows.

□

Assume further that $n \in N_{LG}(LT_0)^{\mathrm{inv}\circ\theta}$ has the form $n = \bar{n}t^\lambda$ where $\bar{n} \in N_G(T_0)$ and $\lambda \in X_*(T_0)$ satisfy $\theta(\bar{n}) = \bar{n}^{-1}$, $\mathrm{Ad}_{\bar{n}} t^{-\lambda} = t^{\theta(\lambda)}$. Let $\psi_{\bar{n}} = \mathrm{Ad}_{\bar{n}} \circ \theta : T_0 \to T_0$. Choose a cocharacter $\mu \in X_*(T_0^{\psi_{\bar{n}}})$ as in [MS98, §6.4], so that $\langle \mu, \alpha \rangle < 0$ for all positive roots $\alpha > 0$ with $\psi_{\bar{n}} \alpha > 0$. Let $m = \max\{1, \langle \mu, \alpha \rangle + 1, \forall \alpha \in \Phi\}$ where Φ is the set of roots of G. Consider the following map on LG:

$$\phi_s : \gamma(t) \mapsto s^{\mu - m\theta(\lambda)} \gamma(s^{-2m} t) s^{-\theta(\mu) + m\lambda}. \tag{14}$$

Lemma 55. *Let* $n, n' \in N_{LG}(LT_0)^{\mathrm{inv}\circ\theta}$ *and assume* $n = \bar{n}t^\lambda$ *as above.*

(i) *The two maps* ϕ_s *and* $\mathrm{inv} \circ \theta$ *commute with each other.*
(ii) ϕ_s *restricts to a* $\mathbb{G}_m$*-action* ρ_s *on* $W^n, W^n_{n'}, W^n_{\leq n'}$. *The* $\mathbb{G}_m$*-action* ρ_s *on* W^n *and* $W^n_{\leq n'}$ *is contrating with unique fixed point* n.

Proof. Part (i) is a direct computation. Proof of (ii). First we check that

$$\phi_s(n) = s^{\mu - m\theta(\lambda)} \bar{n} t^\lambda s^{-2m\lambda} s^{-\theta(\mu) + m\lambda} = \bar{n} t^\lambda s^{\mathrm{Ad}_{\bar{n}^{-1}} \mu - m \mathrm{Ad}_{\bar{n}^{-1}} \theta(\lambda) - 2m\lambda - \theta(\mu) + m\lambda} = n.$$

Here we used that on $X_*(T_0^{\psi_{\bar{n}}}) \subset X_*(T_0)$, $\mathrm{Ad}_{\bar{n}^{-1}}$ acts as θ. Next we show that ϕ_s acts on W^n and contracts it to n. Write $\gamma n \in W^n$. Then

$$\phi_s(\gamma n) = (\mathrm{Ad}_{s^{\mu - m\theta(\lambda)}} \gamma(s^{-2m} t)) n.$$

It suffices to show that the action $\mathrm{Ad}_{s^{\mu - m\theta(\lambda)}} \circ r_{s^m}$ acts on $I_0^{--} \cap \psi_n I_0^{--}$ and contracts it to 1, where $r_s : \gamma(t) \mapsto \gamma(s^{-2} t)$ is the rotation on LG. Since $I_0^{--} \cap \psi_v I_0^{--}$ is stable under the adjoint action of T_0 and ϕ_s contracts elements in $L^{--}T_0$ to 1, it suffices to show the weight of s in each affine root subgroup contained in $I_0^{--} \cap \psi_n I_0^{--}$ is positive. To this end, we compute the weights of Ad_{s^μ} and $\mathrm{Ad}_{s^{-\theta(\lambda)}} \circ r_{s^2}$ separately.

Claim: The weight of s with respect to the action $\mathrm{Ad}_{s^{-\theta(\lambda)}} \circ r_s$ on any affine root subgroup $U_{\tilde{\alpha}} \subset I_0^{--} \cap \psi_n I_0^{--}$ is positive, unless when $\tilde{\alpha} = \alpha \in \Phi$ is a negative finite root such that $\psi_{\bar{v}}(\alpha) < 0$, where the weight is still non-negative.

The positivity of weights of $\mathrm{Ad}_{s^{\mu-m\theta(\lambda)}} \circ r_{s^m}$ then follows immediately from the claim and the definitions of m and μ.

To see the claim, let $\tilde{\alpha} = \alpha + m_\alpha$ be an affine root, where $\alpha \in \Phi$ is the finite part, m_α is the imaginary part. The weight of $\mathrm{Ad}_{s^{-\theta(\lambda)}} \circ r_s$ on $U_{\tilde{\alpha}}$ is

$$\langle -\theta(\lambda), \alpha \rangle - 2m_\alpha = -\langle \lambda, \theta(\alpha) \rangle - 2m_\alpha.$$

When $U_{\tilde{\alpha}} \subset I_0^{--} \cap \psi_n I_0^{--}$, $\tilde{\alpha}$ satisfies $\tilde{\alpha} = \alpha + m_\alpha \in \Phi_{\mathrm{aff}}(I_0^{--})$ and

$$\psi_n(\tilde{\alpha}) = \psi_{\bar{n}}(\alpha) + (m_\alpha + \langle \lambda, \theta(\alpha) \rangle) \in \Phi_{\mathrm{aff}}(I_0^{--}),$$

from which we easily deduce that the weight $-\langle \lambda, \theta(\alpha) \rangle - 2m_\alpha$ is as in the claim.

We can conclude the proof by noticing that the orbit $LG_{w'}$ is stable under ϕ_s. □

We preserve the set ups in Lemma 55. Consider the fixed points

$$L^n = (W^n)^{\mathrm{inv} \circ \theta}, \quad L^n_{n'} = (W^n_{n'})^{\mathrm{inv} \circ \theta}, \quad L^n_{\leq n'} = (W^n_{\leq n'})^{\mathrm{inv} \circ \theta}. \tag{15}$$

Lemma 43 (2) implies there is an isomorphism

$$L^n \cong (I_0^{--} \cap \mathrm{Ad}_n I_0^{--})^{\mathrm{inv} \circ \psi_n}. \tag{16}$$

Using Lemma 55, the same proof as in the spherical case (see Lemma 45) shows:

Lemma 56. (i) *L^n is a connected ind-scheme of ind-finite type and formally smooth.*
(ii) *$L^n_{n'}$ is smooth.*
(iii) *$L^n_{\leq n'}$ is of finite type, affine, and connected.*
(iv) *The $\mathbb{G}_m$-action ρ_s on W^n restricts to a $\mathbb{G}_m$-action on $L^\lambda, L^n_\mu, L^n_{\leq n'}$. The $\mathbb{G}_m$-action ρ_s on L^n and $L^n_{\leq n'}$ is contrating with unique fixed point n.*

6.3.2. *The contracting slices S^v and $S^v_{\leq u}$.* Denote by $\mathcal{O}_v$ the I_0-orbit on LX parametrized by $v \in \mathscr{V} = L^+T_0 \backslash \mathcal{V} / LK$ in Lemma 23. According to Lemma 25, we can find a lifting $\dot{v} \in \mathcal{V}$ of v such that $x_v = \dot{v}LK \in \mathcal{O}_v$ and $n_v = \tau(x_v) = \tau(\dot{v}) = \bar{n}_v t^\lambda$ where $\bar{n}_v \in N_G(T_0)$ and $\lambda \in X_*(T_0)$ satisfy $\theta(\bar{n}_v) = \bar{n}_v^{-1}$, $\mathrm{Ad}_{\bar{n}_v} t^{-\lambda} = t^{\theta(\lambda)}$.

Thus we are in the setting of Lemma 54 and Lemma 55 and we denote by $\psi_v = \mathrm{Ad}_{n_v} \circ \theta$ and $\psi_{\bar{v}} = \mathrm{Ad}_{\bar{n}_v} \circ \theta$ the involutions in *loc. cit.*.

Definition 57. (i) *For any $v \in \mathscr{V} = L^+T_0 \backslash \mathcal{V} / LK$ and a lifting $\dot{v} \in \mathcal{V}$ as above, we define $S^v \subset LX$ as the following fiber product*

$$\begin{array}{ccc} S^v & \longrightarrow & LX \\ \downarrow & & \downarrow \tau \\ W^v & \longrightarrow & LG. \end{array}$$

Here $W^v = W^{n_v} = (I_0^{--} \cap \mathrm{Ad}_{n_v} I_0^{--}) n_v$ is the slice in Lemma 53.
(ii) *For any $v, u \in L^+T_0 \backslash \mathcal{V} / LK$, let $S^v_u = S^v \cap \mathcal{O}_u$ and $S^v_{\leq u} = S^v \cap \overline{\mathcal{O}}_u$.*

It follows from Lemma 54 that there is an isomorphism

$$S^v \cong (I_0^{--} \cap \mathrm{Ad}_n I_0^{--})^{\mathrm{inv} \circ \psi_v}.$$

The following proposition implies that S^v defines a transversal contracting slice to the orbit $\mathcal{O}_v$ inside LX at x_v, parallel to Proposition 47 in the spherical case:

Proposition 58. (i) *The embedding τ induces an isomorphism $S^v \cong L^{n_v}$. In particular, S^v is a connected ind-scheme of ind-finite type and formally smooth.*

(ii) *Choose a cocharacter $\mu \in X_*(T_0^{\psi_{\bar{v}}})$ so that $\langle \mu, \alpha \rangle < 0$ for all positive root $\alpha > 0$ with $\psi_{\bar{v}}\alpha > 0$. Let $m = \max\{1, \langle \mu, \alpha \rangle + 1, \forall \alpha \in \Phi\}$ where Φ is the set of roots of G. Then $\phi_s^v : \gamma(t) \mapsto s^{\mu - m\theta(\lambda)}\gamma(s^{-2m}t)$ defines a $\mathbb{G}_m$-action on S^v, which fixes x_v, and contracts S^v to x_v.*

(iii) *The multiplication map $m : I_0 \times S^v \to LX$ is formally smooth.*

(iv) *We have $S^v \cap \mathcal{O}_v = \{x_v\}$ and $S^v \subset LX$ is transversal to $\mathcal{O}_v$ at x_v.*

(v) *All the $\mathbb{G}_m$-actions ϕ_s^v preserve all the I_0-orbits $\mathcal{O}_u$. Any I_0-equivariant local system on $\mathcal{O}_u$ is $\mathbb{G}_m$-equivariant with respect to the action ϕ_s^v.*

Proof. In view of Lemma 55 and 56, the same arguments as in Proposition 47 in the spherical case apply to the Iwahori case, replacing $L^+\mathfrak{g}, L^{<0}\mathfrak{g}$ with $\mathrm{Lie}(I_0), \mathrm{Lie}(I_0^{--})$. Part (iii) can also be deduced from the well-known formally smoothness of the multiplication map

$$m : I_0 \times W^n \times I_0 \to LG$$

by taking $(\mathrm{inv} \circ \theta)$-fixed subspaces as in Lemma 48. □

Corollary 59. *Let $v, u \in \mathscr{V}$.*

(i) *The multiplication map $m : I_0 \times S^v_{\leq u} \to \mathcal{O}_u$ is formally smooth.*

(ii) *There is a $\mathbb{G}_m$-action on $S^v_{\leq u}$, which fixes x_v, and contracts $S^v_{\leq u}$ to x_v.*

(iii) *$S^v_u \neq \emptyset$ if and only if $v \leq u$.*

Proof. Same proof as in the spherical case Corollary 49 using Proposition 58. □

6.4. **Mars-Springer slices.** Note that the evaluation map $ev : L^+X \to X$ induces a a bijection between I_0-orbits on L^+X and B_0-orbits on X. Let $\mathcal{O}_u \subset L^+X$ be the unique open I_0-orbit such that $\overline{\mathcal{O}}_u = L^+X$. For any I_0-orbit $\mathcal{O}_v \subset L^+X$ let $\mathcal{O}_{v,X} \subset X$ be the corresponding B_0-orbit. We have $\tau(x_v) = n_v = \bar{n}_v = \tau(\bar{x}_v) \in N_G(T_0)^{\mathrm{inv} \circ \theta}$ and let

$$S_{\bar{x}_v} := (U_0^- \cap \mathrm{Ad}_{\bar{n}_v} U_0^-)\bar{x}_v \subset X$$

be the Mars-Springer slice of $\mathcal{O}_{v,X}$ at $\bar{x}_v := ev(x_v) \in \mathcal{O}_{v,X}$ introduced in [MS98, Section 6.4].

Lemma 60. *The evaluation map $ev : L^+X \to X$ induces an isomorphism $S^v_{\leq u} \cong S_{\bar{x}_v}$*

Proof. It suffices to show that the evaluation map $ev : L^+G \to G$ indcues an isomorphism $\tau(S^v_{\leq u}) \cong \tau(S_{\bar{x}_v})$ where $\tau : X \to G$ is the embedding. We have $ev : I_0^{--} \cap L^+G \cong U_0^-$ and hence isomorphisms

$$\tau(S^v_{\leq u}) \cong \tau(S^v \cap L^+X) \cong (I_0^{--} \cap \mathrm{Ad}_{\bar{n}_v} I_0^{--})\bar{n}_v)^{\mathrm{inv} \circ \theta} \cap L^+G \overset{ev}{\cong} (U_0^- \cap \mathrm{Ad}_{\bar{n}_v} U_0^-)\bar{n}_v)^{\mathrm{inv} \circ \theta} \cong \tau(S_{\bar{x}_v})$$

where the last isomorphism follows from the fact that $(U_0^- \cap \mathrm{Ad}_{\bar{n}_v} U_0^-)\bar{n}_v)^{\mathrm{inv} \circ \theta} \cong (U_0^- \cap \mathrm{Ad}_{\bar{n}_v} U_0^-)^{\mathrm{inv} \circ \psi_{\bar{v}}}$ is a single $(U_0^- \cap \mathrm{Ad}_{\bar{n}_v} U_0^-)$-orbit under the twsited conjugation action $u \cdot v = uv\psi_{\bar{v}}(u)^{-1}$ (see, e.g., [Ric82, Proposition 9.1]). The lemma follows.

□

7. Lagrangians

We show that the slices $S^0_{\leq\lambda}$ are Lagrangians inside the affine Grassmannian slices $W^0_{\leq\lambda}$. Along the way, we prove a combinatorial formula for the codimensions of spherical orbits. The proof uses real groups and quasi-maps [CN23] and a spreading out argument. The codimension formula obtained in this section provides a simple way to check the semisimplicity criterion of relative Satake category in Section 10.1.

7.1. **Quasi-maps.** Consider the stack $Z = LG^+\backslash LG/K[t^{-1}]$ of quasi-maps classifying a G-bundle $\mathcal{E}$ on $\mathbb{P}^1$ and a section $\phi : \mathbb{P}^1 - \{0\} \to \mathcal{E} \times^G X$, i.e., a K-reduction $\mathcal{E}_K$ of $\mathcal{E}|_{\mathbb{P}^1-0}$ (here t is the local coordinate at 0). We have a natural map

$$f : Z = L^+G\backslash LG/K[t^{-1}] \to L^+G\backslash LX \tag{17}$$

between ind-stacks. For any $\lambda \in \Lambda^+_S$ we write

$$Z_\lambda = (f)^{-1}(L^+G\backslash LX_\lambda), \quad \overline{Z}_\lambda = (f)^{-1}(L^+G\backslash \overline{LX}_\lambda) \tag{18}$$

for the pre-images of the corresponding orbits and orbit closures and

$$f_\lambda := f|_{\overline{Z}_\lambda} : \overline{Z}_\lambda \to L^+G\backslash \overline{LX}_\lambda$$

for the induced map.

Lemma 61. *Z is an ind-stack ind-locally of finite type and Z_λ, $\overline{Z}_\lambda$ are stacks locally of finite type.*

Proof. This is [GN10, Section 3.4] and [BFT25, Lemma 3.5.2]. □

Recall the notion of placid morphisms in A.3.

Lemma 62. *The morphisim $f_\lambda : \overline{Z}_\lambda \to L^+G\backslash \overline{LX_\lambda}$ is placid.*

Proof. Consier the following Cartesian diagrams

$$\begin{array}{ccccc}
\overline{Z}_\lambda & \xleftarrow{\pi} & \overline{Z}_\lambda \times_{L^+G\backslash\overline{LX_\lambda}} \overline{LX_\lambda} & \xrightarrow{i} & LG/K[t^{-1}] \\
\downarrow{\scriptstyle f_\lambda} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
L^+G\backslash\overline{LX_\lambda} & \xleftarrow{\pi} & \overline{LX}_\lambda & \xrightarrow{i} & LX
\end{array}$$

where π are pro-smooth coverings, i are the natural inclusions, and b is the natural map

$$b : LG/K[t^{-1}] \to LG/LK \to LX.$$

Note that $\overline{Z}_\lambda \times_{L^+G\backslash\overline{LX_\lambda}} \overline{LX_\lambda}$ (resp. $LG/K[t^{-1}]$) is a L^+G-torsor over $\overline{Z}_\lambda$ (resp. Z) and the finiteness results in Lemma 61 implies it is a placid scheme (resp. a placid ind-scheme). Since π is a pro-smooth covering, to show the placidness of f_λ, it suffices to show that the map a is placid. From Lemma 100, it suffices to show that a is formally smooth. Since a is the base change of b, we reduce to show that b is formally smooth. Since $K[t^{-1}]$ and LG are formally smooth, it follows that $LG/K[t^{-1}]$ is a formally smooth placid ind-scheme and the formally smoothness of b follows from Lemma 96 and the fact that the tangent morphism $db : T_{LG/K[t^{-1}]} \to b^*T_{LX}$ is surjective. □

Proposition 63. *For any $\lambda \leq \mu \in \Lambda_S^+$, the codimension function* $\mathrm{codim}_{\overline{LX}_\lambda/\overline{LX}_\mu} : \overline{LX}_\lambda \to \mathbb{Z}$ *is constant and is equal to* $\dim(Z_\mu) - \dim(Z_\lambda)$.

Proof. Consider the Cartesian diagram

$$\begin{array}{ccc} \overline{Z}_\lambda & \longrightarrow & \overline{Z}_\mu \\ \downarrow{\scriptstyle f_\lambda} & & \downarrow{\scriptstyle f_\mu} \\ L^+G\backslash\overline{LX}_\lambda & \longrightarrow & L^+G\backslash\overline{LX}_\mu \end{array} .$$

By Lemma 62 the vertical map f_μ is placid and A.4 (7) implies

$$\dim_{\overline{Z}_\lambda/\overline{Z}_\mu} = f_\lambda^* \dim_{(L^+G\backslash\overline{LX}_\lambda)/(L^+G\backslash\overline{LX}_\mu)} . \tag{19}$$

Since $\overline{LX}_\lambda, \overline{Z}_\lambda, \lambda \in \Lambda_S^+$ are equidimensional placid, A.4 (6) implies that both dimension fucntions $\dim_{\overline{LX}_\lambda/\overline{LX}_\mu}$ and $\dim_{(L^+G\backslash\overline{LX}_\lambda)/(L^+G\backslash\overline{LX}_\mu)}$ are constant functions with the same value and (19) implies the common value is given by

$$\dim_{\overline{LX}_\lambda/\overline{LX}_\mu} = \dim_{(L^+G\backslash\overline{LX}_\lambda)/(L^+G\backslash\overline{LX}_\mu)} = \dim_{\overline{Z}_\lambda/\overline{Z}_\mu} = \dim(Z_\lambda) - \dim(Z_\mu).$$

Since $\mathrm{codim}_{\overline{LX}_\lambda/\overline{LX}_\mu} = -\dim_{\overline{LX}_\lambda/\overline{LX}_\mu}$, the porposition follows. □

7.2. **Codimension for spherical orbits.** In this section we assume LX is connected (e.g., when G is a simply connected semisimple group).

Proposition 64. $\mathrm{codim}_{\overline{LX}_\lambda/\overline{LX}_\mu} : \overline{LX}_\lambda \to \mathbb{Z}$ *is constant with value equal to* $\langle \rho, \mu - \lambda \rangle$.

Proof. Since $\pi_0(LX) \cong \Lambda_S/\Lambda_S \cap Q = 0$ by assumption, Theorem 18 implies $LX_0 \subset \overline{LX}_\lambda$ for all $\lambda \in \Lambda_S^+$. We claim that $\mathrm{codim}_{\overline{LX}_0/\overline{LX}_\mu} : \overline{LX}_0 \to \mathbb{Z}$ is equal to $\langle \rho, \mu \rangle$. Then $\mathrm{codim}_{\overline{LX}_\lambda/\overline{LX}_\mu} = \mathrm{codim}_{\overline{LX}_0/\overline{LX}_\mu} - \mathrm{codim}_{\overline{LX}_0/\overline{LX}_\lambda}$ is the constant function with value $\langle \rho, \mu - \lambda \rangle$.

Proof of the claim. By Proposition (63), we need to show that $\dim(Z_\mu) - \dim(Z_0) = \langle \rho, \mu \rangle$. We have $Z_0 = \overline{Z}_0 \cong L^+K\backslash LK/K[t^{-1}] \cong \mathrm{Bun}_K(\mathbb{P}^1)$ and hence $\dim(Z_0) = -\dim(K)$. On the other hand, the open orbit $L^{<0}G \subset L^+G\backslash LG$ gives rise to an open dense embedding $L^{<0}X/K \cong (L^{<0}G/L^{<0}K)/K \subset L^+G\backslash LG/K[t^{-1}]$ where K acts on $L^{<0}X$ be the conjugation action. Note that we have the following Cartesian diagrams

$$\begin{array}{ccccccc} (L^{<0}X \cap \overline{LX}_\mu)/K & \xrightarrow{j} & \overline{Z}_\mu & \longrightarrow & L^+G\backslash\overline{LX}_\mu & \xrightarrow{\tau} & \overline{LG}_\mu/\operatorname{Ad}_\theta(L^+G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L^{<0}X/K & \xrightarrow{j} & L^+G\backslash LG/K[t^{-1}] & \longrightarrow & L^+G\backslash LX & \xrightarrow{\tau} & LG/\operatorname{Ad}_\theta(L^+G) \end{array}$$

where the vertical maps are natural inclusions, the maps j are open dense embeddings, and $LG/\operatorname{Ad}_\theta(L^+G)$ is the quotient of LG by the θ-twisted conjugation action. Since $L^{<0}X \cap \overline{LX}_\mu = L^0_{\leq\mu}$ and $(L^{<0}X \cap \overline{LX}_\mu)/K \subset \overline{Z}_\mu$ is open dense, Lemma 65 below implies

$$\dim(Z_\mu) = \dim((L^{<0}X \cap \overline{LX}_\mu)/K) = \dim(L^0_\mu/K) = \langle \rho, \mu \rangle - \dim(K).$$

All together, we obtain

$$\dim(Z_\mu) - \dim(Z_0) = (\langle \rho, \mu \rangle - \dim(K)) - (-\dim(K)) = \langle \rho, \mu \rangle.$$

The claim follows. □

Lemma 65. *We have* $\dim(L^0_\mu) = \langle \rho, \mu \rangle$.

Proof. Pick a mixed characteristic DVR A between $\mathbb{Z}[1/2]$ and $\mathbb{C}$ with residue field k and a smooth model G_A of G over A. According to the Isomorphism Theorem in [Con14, Theorem 6.1.17], one can find an involution $\theta_A : G_A \to G_A$ such that $\theta_A \otimes k = s\theta s^{-1}$ for some $s \in T$.

Note that $W^0_\mu = L^{<0}G \cap LG_\mu$ admits a smooth model $W^0_{\mu,A} = L^{<0}G_A \cap LG_{\mu,A}$ over A and, by [Gro70, Lemma 9.4], the fixed points $L^0_{\mu,A} = (W^0_{\mu,A})^{\mathrm{inv}\circ\theta_A}$ is smooth over A with special fiber $(L^0_{\mu,A}) \otimes_A k \cong L^0_\mu$. Thus we have $\dim(L^0_\mu) = \dim((L^0_{\mu,A}) \otimes_A k) = \dim((L^0_{\mu,A}) \otimes_A \mathbb{C})$ and we reduce to show $\dim((L^0_{\mu,A}) \otimes_A \mathbb{C}) = \langle \rho, \mu \rangle$.

Let $G_\mathbb{R}$ be the real form associated to the involution $\theta_\mathbb{C} = \theta_A \otimes \mathbb{C}$ on the complex reductive group $G_\mathbb{C} = G_A \otimes \mathbb{C}$. Let $\mathrm{Gr}_\mathbb{R} = LG_\mathbb{R}/L^+G_\mathbb{R}$ be the real affine Grassmannian for $G_\mathbb{R}$. For any real dominant weight $\mu \in \Lambda^+_S$, we denote by $\mathrm{Gr}_{\mathbb{R},\mu} = LG_\mathbb{R} \cdot [t^\mu]$ the corresponding real spherical orbit and its intersection $\mathrm{Gr}^0_{\mathbb{R},\mu} := \mathrm{Gr}_{\mathbb{R},\mu} \cap L^{<0}G_\mathbb{R} \cdot [t^0]$ with the open $L^{<0}G_\mathbb{R}$-orbit $L^{<0}G_\mathbb{R} \cdot [t^0] \subset \mathrm{Gr}_\mathbb{R}$, see [Nad05, Section 3.6]. According to [CN23, Theorem 6.9], there is a real analytic isomorphism

$$(L^0_{\mu,A}) \otimes \mathbb{C} \cong (L^{<0}G_\mathbb{C} \cap LG_{\mu,\mathbb{C}})^{\mathrm{inv}\circ\theta_\mathbb{C}} \cong \mathrm{Gr}^0_{\mathbb{R},\mu}$$

and hence by [Nad05, Proposition 3.6.1] we have $\dim((L^0_{\mu,A}) \otimes \mathbb{C}) = \frac{1}{2}\dim_\mathbb{R}(\mathrm{Gr}^0_{\mathbb{R},\mu}) = \langle \rho, \mu \rangle$. The lemma follows.

□

7.3. **Lagrangian slices.** Recall the affine Grassmannian slice $W^\lambda_{\leq\mu}$ has a natural Poisson structure and W^λ_ν for $\lambda \leq \nu \leq \mu$ are symplectic leaves of dimension $\dim W^\lambda_\nu = 2\langle \rho, \nu - \lambda \rangle$ (see, e.g., [KWWY11]). As an application of the co-dimension formula, we show that, when $\lambda = 0$, the fixed points subvariety $L^0_\mu = (W^0_\mu)^{\mathrm{inv}\circ\theta} \subset W^0_\mu$ is a Lagrangian subvariety of dimension $\dim(L^0_\mu) = \langle \rho, \mu \rangle$.

We preserve the setup in Section 7.2. We further assume p is large enough so that there is a non-degenerate symmetric bilinear invariant form $(\ ,\ )$ on $\mathfrak{g}$ that is also θ-invariant.

7.3.1. *Poisson structures on loop groups and affine Grassmannians.* It is well-known that $(L\mathfrak{g}, L^{<0}\mathfrak{g}, L^+\mathfrak{g})$ form a *Manin triple*, which we review. Fix a non-degenerate symmetric bilinear invariant form $(\ ,\ )$ on $\mathfrak{g}$ that is also θ-invariant. Extend it F-linearly to $L\mathfrak{g} = \mathfrak{g}(F)$. Composing with residue map, we get a k-valued inner product on $L\mathfrak{g}$, which we still denote by $(\ ,\ )$. Then $L^{<0}\mathfrak{g}, L^+\mathfrak{g}$ are both Lagrangian subspaces of $L\mathfrak{g}$ with respect to $(\ ,\ )$. Fix a basis e^*_i of $L^{<0}\mathfrak{g}$ and a basis e_i of $L^+\mathfrak{g}$ such that $(e^*_i, e_j) = \delta_{ij}$. Define r-matrix

$$r = \sum_i e^*_i \wedge e_i \in \wedge^2 L\mathfrak{g},$$

where the formal sum is understood as an element of $\mathfrak{g}(\!(t^\pm)\!) \otimes \mathfrak{g}(\!(t^\pm)\!)$.

The definition of r is independent of the choice of orthonormal basis. Since θ acts continuously on $L^{<0}\mathfrak{g}$ and $L^+\mathfrak{g}$, we have

$$\theta(r) = r.$$

Define a bi-vector field $\pi \in \Gamma(LG, \wedge^2 TLG)$ by

$$\pi(g) := R_{g*}r - L_{g*}r,$$

where $g \in LG$, L_g, R_g are left and right translations by g. Then π gives a Poisson bi-vector and defines a Poisson structure on LG. Moreover, $L^{<0}G, L^+G$ are Poisson subgroups of LG.

There are two ways to define a bi-vector field on Gr from r. Let $p : LG \to \mathrm{Gr}$ be the projection map. Define $\bar{\pi}_1 \in \Gamma(\mathrm{Gr}, \wedge^2 T\mathrm{Gr})$ by $\bar{\pi}_1(g) = d_g p(\pi_g)$.

On the other hand, the action of LG on Gr defines a map $\kappa : L\mathfrak{g} \to \Gamma(\mathrm{Gr}, T\mathrm{Gr})$. Let $\bar{\pi}_2 = \kappa(r) \in \Gamma(\mathrm{Gr}, \wedge^2 T\mathrm{Gr})$.

Lemma 66. *We have $\bar{\pi}_1 = \bar{\pi}_2$.*

Proof. At any $g \in LG$, let $\bar{g} = p(g)$. Since $e_i \in L^+\mathfrak{g}$, we have $p(ge^{te_i}) = \bar{g}$. Thus $d_g p(L_{g*} e_i) = 0$, $d_g p(L_{g*} r) = 0$, $\bar{\pi}_1(g) = dp(R_{g*} r)$. Also, for $a \in L\mathfrak{g}$, $\kappa(a)(\bar{g})$ is represented by curve $p(e^{ta} g)$, i.e. $\kappa(a)(\bar{g}) = d_g p(R_{g*} a)$. Thus $\bar{\pi}_1(\bar{g}) = \bar{\pi}_2(\bar{g}) = d_g p(R_{g*} r)$. □

Denote $\bar{\pi} = \bar{\pi}_1 = \bar{\pi}_2$. From [LY08, Theorem 2.3], we know $\bar{\pi}$ defines a Poisson structure on Gr, and both L^+G-orbits $p(LG_\lambda) = L^+G \cdot [t^\lambda]$ and $L^{<0}G$-orbits $p(W^\lambda) = L^{<0}G \cdot [t^\lambda]$ are Poisson subvarieties. The intersections of L^+G-orbits and $L^{<0}G$-orbits, when nonempty, are symplectic leaves, c.f. [LY08, Corollary 2.9], [KWWY11, Theorem 2.5].

7.3.2. *Lagrangian slices.* The projection $p : LG \to \mathrm{Gr} = LG/L^+G$ restricts to an embedding on W^λ, thus $p : W^\lambda_\mu \xrightarrow{\sim} L^{<0}G \cdot [t^\lambda] \cap L^+G \cdot [t^\mu]$. The pullback induces a symplectic structure on W^λ_μ.

Lemma 67. *$L^\lambda_\mu \subset W^\lambda_\mu$ is a coisotropic subvariety.*

Proof. This is equivalent to that the ideal of L^λ_μ in W^λ_μ is closed under Poisson bracket. For this, it suffices to prove for the embedding $L^\lambda \subset W^\lambda$, where the Poisson structure on W^λ is pullback from $p : W^\lambda \xrightarrow{\sim} L^{<0}G \cdot [t^\lambda] \subset \mathrm{Gr}$.

Denote $\sigma = \mathrm{inv} \circ \theta$, $L^\lambda = (W^\lambda)^\sigma$. For any $g \in L^\lambda$, $g \in \tau(LX) = \tau(LG/LK)$, so we can write $g = h\theta(h)^{-1}$ for some $h \in LG$. For $x \in L\mathfrak{g}$, if $e^{tx} g \in (LG)^\sigma$, then $x = -\mathrm{Ad}_g(d\theta(x))$, $d\theta(\mathrm{Ad}_{h^{-1}} x) = -\mathrm{Ad}_{h^{-1}} x$, $x \in \mathrm{Ad}_h L\mathfrak{p}$. Thus

$$T_g(L^\lambda) = R_{g*}(L^{<0}\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda} L^{<0}\mathfrak{g} \cap \mathrm{Ad}_h L\mathfrak{p}).$$

Under the inner product $(\ ,\ )$, the conormal space $(N^*_{L^\lambda} W^\lambda)(g) = \{\phi \in T^*_g W^\lambda \mid \phi(T_g L^\lambda) = 0\}$ can be identified with $N_g := R_{g*}(L^+\mathfrak{g} \cap \mathrm{Ad}_{t^\lambda} L^+\mathfrak{g} \cap \mathrm{Ad}_h L\mathfrak{k})$.

In view of Lemma 66, to show the coisotropicity, we need to show that the contraction of conormal bundle with bivector π

$$N^*_{L^\lambda} W^\lambda \hookrightarrow T^* W^\lambda|_{L^\lambda} \xrightarrow{\pi} TW^\lambda|_{L^\lambda}$$

lands inside TL^λ. Since $T_g L^\lambda = \{x \in T_g W^\lambda \subset T_g LG \mid (x, N_g) = 0\}$, it remains to show that for any $g = h\theta(h)^{-1} \in L^\lambda$ and $x, y \in N_g$, $(\pi_g, x \wedge y) = 0$. Write $x = R_{g*}\mathrm{Ad}_h a$, $y = R_{g*}\mathrm{Ad}_h b$, where $a, b \in L\mathfrak{k}$. Since $(\ ,\ )$ is LG-invariant, we have

$$(\pi_g, x \wedge y) = (R_{g*} r - L_{g*} r, R_{g*}\mathrm{Ad}_h(a \wedge b)) = (\mathrm{Ad}_{h^{-1}} r - \mathrm{Ad}_{\theta(h^{-1})} r, a \wedge b).$$

Recall $\theta(r) = r$. Note that θ acts trivially on $a, b \in L\mathfrak{k}$ and $(\ ,\ )$ is θ-invariant. Thus

$$(\mathrm{Ad}_{\theta(h^{-1})} r, a \wedge b) = (\mathrm{Ad}_{h^{-1}} r, a \wedge b),$$

which completes the proof. □

Proposition 68. *$L^0_\mu \subset W^0_\mu$ is a Lagrangian subvariety.*

Proof. From Lemma 65 and Lemmra 68, we see that $L^0_\mu \subset W^0_\mu$ is coisotropic subvarities of dimension $\dim(L^0_\mu) = \frac{1}{2}\dim(W^\lambda_\mu) = \langle \rho, \mu \rangle$, and hence is a Lagrangian. □

Remark 69. Specializing Proposition 68 to the case in Proposition 51, we obtain (special case of) a well-known result of Kostant-Rallis saying that the symmetric nilpotent orbits $\mathcal{O}_{\mathfrak{p},\lambda}$ are Lagrangian subvarieties of the nilpotent orbits $\mathcal{O}_\lambda$ with respect to the Kostant-Kirillov symplectic forms.

8. Affine Hecke modules

Let $\mathscr{D}$ be the set of pairs (ξ, v) where $v \in \mathscr{V}$ and ξ is an I_0-equivariant local system on the I_0-orbit $\mathcal{O}_v \subset LX$. Let M be the free $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module with basis indexed by $\mathscr{D}$. In this section we endow M with a module structure over the affine Hecke algebra of G and an anti-linear involution $D_\delta : M \to M$ compatibile with the module structure. The construction is geometric and relies on the ℓ-adic sheaf theory on placid (ind)schemes, where ℓ is a prime number $p \neq \ell$ (see Appendix A.4 for a review).

For the rest of the paper, we will fix a dimension theory δ on LX (see Proposition 63). Note that, for any $v \in \mathscr{V}$ (resp. $\lambda \in \Lambda_S^+$), the function $\delta_{\overline{\mathcal{O}}_v} : \overline{\mathcal{O}}_v \to \mathbb{Z}$ (resp. $\delta_{\overline{LX}_\lambda} : \overline{LX}_\lambda \to \mathbb{Z}$) is constant and we will write $\delta(v)$ (resp. $\delta(\lambda)$) for its value to be called the dimension of $\mathcal{O}_v$ (resp. LX_λ).

8.1. Affine Hecke module categories.

8.1.1. *Equivariant sheaves on LG and LX.* Recall the Iwahori decomposition $LG = \bigsqcup_{w \in \widetilde{\mathrm{W}}} I_0 w I_0$ of the loop group into $I_0 \times I_0$-orbits. Here $\widetilde{\mathrm{W}} \simeq N_{LG}(LT_0)/L^+T_0$ is the extended Weyl group. Write $LG_w = I_0 w I_0$ and $\overline{LG}_w$ be the orbit closure. We denote by $D(I_0\backslash LG/I_0)$ the dg category of $I_0 \times I_0$-equivariant constructible complexes on LG, known as the affine Hecke category. It has a Verdier duality functor $\mathbb{D} : D(I_0\backslash LG/I_0) \cong D(I_0\backslash LG/I_0)^{op}$ and a monoidal structure given by the convolution product: consider the maps

$$LG/I_0 \times LG/I_0 \overset{p\times \mathrm{id}}{\leftarrow} LG \times LG/I_0 \overset{q}{\to} LG \times^{I_0} LG/I_0 \overset{m}{\to} LG/I_0$$

where p and q are the natural quotient maps and m is the multiplication map. Given $\mathcal{M}_1, \mathcal{M}_2 \in D(I_0\backslash LG/I_0)$ we have

$$\mathcal{M}_1 \star \mathcal{M}_2 := m_!(\mathcal{M}_1 \tilde{\boxtimes} \mathcal{M}_2) \tag{20}$$

where $\mathcal{M}_1 \tilde{\boxtimes} \mathcal{M}_2 \in D(I_0\backslash LG \times^{I_0} LG/I_0)$ is the unique complex such that $(p \times \mathrm{id})^*(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \cong q^*(\mathcal{M}_1 \tilde{\boxtimes} \mathcal{M}_2)$.

We denote by

$$D(I_0\backslash LX) \cong \mathrm{colim}_{\underline{v},!}\, D(I_0\backslash \overline{\mathcal{O}}_{\underline{v}})$$

the dg category I_0-equivariant constructible complexes on the I_0-placid ind-scheme $LX \cong \mathrm{colim}_{\underline{v}} \overline{\mathcal{O}}_{\underline{v}}$. We have the affine Hecke action, denoted by $\star$, of the affine Hecke category $D(I_0\backslash LG/I_0)$ on $D(I_0\backslash LX)$ defined as follows. Consider the maps

$$LG/I_0 \times LX \overset{p\times \mathrm{id}}{\leftarrow} LG \times LX \overset{q}{\to} LG \times^{I_0} LX \overset{a}{\to} LX$$

where a is the action map. Given $\mathcal{M} \in D(I_0\backslash LG/I_0)$ and $\mathcal{F} \in D(I_0\backslash LX)$ we have

$$\mathcal{M} \star \mathcal{F} := a_!(\mathcal{M} \tilde{\boxtimes} \mathcal{F}) \tag{21}$$

where $\mathcal{M} \tilde{\boxtimes} \mathcal{F} \in D(I_0\backslash LG \times^{I_0} LX)$ is the unique complex such that $(p \times \mathrm{id})^*(\mathcal{M} \boxtimes \mathcal{F}) \cong q^*(\mathcal{M} \tilde{\boxtimes} \mathcal{F})$.

The affine Hecke category $D(I_0\backslash LG/I_0)$ has a standard t-structure with abelian heart $\mathrm{Shv}(I_0\backslash LG/I_0)$ the abelian category of $I_0\times I_0$-equivariant constructible sheaves on LG and a perverse t-structure with abelian heart $\mathrm{Perv}(I_0\backslash LG/I_0)$ the abelian category $I_0\times I_0$-equivariant perverse sheaves on LG. We denote by $\mathscr{H}^n : D(I_0\backslash LG/I_0)\to \mathrm{Shv}(I_0\backslash LG/I_0)$ and ${}^p\mathscr{H}^n : D(I_0\backslash LG/I_0)\to \mathrm{Perv}(I_0\backslash LG/I_0)$ the corresponding cohomological functors.

The category $D(I_0\backslash LX)$ also has a standard t-structure with abelian heart $\mathrm{Shv}(I_0\backslash LX)$ the I_0-equivariant constructible sheaves on LX. According to A.7, the dimension theory δ on LX gives rise to a $*$-adapted perverse t-structure on $D(I_0\backslash LX)$ with abelian heart $\mathrm{Perv}_\delta(I_0\backslash LX)$ called the abelian category of I_0-equivariant perverse on LX. We denote by $\mathscr{H}^n : D(I_0\backslash LX)\to \mathrm{Shv}(I_0\backslash LX)$ and ${}^p\mathscr{H}^n_\delta : D(I_0\backslash LG/I_0)\to \mathrm{Perv}_\delta(I_0\backslash LX)$ the corresponding cohomological functors.

8.1.2. *Verdier duality.* Denote by $\mathbb{D} : D(I_0\backslash LG/I_0)\cong D(I_0\backslash LG/I_0)^{op}$ the Verdier duality functor. By Lemma 104, the dimension theory δ gives rise to a Verdier duality functor

$$\mathbb{D}_\delta : D(I_0\backslash LX)\cong D(I_0\backslash LX)^{op}$$

which is t-exact with respect to the $*$-adapted perverse t-structure above and satisfying $\mathbb{D}_\delta^2\cong \mathrm{Id}$.

Proposition 70. (i) *For any $\mathcal{M}\in D(I_0\backslash LG/I_0)$ and $\mathcal{F}\in D(I_0\backslash LX)$, there is an isomorphism $\mathbb{D}_\delta(\mathcal{M}\star\mathcal{F})\cong \mathbb{D}(\mathcal{M})\star\mathbb{D}_\delta(\mathcal{F})$*

(ii) *Assume both $\mathcal{M}\in D(I_0\backslash LG/I_0)$ and $\mathcal{F}\in D(I_0\backslash LX)$ are semi-simple complexes. Then the convolution $\mathcal{M}\star\mathcal{F}$ is semi-simple.*

Proof. There are $w\in\widetilde{\mathrm{W}}$, $v\le v'\in\mathscr{V}$ such that $\mathcal{M}\in D(I_0\backslash\overline{LG}_w/I_0)$, $\mathcal{F}\in D(I_0\backslash\overline{\mathcal{O}}_v)$, and $\mathcal{M}\star\mathcal{F}\in D(I_0\backslash\overline{\mathcal{O}}_{v'})$. One can find étale covers $U_{v'}\to\overline{\mathcal{O}}_{v'}$ (resp. $W\to\overline{LG}_w\times\overline{\mathcal{O}}_v$) admitting placid presentations, fitting into the following commutative diagram

$$\begin{array}{ccccccc}
\overline{LG}_w/I_0\times U_v & \xleftarrow{p} & W & \xrightarrow{q} & V & \xrightarrow{a} & U_{v'}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\overline{LG}_w/I_0\times\overline{\mathcal{O}}_v & \xleftarrow{p} & \overline{LG}_w\times\overline{\mathcal{O}}_v & \xrightarrow{q} & \overline{LG}_w\times^{I_0}\overline{\mathcal{O}}_v & \xrightarrow{a} & \overline{\mathcal{O}}_{v'}
\end{array}.$$

Here $V=U'\times_{\overline{\mathcal{O}}_{v'}}(\overline{LG}_w\times^{I_0}\overline{\mathcal{O}}_v)\to\overline{LG}_w\times^{I_0}\overline{\mathcal{O}}_v$ and $U_v=U_{v'}\times_{\overline{\mathcal{O}}_{v'}}\overline{\mathcal{O}}_v\to\overline{\mathcal{O}}_v$ are the étale covers obtained by the base changes. Let $a^{[n]}:U_{v'}^{[n]}\to V^{[n]}$ be the map between the the Čech nerves associated to the smooth covers $U_{v'}\to I_0\backslash\overline{\mathcal{O}}_{v'}$ and $V\to I_0\backslash\overline{LG}_w\times^{I_0}\overline{\mathcal{O}}_v$. Then by (A.1) (4) there are placid presentations $U_{v'}^{[n]}\cong\lim_j U_{v',j}^{[n]}$, $U_v^{[n]}\cong\lim_j U_{v,j}^{[n]}$, $V^{[n]}\cong\lim_j V_j^{[n]}$, such that $U_v^{[n]}\to U_{v'}^{[n]}$ and $a^{[n]}:V^{[n]}\to U_{v'}^{[n]}$ are base changes from $U_{v,j_0}^{[n]}\to U_{v',j_0}^{[n]}$ and $a_j^{[n]}:V_j^{[n]}\to U_{v',j}^{[n]}$ for some j.

Proof of (i). We have

$$\mathcal{M}\tilde{\boxtimes}\mathcal{F}\cong\lim^*_{[n]}\mathrm{colim}^*_j\,\mathcal{F}^{[n]}_{V,j}\in D(I_0\backslash\overline{LG}_w\times^{I_0}\overline{\mathcal{O}}_v)\cong\lim^*_{[n]}\mathrm{colim}^*_j\,D(V_j^{[n]}) \tag{22}$$

and it follows that

$$\mathcal{M}\star\mathcal{F}=a_!\mathcal{M}\tilde{\boxtimes}\mathcal{F}\cong a_!(\lim^*_{[n]}\mathrm{colim}^*_j\,\mathcal{F}^{[n]}_{V,j})\cong\lim^*_{[n]}\mathrm{colim}^*_j\,a^{[n]}_{j,!}(\mathcal{F}^{[n]}_{V,j}) \tag{23}$$

$$\in D(I_0\backslash\overline{\mathcal{O}}_v)\cong\lim^*_{[n]}\mathrm{colim}^*_j\,D(U^{[n]}_{v',j}).$$

From the construction of Verdier duality in (104), we have

$$\mathbb{D}_\delta(\mathcal{M}\star\mathcal{F}) \overset{(23)}{\cong} \mathbb{D}_\delta(\lim^*_{[n]}\operatorname{colim}^*_j a^{[n]}_{j,!}(\mathcal{F}^{[n]}_{V,j})) \cong \lim^*_{[n]}\operatorname{colim}^*_j(\mathbb{D}a^{[n]}_{j,!}((\mathcal{F}^{[n]}_{V,j})\langle -2\dim(U^{[n]}_{v',j}+2\delta_{v'})\rangle)) \cong$$
$$\cong \lim^*_{[n]}\operatorname{colim}^*_j(a^{[n]}_{j,!}(\mathbb{D}(\mathcal{F}^{[n]}_{V,j})\langle -2\dim(U^{[n]}_{v',j}+2\delta_{v'})\rangle)) \cong$$
$$\cong a_!(\lim^*_{[n]}\operatorname{colim}^*_j(\mathbb{D}(\mathcal{F}^{[n]}_{V,j})\langle -2\dim(U^{[n]}_{v',j}+2\delta_{v'})\rangle))).$$

We claim that there is an isomorphism

$$\mathbb{D}\mathcal{M}\tilde{\boxtimes}\mathbb{D}_\delta\mathcal{F} \cong \lim^*_{[n]}\operatorname{colim}^*_j(\mathbb{D}(\mathcal{F}^{[n]}_{V,j})\langle -2\dim(U^{[n]}_{v',j}+2\delta_{v'})\rangle)) \in D(I_0\backslash\overline{LG}_w \times^{I_0} \overline{\mathcal{O}}_v)$$

and it follows that

$$\mathbb{D}_\delta(\mathcal{M}\star\mathcal{F}) \cong a_!(\mathbb{D}\mathcal{M}\tilde{\boxtimes}\mathbb{D}_\delta\mathcal{F}) \cong \mathbb{D}\mathcal{M}\star\mathbb{D}_\delta\mathcal{F}.$$

Part (i) follows.

Proof of the claim. By definition the complex $\mathcal{M}\tilde{\boxtimes}\mathcal{F}$ is characterized by the isomorphism $p^*(\mathcal{M}\boxtimes\mathcal{F}) \cong q^*(\mathcal{M}\tilde{\boxtimes}\mathcal{F})$. There are isomorphisms

$$(p^{[n]}_j)^*(\mathcal{M}\boxtimes\mathcal{F}^{[n]}_j) \cong (q^{[n]}_j)^*(\mathcal{F}^{[n]}_{V,j}) \tag{24}$$

inducing

$$p^*(\mathcal{M}\boxtimes\mathcal{F}) \cong \lim^*_{[n]}\operatorname{colim}^*_j(p^{[n]}_j)^*(\mathcal{M}\boxtimes\mathcal{F}^{[n]}_j) \cong \lim^*_{[n]}\operatorname{colim}^*_j(q^{[n]}_j)^*(\mathcal{F}^{[n]}_{V,j}) \cong q^*(\mathcal{M}\tilde{\boxtimes}\mathcal{F})$$

where $\mathcal{F} \cong \lim^*_{[n]}\operatorname{colim}^*_j \mathcal{F}^{[n]}_j \in D(I_0\backslash\overline{\mathcal{O}}_v) \cong \lim^*_{[n]}\operatorname{colim}^*_j D(U^{[n]}_{v,j})$ and

$$p^{[n]} \cong \lim_j p^{[n]}_j : W^{[n]} \cong W^{[n]}_j \to \overline{LG}_w/I_0 \times U^{[n]}_v \cong \lim_j \overline{LG}_w/I_0 \times U^{[n]}_{v,j},$$
$$q^{[n]} \cong \lim_j q^{[n]}_j : W^{[n]} \cong \lim_j W^{[n]}_j \to V^{[n]} \cong \lim_j V^{[n]}_j$$

are the maps between Čech nerves. Since $f^*\mathbb{D}\langle -2\dim(X)\rangle \cong \mathbb{D}\langle -2\dim(Y)\rangle f^*$ for a smooth map $f : X \to Y$ between finite type schemes, we conclude from (24) that

$$p^{[n]*}_j(\mathbb{D}\mathcal{M}\boxtimes\mathbb{D}\mathcal{F}^{[n]}_j\langle -2\dim(U^{[n]}_{v,j}) - 2l(w)\rangle) \cong q^{[n],*}_j(\mathbb{D}\mathcal{F}^{[n]}_{V,j})\langle -2\dim(V^{[n]}_j)\rangle. \tag{25}$$

On the other hand, the discussion of dimension functions in (A.4) implies that

$$\dim_a = \dim(V^{[n]}_j) - \dim(U^{[n]}_{v',j}) = \delta_v + l(w) - \delta_{v'}. \tag{26}$$

All together we get

$$\begin{aligned} p^*(\mathbb{D}\mathcal{M}\boxtimes\mathbb{D}_\delta\mathcal{F}) &\cong \lim^*_{[n]}\operatorname{colim}^*_j(p^{[n]*}(\mathbb{D}\mathcal{M}\boxtimes\mathbb{D}\mathcal{F}^{[n]}_j\langle -2\dim(U^{[n]}_{v,j}) + 2\delta_v\rangle)) \\ &\overset{(25)}{\cong} \lim^*_{[n]}\operatorname{colim}^*_j(q^{[n],*}(\mathbb{D}\mathcal{F}^{[n]}_{V,j})\langle -2\dim(V^{[n]}_j) + 2\delta_v + 2l(w)\rangle) \\ &\overset{(26)}{\cong} \lim^*_{[n]}\operatorname{colim}^*_j(q^{[n],*}(\mathbb{D}\mathcal{F}^{[n]}_{V,j})\langle -2\dim(U^{[n]}_{v',j}) + 2\delta_{v'}\rangle) \\ &\cong q^*(\lim^*_{[n]}\operatorname{colim}^*_j(\mathbb{D}\mathcal{F}^{[n]}_{V,j})\langle -2\dim(U^{[n]}_{v',j}) + 2\delta_{v'}\rangle). \end{aligned}$$

The desried claim follows.

Proof of (ii). To show that $\mathcal{M}\star\mathcal{F}$ is semi-simple it suffices to check that each $a^{[n]}_{j,!}(\mathcal{F}^{[n]}_{V,j})$ in (23) is semi-simple. Note that since $\mathcal{M}\tilde{\boxtimes}\mathcal{F}$ is semi-simple it implies each $\mathcal{F}^{[n]}_{V,j} \in D(V^{[n]}_j)$ is semi-simple. On the other hand, since a proper it follows from [Gro65, Theorem 8.10.5]

that each $a_j^{[n]}$ is proper, and the decomposition theorem implies each $a_{j,!}^{[n]}(\mathcal{F}_{V,j}^{[n]}) \in D(U_{v',j}^{[n]})$ is semi-simple. Part (ii) follows.

□

8.1.3. *Equivariant local systems.* We shall give a description of equivariant local systems on the orbits LG_w and $\mathcal{O}_v$. Denote by $\pi_{T_0} : I_0 \to I_0/I_0^+ \simeq T_0$ the quotient map. For an orbit $\mathcal{O}_v$, note that T_0 is stable under involution ψ_v. Thus $\pi_{T_0}(I_0 \cap (LG)^{\psi_v}) = T_0^{\psi_v}$. Denote $T_v := T_0^{\psi_v}$. The T_0-equivariant rank one local systems on T_0/T_v are given by characters $X(\pi_0(T_v))$. For $\xi \in X(\pi_0(T_v))$, denote the corresponding local system on T_0/T_v by $\mathcal{L}_\xi$.

Lemma 71. (i) *Any $I_0 \times I_0$-equivariant local system $\mathcal{L}_w$ on the $I_0 \times I_0$-orbit LG_w is isomorphic to the constant local system $\mathcal{L}_w \cong \overline{\mathbb{Q}}_{\ell,LG_w}$ with the canonical equivariant structure.*

(ii) *Any I_0-equivariant local system on an I_0-orbit $\mathcal{O}_v$ is a direct sum of rank one local systems of the form $\mathcal{L}_{\xi,v} := \pi_v^*\mathcal{L}_\xi$, $\xi \in X(\pi_0(T_v))$, where $\pi_v : \mathcal{O}_v = I_0 x_v \simeq I_0/I_0 \cap (LG)^{\psi_v} \to T_0/T_v$.*

Proof. Part (i) follows from the standard fact that points in LG_w have connected $I_0 \times I_0$-stabilizers. Proof of (ii). Let $C_v = I_0 \cap (LG)^{\psi_v}$, $D_v = \pi_{T_0}(C_v) = T^{\psi_v}$. Denote their neutral components by C_v^0, D_v^0. The map $C_v \to D_v$ has fibers isomorphic to $I_0^+ \cap (LG)^{\psi_v}$. Since I_0^+ is pro-unipotent, $I_0^+ \cap (LG)^{\psi_v}$ is connected. Thus the induced map $C_v/C_v^\circ \to D_v/D_v^\circ$ is an isomorphism. As a consequence, C_v/C_v^0 is a commutative finite group. Its representations give the equivariant local systems. □

Definition 72. *Let $\mathscr{D}$ be the set of pairs (ξ, v) where $v \in \mathscr{V}$ and ξ is an I_0-equivariant local system on the I_0-orbit $\mathcal{O}_v \subset LX$.*

8.2. **Affine Lusztig-Vogan modules.** We assume G, B_0, T_0, θ are defined over a finite field $\mathbb{F}_q$ and T_0 is split over $\mathbb{F}_q$. There are Frobenius morphisms $F : LG \to LG$, $F : LX \to LX$, etc. Consider the group $A = \overline{\mathbb{Q}}_\ell^\times/\mu$ where $\mu \subset \overline{\mathbb{Q}}_\ell^\times$ is the subgroup of roots of 1. Let $\mathbb{Z}[A]$ be the group ring of A. We fix a square root $q^{\frac{1}{2}}$ of q and identify $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \subset \mathbb{Z}[A]$ sending $q^{\frac{1}{2}}$ to its image in A.

8.2.1. *The affine Hecke module M.* We follow closely the discussions in [Lus84, LV83, MS98]. For any positive integer n, let $\mathrm{Shv}(I_0\backslash LX)^{F^n}$ be the abelian category consisting of pairs $(\mathcal{F}, \Phi_n)$ where $\mathcal{F}$ is an I_0-equivariant sheaf on LX and $\Phi_n : (F^n)^*\mathcal{F} \cong \mathcal{F}$ is an isomorphism. For any positive integer m divisible by n, we have a natural exact functor $\mathrm{Shv}(I_0\backslash LX)^{F^n} \to \mathrm{Shv}(I_0\backslash LX)^{F^m}$ sending $(\mathcal{F}, \Phi_n)$ to $(\mathcal{F}, (\Phi_n)^{m/n})$ and we define

$$\mathrm{Shv}(I_0\backslash LX)^{\mathrm{Weil}} := \mathrm{colim}_n \mathrm{Shv}(I_0\backslash LX)^{F^n}.$$

Thus $\mathrm{Shv}(I_0\backslash LX)^{\mathrm{Weil}}$ is an abelian category with objects consisting of pairs $(\mathcal{F}, \Phi)$ where Φ, to be called a Weil structure on $\mathcal{F}$, is a system of isomorphism $\Phi_n : (F^n)^*\mathcal{F} \cong \mathcal{F}$ for sufficiently divisible n such that $(\Phi_n)^{m/n} = \Phi_m$ and two systems $(\mathcal{F}, \Phi)$ and $(\mathcal{F}, \Phi')$ are isomorphic if there exists a multiple m of n_0 and n_0' such that $\Phi_m = \Phi_m'$. Given an object $(\mathcal{F}, \Phi) \in \mathrm{Shv}(I_0\backslash LX)^{\mathrm{Weil}}$ and a point $\gamma \in LX$, we have $\gamma \in (LX)^{F^n}$ for sufficiently divisible n and hence an endomorphism $\Phi_n : \mathcal{F}_x \cong \mathcal{F}_x$ of the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. We can define the Frobenius eigenvalues of Φ on $\mathcal{F}_x$, as elements in the group $A = \overline{\mathbb{Q}}_\ell^\times/\mu$, as being the n-th roots of the eigenvalues of Φ_n for sufficiently divisible n. We define $\mathcal{C}_{LX} \subset \mathrm{Shv}(I_0\backslash LX)^{\mathrm{Weil}}$

to be the full subcategory consisting of $(\mathcal{F}, \Phi)$ such that the eigenvalues of Φ lands in the subgroup $q^{\frac{\mathbb{Z}}{2}} \subset A$. We denote by $M = \mathcal{K}(\mathcal{C}_{LX})$ the Grothendieck group of $\mathcal{C}_{LX}$.

For any equivariant local system $\mathcal{L}_{\xi,v}$ on the orbit $\mathcal{O}_v$ we have the extension by zero sheaf (i.e., the standard object)

$$j_{v,!}\mathcal{L}_{\xi,v} \in \mathrm{Shv}(I_0\backslash LX)$$

where $j_v : \mathcal{O}_v \to LX$ is the inclusion map (see (48)). Since ξ is a finite order character, for a sufficiently divisible n we have an isomorphism $(F^n)^* j_{v,!}\mathcal{L}_{\xi,v} \cong j_{v,!}\mathcal{L}_{\xi,v}$ and hence a canonical Weil structure Φ on $j_{v,!}\mathcal{L}_{\xi,v}$ such that the Frobenius eigenvalue of Φ on non-zero stalks is equal to 1 (note that $\mathcal{L}_{\xi,v}$ is of rank 1). We denote by $[\mathcal{L}_{\xi,v}] = [(j_{v,!}\mathcal{L}_{\xi,v}, \Phi)] \in M$ the class of $(j_{v,!}\mathcal{L}_{\xi,v}, \Phi) \in \mathcal{C}_{LX}$. Then M is a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module with basis $[\mathcal{L}_{\xi,v}], (\xi, v) \in \mathscr{D}$ where the multiplication of $q^{\frac{1}{2}}$ on $[\mathcal{L}_{\xi,v}]$ is given by $q^{\frac{1}{2}}[\mathcal{L}_{\xi,v}] = [(j_{v,!}\mathcal{L}_{\xi,v}, q^{\frac{1}{2}}\Phi)]$ where $q^{\frac{1}{2}}\Phi$ is the Weil structure on $j_{v,!}\mathcal{L}_{\xi,v}$ with eigenvalue $q^{\frac{1}{2}}$ on the non-zero stalks of $j_{v,!}\mathcal{L}_{\xi,v}$. Note that $(j_{v,!}\mathcal{L}_{\xi,v}, q^{-n}\Phi)$ can be identified with the Tate twist $\mathcal{L}_{\xi,v}(n)$ and hence we have $q^{-n}[\mathcal{L}_{\xi,v}] = [\mathcal{L}_{\xi,v}(n)]$.

We similarly define abelian category $\mathrm{Shv}(I_0\backslash LG/I_0)^{\mathrm{Weil}} = \mathrm{colim}_n \mathrm{Shv}(I_0\backslash LG/I_0)^{F^n}$ and the full subcategory $\mathcal{C}_{LG} \subset \mathrm{Shv}(I_0\backslash LG/I_0)^{\mathrm{Weil}}$. We denote by $H = \mathcal{K}(\mathcal{C}_{LG})$ the Grothendieck group of $\mathcal{C}_{LG}$. For any orbit LG_w, we have the extension by zero $j_{w,!}\mathcal{L}_w$ of the constant local system $\mathcal{L}_w$ on $j_w : LG_w \to LG$ and H is a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module with basis $[\mathcal{L}_w], w \in \widetilde{\mathrm{W}}$.

The convolution map makes H into an algebra over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$, known as the affine Hecke algebra of G, and the convolution action makes M into an H-module:

$$[\mathcal{L}_w][\mathcal{L}_{\xi,v}] = \sum_i (-1)^i [\mathscr{H}^i(\mathcal{L}_w \star \mathcal{L}_{\xi,v})]. \tag{27}$$

Finally, the Verdier duality functors induce involutions D and D_δ on H and M given by

$$D[\mathcal{H}] = \sum_i (-1)^i [\mathscr{H}^i \mathbb{D}(\mathcal{H})], \quad D_\delta[\mathcal{M}] = \sum_i (-1)^i [\mathscr{H}^i \mathbb{D}_\delta(\mathcal{M})],$$

and it follows from Proposition 70 that

$$D_\delta([\mathcal{H}][\mathcal{M}]) = D([\mathcal{H}]) D_\delta([\mathcal{M}]).$$

8.2.2. *The affine Hecke module M'.* For any positive integer n, we define $\mathrm{Perv}(I_0\backslash LX)^{F^n}$ to be the abelian category consisting of pairs $(\mathcal{P}, \Phi_n)$ where $\mathcal{P} \in \mathrm{Perv}_\delta(I_0\backslash LX)$ and $\Phi_n : (F^n)^*\mathcal{F} \cong \mathcal{F}$ is an isomorphism. For any positive integer m divisible by n, we have a natural exact functor $\mathcal{A}_{LX,n} \to \mathcal{A}_{LX,m}$ sending $(\mathcal{P}, \Phi_n)$ to $(\mathcal{P}, (\Phi_n)^{m/n})$ and we define

$$\mathrm{Perv}_\delta(I_0\backslash LX)^{\mathrm{Weil}} := \mathrm{colim}_n \mathrm{Perv}_\delta(I_0\backslash LX)^{F^n}.$$

Thus $\mathrm{Perv}_\delta(I_0\backslash LX)^{\mathrm{Weil}}$ is an abelian category with objects consisting of pairs $(\mathcal{P}, \Phi)$ where Φ, to be called a Weil structure on $\mathcal{P}$, is a system of isomorphism $\Phi_n : (F^n)^*\mathcal{P} \cong \mathcal{P}$ for sufficiently divisible n such that $(\Phi_n)^{m/n} = \Phi_m$ and two systems $(\mathcal{P}, \Phi)$ and $(\mathcal{P}, \Phi')$ are isomorphic if there exists a multiple m of n_0 and n_0' such that $\Phi_m = \Phi'_m$. Given an object $(\mathcal{F}, \Phi) \in \mathrm{Perv}_\delta(I_0\backslash LX)^{\mathrm{Weil}}$ and a point $\gamma \in LX$, we have $\gamma \in (LX)^{F^n}$ for sufficiently divisible n and hence an endomorphism $\mathscr{H}^i\Phi_n : \mathscr{H}^i\mathcal{P}_x \cong \mathscr{H}^i\mathcal{P}_x$ of the stalk $\mathscr{H}^i\mathcal{P}_x$ of $\mathscr{H}^i\mathcal{P}$ at x. We can define the Frobenius eigenvalues of Φ on $\mathscr{H}^i\mathcal{P}_x$, as elements in the group $A = \overline{\mathbb{Q}}_\ell^\times/\mu$, as being the n-th roots of the eigenvalues of $\mathscr{H}^i\Phi_{n,x}$ for sufficiently divisible n. We define $\mathcal{A}_{LX} \subset \mathrm{Perv}_\delta(I_0\backslash LX)^{\mathrm{Weil}}$ to be the full subcategory consisting of $(\mathcal{P}, \Phi)$ such that

the eigenvalues of Φ lands in $q^{\frac{\mathbb{Z}}{2}} \subset A$. We denote by $M' = \mathcal{K}(\mathcal{A}_{LX})$ the Grothendieck group of $\mathcal{A}_{LX}$.

For any equivariant local system $\mathcal{L}_{\xi,v}$ on the orbit $\mathcal{O}_v$ we have the IC-extension in (50):

$$\mathrm{IC}_{\xi,v} = (j_v)_{!,*}\mathcal{L}_{\xi,v}[\delta(v)] \in \mathrm{Perv}_\delta(I_0\backslash LX)$$

where $j_v : \mathcal{O}_v \to LX$ is the inclusion map and $\delta(v)$ is the dimension function of $\mathcal{O}_v$ fixed at the beginning of §8. There is a canonical Weil structure Φ on $\mathrm{IC}_{\xi,v}$ such that the Frobenius eigenvalue of $\mathscr{H}^{-\delta(v)}\Phi_{\xi,v}$ on the stalks of $\mathscr{H}^{-\delta(v)}(\mathrm{IC}_{\xi,v}|_x) \cong \mathcal{L}_{\xi,v}|_x, x \in \mathcal{O}_v$ is equal to 1. It follows from [BBDG18, Section 5] that $(\mathrm{IC}_{\xi,v}, \Phi) \in \mathcal{A}_{LX}$ and we denote by $[\mathrm{IC}_{\xi,v}] = [(\mathrm{IC}_{\xi,v}, \Phi)] \in M'$. Then M' is a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module with basis $[\mathrm{IC}_{\xi,v}]$ where the multiplication of $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ on $[\mathrm{IC}_{\xi,v}]$ is given by $q^{\frac{1}{2}}[\mathrm{IC}_{\xi,v}] = [(\mathrm{IC}_{\xi,v}, q^{\frac{1}{2}}\Phi]$ where $q^{\frac{1}{2}}\Phi$ is the Weil structure on $\mathrm{IC}_{\xi,v}$ with eigenvalue $q^{\frac{1}{2}}$ on the stalks of $\mathscr{H}^{-\delta(v)}(\mathrm{IC}_{\xi,v}|_x), x \in \mathcal{O}_v$. Note that $(\mathrm{IC}_{\xi,v}, q^{-n}\Phi)$ can be identify with the Tate twist $\mathrm{IC}_{\xi,v}(n)$ and hence we have $q^{-n}[\mathrm{IC}_{\xi,v}] = [\mathrm{IC}_{\xi,v}(n)]$.

We have similarly defined category $\mathcal{A}_{LG} \subset \mathrm{Perv}(I_0\backslash LG/I_0)$ and denote by $H' = \mathcal{K}(\mathcal{A}_{LG})$ the Grothendieck group of $\mathcal{A}_{LG}$. For any orbit LG_w, we have the IC-extension IC_w of the constant local system $\mathcal{L}_w$ on $j_w : LG_w \to LG$ and H' is a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module with basis $[\mathrm{IC}_w]$.

The convolution map makes H' into an algebra over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ and the convolution action makes M' an H'-module:

$$[\mathrm{IC}_w][\mathrm{IC}_{\xi,v}] = \sum_i (-1)^i [{}^p\mathscr{H}^i_\delta(\mathrm{IC}_w \star \mathrm{IC}_{\xi,v})]. \tag{28}$$

Finally, the Verdier duality functors induce involutions D and D_δ on H' and M' given by

$$D[\mathcal{H}] = \sum_i (-1)^i [{}^p\mathscr{H}^i\mathbb{D}(\mathcal{H})], \quad D_\delta[\mathcal{P}] = \sum_i (-1)^i [{}^p\mathscr{H}^i_\delta\mathbb{D}_\delta(\mathcal{P})],$$

and it follows from Lemma 21 that

$$D_\delta([\mathcal{H}][\mathcal{P}]) = D([\mathcal{H}])D_\delta([\mathcal{P}]).$$

8.2.3. *Two bases.* Consider the map $h : M' \to M$ sending $[\mathcal{A}] \to \sum_{i\in\mathbb{Z}}(-1)^i[\mathscr{H}^i(\mathcal{A})]$, where each $\mathscr{H}^i(\mathcal{A})$ is equipped with the induced Weil structure $\mathscr{H}^i\Phi_n : \mathscr{H}^i(F^n)^*(\mathcal{A}) \cong \mathscr{H}^i(\mathcal{A}))$. Since

$$h([\mathrm{IC}_{\xi,v}]) \in (-1)^{\delta(v)}[\mathcal{L}_{\xi,v}] + \sum_{\bar{u}<\bar{v}} \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}][\mathcal{L}_{\eta,u}]$$

it follows that $h : M' \cong M$ is bijective. Similarly, we have a bijection $h_G : H' \cong H$ sending $h_G([\mathcal{A}]) = \sum_{i\in\mathbb{Z}}(-1)^i[\mathscr{H}^i(\mathcal{A})]$.

Lemma 73. *(i) For any $a, b \in H'$ we have $h_G(ab) = h_G(a)h_G(b)$. (ii) For any $a \in H'$ and $m \in M'$, we have $h(am) = h_G(a)h(m)$. (iii) $D_\delta \circ h = h \circ D$.*

Proof. The proof is the same as in the finite dimensional case [MS98, Proposition 3.2.8 and 3.3.2].

□

The collection $\{h_G[\mathrm{IC}_w]\}_{w\in\widetilde{\mathrm{W}}}$ forms a new basis of H, known as the Kazhdan-Lusztig basis of H, and the transition matrix between the new basis and the standard basis $\{[\mathcal{L}_w]\}_{w\in\widetilde{\mathrm{W}}}$ is given by the Affine Kazhdan-Lusztig polynomials for the extended affine Weyl group $\widetilde{\mathrm{W}}$.

Similarly, the collection $\{h[\mathrm{IC}_{\xi,v}]\}_{(\xi,v)\in\mathscr{D}}$ forms a new basis of M and the transition matrix between the new basis and the standard basis $\{[\mathcal{L}_{\xi,v}]\}_{(\xi,v)\in\mathscr{D}}$ is given by a certain class of polynomials to be called the *Affine Kazhdan-Lusztig-Vogan polynomials*. In Section 9, we will provide a characterization of those polynomials and an algorithm to compute them.

8.3. **Simple reflection actions.** For each affine simple root α, we provide an explicit formula for the action of the simple reflection $[\mathcal{L}_{s_\alpha}] \in H$ on M.

To state the results, we collect some notations from [MS98]. First, note that the action of s_α preserves T_0. For $\xi \in X(\pi_0(T_0^{\psi_0}))$, $s_\alpha\xi \in X(\pi_0(s_\alpha \cdot T_0^{\psi_0}))$. Next, we need a technical construction when v is of type IIIb, IVb for s_α. For any torus T, define $\hat{X}(T) = X^*(T) \otimes_{\mathbb{Z}} (\mathbb{Z}_{(p)}/\mathbb{Z})$ where $\mathbb{Z}_{(p)}$ is the localization at (p). Then $\hat{X}(T)$ is isomorphic to the group of tame rank one local systems on T via Kummer local systems. For $\xi \in \hat{X}(T)$, denote the associated local system by $\mathcal{L}_\xi$. For a surjective map $\phi : T \to T'$ between tori, by Lemma 2.1.5 of *loc. cit.* we have exact sequence

$$0 \to X(\pi_0(\ker(\phi))) \to \hat{X}(T') \xrightarrow{\hat{\phi}} \hat{X}(T) \to \hat{X}(\ker(\phi)^0) \to 0$$

where $\ker(\phi)^\circ$ is the neutral component of $\ker(\phi)$. For examples, if $T = T_0$, $T' = T_0/T_v$, $\xi \in X(\pi_0(T_v))$, then $\hat{\phi}_v\xi = 0$. Thus since we restrict ourselves to I_0-equivariant local systems without weights, we always have $\hat{\phi}_v\xi = 0$. In the case of $\mathcal{O}_v$ being of type IIIb or IVb for s_α, let $\mathcal{O}_{v'} \neq \mathcal{O}_v$ be the closed orbit in $P_\alpha x_v$ as in Lemma 28. From §4.2.3, §4.2.4 of [MS98] we see $T_0 = T_{v'}\ker(\alpha)$ in type IIIb (resp. $\mathrm{Im}(\check{\alpha}) \subset T_v$ in type IVb), so that $T_{v'}/T_{v'} \cap \ker(\alpha) \xrightarrow{\alpha} \mathbb{G}_m$ is an isomorphism. In particular, $T_v \cap \ker(\alpha)$ is connected. Also, in both two types $T_v \cap \ker(\alpha) = T_{v'} \cap \ker(\alpha)$. Applying Lemma 2.1.5 of *loc. cit.* to surjection $T_0/T_{v'} \to T_0/T_{v'} \cap \ker(\alpha)$, we get

$$0 \to \hat{X}(T_0/T_{v'}) \to \hat{X}(T_0/T_v \cap \ker(\alpha)) \to \hat{X}(\mathbb{G}_m) \to 0.$$

As in §4.2.7 of *loc. cit.*, denote by $a_{v'}$ the composition

$$a_{v'} : \hat{X}(T_0/T_v) \to \hat{X}(T_0/T_v \cap \ker(\alpha)) \to \hat{X}(\mathbb{G}_m).$$

In type IIIb, similarly define $a_{v''}$ which will satisfy $a_{v''}(\xi) = -a_{v'}(s\xi)$. Since we are in the situation $\hat{\phi}_v\xi = 0$, from §4.2.7 we always have $a_{v'}(\xi) = -a_{v''}(\xi)$ in type IIIb and $2a_{v'}(\xi) = 0$ in type IVb.

By the same discussion as in §4.1.3 of *loc. cit.*, we have

Proposition 74. *For α any simple affine root, the action of $[\mathcal{L}_{s_\alpha}] \in H$ on M is given by the following formula.*

(29)

$$[\mathcal{L}_{s_\alpha}][\mathcal{L}_{\xi,v}] = \begin{cases} q[\mathcal{L}_{\xi,v}], & \mathcal{O}_v \text{ is of type I for } s_\alpha, \\ [\mathcal{L}_{s_\alpha\xi,v'}], & \mathcal{O}_v \text{ is of type IIa for } s_\alpha, \\ (q-1)[\mathcal{L}_{\xi,v}] + q[\mathcal{L}_{s_\alpha\xi,v'}], & \mathcal{O}_v \text{ is of type IIb for } s_\alpha, \\ [\mathcal{L}_{s_\alpha\xi,v'}] + [\mathcal{L}_{s_\alpha\xi,v''}], & \mathcal{O}_v \text{ is of type IIIa for } s_\alpha, \\ (q-2)[\mathcal{L}_{\xi,v}] + (q-1)([\mathcal{L}_{\xi,v'}] + [\mathcal{L}_{\xi,v''}]), & \mathcal{O}_v \text{ is of type IIIb for } s_\alpha,\ a_{v'}(\xi) = 0, \\ -[\mathcal{L}_{\xi,v}], & \mathcal{O}_v \text{ is of type IIIb for } s_\alpha,\ a_{v'}(\xi) \neq 0, \\ [\mathcal{L}_{\xi,v}] + [\mathcal{L}_{\xi_1,v'}] + [\mathcal{L}_{\xi_2,v'}], & \mathcal{O}_v \text{ is of type IVa for } s_\alpha, \\ (q-1)[\mathcal{L}_{\xi,v}] - [\mathcal{L}_{\xi',v}] + (q-1)[\mathcal{L}_{\bar{\xi},v'}], & \mathcal{O}_v \text{ is of type IVb for } s_\alpha,\ a_{v'}(\xi) = 0, \\ -[\mathcal{L}_{\xi,v}], & \mathcal{O}_v \text{ is of type IVb for } s_\alpha,\ a_{v'}(\xi) \neq 0. \end{cases}$$

In the above, in type IVa, ξ_1, ξ_2 are preimages of the image of ξ in $\hat{X}(T_0/T_v \cap \ker(\alpha))$, c.f. §4.2.6 of *loc. cit.*; in type IVb where $a_{v'}(\xi)$, ξ' is the element other than ξ that has the same image as ξ in $\hat{X}(T_0/T_v \cap \ker(\alpha))$, and $\bar{\xi} \in \hat{X}(T_0/T_{v'})$ is the unique element that maps to the image of ξ in $\hat{X}(T_0/T_v \cap \ker(\alpha))$.

Proof. It follows essentially from the results in [MS98, §4.1.3]. The key observation here is that the computation of simple Hecke action in *loc. cit.* only uses information about the Levi subgroup of the parahoric subgroup and the stabilizer of the orbit. We demonstrate this by carrying it out when $\mathcal{O}_v$ is of type IIb for s_α, following §4.3.7 and §4.3.9 of *loc. cit.*.

We chosen an isomorphism $u_{\pm\alpha} : k \simeq U_{\pm\alpha}$ and defined element $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ as in §4.1.1 of *loc. cit.*, representing s_α. In the following, we choose orbit representatives $\tilde{v} \in \mathcal{O}_v$ such that the stabilizers $H_{\alpha,\tilde{v}}$ are exactly as in Lemma 28. Note that they are not necessarily the same as the representatives $\dot{v}$ we used earlier. In the case of $\mathcal{O}_v$ being type IIb for s_α, $P_\alpha\tilde{v} = \mathcal{O}_v \sqcup \mathcal{O}_{v'} = I_0\tilde{v} \sqcup I_0\tilde{v}'$ in which $I_0\tilde{v}$ is open and $I_0\tilde{v}'$ has codimension one. Also, $n_\alpha\tilde{v} \in I_0\tilde{v}'$ and $n_\alpha u_\alpha(x)\tilde{v} \in I_0\tilde{v}$ if $x \neq 0$. Here we also choose $\tilde{v}' = n_\alpha\tilde{v}$. Thus $I_0 s_\alpha I_0\tilde{v} = P_\alpha\tilde{v}$. We have isomorphisms

$$\begin{aligned} I_0 s_\alpha I_0 \times^{I_0} I_0\tilde{v} &\simeq \{(pI_0, y) \in I_0 s_\alpha I_0/I_0 \times P_\alpha\tilde{v} \mid p^{-1}y \in I_0\tilde{v}\} \\ &\simeq \{(x, y) \in k \times P_\alpha\tilde{v} \mid n_\alpha^{-1}u_\alpha(-x)y \in I_0\tilde{v}\}, \end{aligned}$$

where the first isomorphism is given by $(p, a) \mapsto (pI_0, pa)$, the second is given by writing $pI_0 = u_\alpha(x)n_\alpha I_0$ and preserving y. The multiplication map μ becomes projection $(x, y) \mapsto y$. By [MS98, Lemma 4.1.5], $U_{-\alpha}\tilde{v} \subset P_\alpha^+\tilde{v}$. From the above discussions we can see $n_\alpha^{-1}u_\alpha(-x)y \in I_0\tilde{v}$ if and only if either $y \in I_0\tilde{v}'$ or $y \in u_\alpha(k\backslash\{x\})P_\alpha^+ T_0\tilde{v}$.

Now given $\mathcal{L}_{\xi,v}$ on $I_0\tilde{v}$, denote by $\widetilde{\mathcal{L}}$ the local system on $I_0 s_\alpha I_0 \times^{I_0} I_0\tilde{v}$ whose pull back to $I_0 s_\alpha I_0 \times I_0\tilde{v}$ is $\underline{\overline{\mathbb{Q}_\ell}} \boxtimes \mathcal{L}_{\xi,v}$. Then

$$[\mathcal{L}_{s_\alpha}][\mathcal{L}_{\xi,v}] = \sum_i (-1)^i [\mathscr{H}^i(\mu_! \widetilde{\mathcal{L}}|_{I_0 v})] + \sum_i (-1)^i [\mathscr{H}^i(\mu_! \widetilde{\mathcal{L}}|_{I_0 v'})].$$

We first compute the first sum. Here $\mu^{-1}(I_0\tilde{v}) = \{(x, u_\alpha(x')P_\alpha^+ T_0\tilde{v}) \mid x \neq x'\}$. By the definition of type IIb, we have $\mathrm{Stab}_{I_0}(\tilde{v}) \subset I_0 \cap \mathrm{Ad}_{n_\alpha} I_0 = T_0 P_\alpha^+$. Thus we have isomorphism $k \times T_0 P_\alpha^+\tilde{v} \simeq I_0\tilde{v}$ given by $(x', y') \mapsto u_\alpha(x')y'$. Denote $z = x - x'$, then $\mu^{-1}(I_0\tilde{v}) \simeq k^\times \times I_0\tilde{v}$ on

which $\mu(z,y)=y$. Denote by pr_2 the projection to second factor. Then $\widetilde{\mathcal{L}}|_{\mu^{-1}(I_0\tilde{v})}\simeq \mathrm{pr}_2^*\mathcal{L}_{\xi,v}$. We conclude

$$\sum_i(-1)^i[\mathscr{H}^i(\mu_!\widetilde{\mathcal{L}}|_{I_0\tilde{v}})]=\sum_i(-1)^i[\mathscr{H}^i(\mathrm{pr}_{2,!}\mathrm{pr}_2^*\mathcal{L}_{\xi,v})]=\sum_i(-1)^iH_c^i(\mathbb{G}_m)[\mathcal{L}_{\xi,v}]=(q-1)[\mathcal{L}_{\xi,v}].$$

Next we compute the second sum over $I_0\tilde{v}'$. Here $x\in k, y\in I_0\tilde{v}'$. We have $\mu^{-1}(I_0\tilde{v}')\simeq k\times I_0\tilde{v}'$. Denote by pr_2 the projection to the second factor. Write $y=tu\tilde{v}'$ where $t\in T_0$, $u\in I_0^+$. Then $n_\alpha^{-1}u_\alpha(-x)y\in P_\alpha^+\mathrm{Ad}_{n_\alpha}(t)\tilde{v}$. The image of (x,y) in T_0/T_v is $\mathrm{Ad}_{n_\alpha}(t)$. Thus the restriction of $\widetilde{\mathcal{L}}$ to $\mu^{-1}(I_0\tilde{v}')\simeq k\times I_0\tilde{v}'$ is $\mathrm{pr}_2^*\mathcal{L}_{s_\alpha\xi,v'}$. We obtain

$$\sum_i(-1)^i[\mathscr{H}^i(\mu_!\widetilde{\mathcal{L}}|_{I_0\tilde{v}'})]=\sum_i(-1)^i[\mathscr{H}^i(\mathrm{pr}_{2,!}\mathrm{pr}_2^*\mathcal{L}_{s_\alpha\xi,v'})]=\sum_i(-1)^iH_c^i(\mathbb{A}^1)[\mathcal{L}_{s_\alpha\xi,v'}]=q[\mathcal{L}_{s_\alpha\xi,v'}].$$

This completes the computation for type IIb. □

To completely determine the Hecke of H on M, we also need to describe the action of $[\mathcal{L}_w]$ for $w\in\Omega$.

Lemma 75. *For $w\in\Omega$, $[\mathcal{L}_w][\mathcal{L}_{\xi,v}]=[\mathcal{L}_{w\xi,wv}]$, where w acts on ξ via conjugation on $\hat{X}(T_0)$ and on $\mathscr{V}$ via left multiplication.*

Proof. Let $\dot{w}\in N_{LG}(LT_0)\cap N_{LG}(I_0)$ be a lift of w. Note that

$$I_0\dot{w}I_0\times^{I_0}\mathcal{O}_v=\dot{w}\times^{I_0}I_0x_v\simeq \dot{w}I_0x_v=I_0\dot{w}x_v.$$

Thus the convolution action is just the direct image of the left multiplication by $\dot{w}$.

The above last equality $\dot{w}I_0x_v=I_0\dot{w}x_v$ induces a conjugation action of $\dot{w}$ on I_0, thus an action on the abelian quotient $T_0\simeq I_0/I_0^+$. So the action of w on ξ is the conjugation action on a character of T_0 that represents ξ. □

8.4. **The polynomials $b_{\eta,u;\xi,v}$ and $c_{\eta,u;\xi,v,i}$.** Consider the partial ordering on I_0-orbits by closure relation and write $u<v$ if $\mathcal{O}_u\subset\overline{\mathcal{O}}_v$ and $u\neq v$.

According to (53), we have $\mathbb{D}_\delta(\mathrm{IC}_{\xi,v})\cong\mathrm{IC}_{-\xi,v}(\delta(v))$ and $\mathbb{D}_\delta(j_{v,!}\mathcal{L}_{\xi,v})\cong j_{v,*}\mathcal{L}_{-\xi,v}[2\delta(v)](\delta(v))$. It follows that

$$D_\delta[\mathcal{L}_{\xi,v}]=q^{-\delta(v)}[\mathcal{L}_{-\xi,v}]+\sum_{u<v}b_{\eta,u;\xi,v}[\mathcal{L}_{\eta,u}]\in M \tag{30}$$

and

$$[\mathscr{H}^i\mathrm{IC}_{\xi,v}]=\delta_{i,-\delta(v)}[\mathcal{L}_{\xi,v}]+\sum_{u<v}c_{\eta,u;\xi,v,i}[\mathcal{L}_{\eta,u}]\in M \tag{31}$$

where $b_{\eta,u;\xi,v},c_{\eta,u;\xi,v,i}\in\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$. The polynomials $b_{\eta,u;\xi,v}$ can be viewed as an affine analog of the $R_{\gamma,\delta}$ in [LV83] and the $c_{\eta,u;\xi,v,i}$ are related to the Affine Kazhdan-Lusztig-Vogan polynomials in Section 9.2.

Apply the bijection $h:M'\cong M$ to equality $D[\mathrm{IC}_{\xi,v}]=q^{-\delta(v)}[\mathrm{IC}_{-\xi,v}]$ and plug in the relations (30), (31). By comparing coefficients of $[\mathcal{L}_{\eta,u}]$ for $u<v$, we obtain

$$\begin{aligned}&\sum_i(-1)^ic_{\eta,u;-\xi,v,i}-q^{\delta(v)-\delta(u)}\sum_i(-1)^i\bar{c}_{-\eta,u;\xi,v,i}\\&=(-1)^{\delta(v)}q^{\delta(v)}b_{\eta,u;\xi,v}+q^{\delta(v)}\sum_{u<z<v}b_{\eta,u;\zeta,z}\sum_i(-1)^i\bar{c}_{\zeta,z;\xi,v,i}\end{aligned} \tag{32}$$

where $\bar{c} \in \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ means inverting $q^{\frac{1}{2}}$.

Lemma 76. *Assume $c_{\eta,u;\xi,v,i} \in \mathbb{N}q^{\frac{1}{2}(i+\delta(v))}$. Then the coefficients $c_{\eta,u;\xi,v,i}$'s can be expressed as a product $a\prod_j b_j$ where $a \in \mathbb{Z}q^{\mathbb{Z}}$, b_j or b_j^{-1} is a monomial of $q^{\frac{1}{2}}$ in some $b_{\eta,u;\xi,v}$.*

Proof. It follows from the same argument as in §3.4.3 of *loc. cit.* by comparing degrees of q on both sides of (32) and descending induction on $\delta(u)$. □

8.5. **An algorithm.** We first describe an algorithm that computes $b_{\eta,u;\xi,v}$. By Lemma 76 and Theorem 81 to be proved later, this also provides a way to compute $c_{\eta,u;\xi,v,i}$ using (32).

Let $\mathcal{O}_v$ be an I_0-orbit, $\xi \in X(\pi_0(T_0))$. If $\mathcal{O}_v = \overline{\mathcal{O}}_v$ is closed, clearly $b_{\eta,u;\xi,v} = 0$. Otherwise by Lemma 31, there exists simple affine root α such that $\mathcal{O}_v$ is of type b for s_α. In view of Corollary 27.(i), we can take induction on $\dim \mathcal{O}_v$.

8.5.1. Case 1: $\mathcal{O}_v$ is of type IIb for some s_α. By Lemma 28 $P_\alpha \mathcal{O}_v = \mathcal{O}_v \sqcup \mathcal{O}_{v'}$ in which $\mathcal{O}_v$ is open. We adopt the trick in [MS98, §5.2]. From (29), we have relation

$$[\mathcal{L}_{s_\alpha}][\mathcal{L}_{\xi,v}] = (q-1)[\mathcal{L}_{\xi,v}] + q[\mathcal{L}_{s\xi,v'}].$$

Apply D to the above. Similar as Proposition 3.3.2, Lemma 4.4.7 of *loc. cit.*, we have

$$D[\mathcal{L}_{s_\alpha}] = q^{-1}([\mathcal{L}_s] + (1-q)), \quad D([\mathcal{L}_{s_\alpha}][\mathcal{L}_{\xi,v}]) = (D[\mathcal{L}_{s_\alpha}])(D[\mathcal{L}_{\xi,v}]).$$

Note that D inverts q. We obtain

$$(q^{-1}[\mathcal{L}_{s_\alpha}] + q^{-1} - 1)D[\mathcal{L}_{\xi,v}] = (q^{-1}-1)D[\mathcal{L}_{\xi,v}] + q^{-1}D[\mathcal{L}_{s\xi,v'}],$$

i.e. $[\mathcal{L}_{s_\alpha}]D[\mathcal{L}_{\xi,v}] = D[\mathcal{L}_{s\xi,v'}]$. Since in the affine Hecke algebra, $[\mathcal{L}_{s_\alpha}]^{-1} = q^{-1}[\mathcal{L}_{s_\alpha}] + q^{-1} - 1$, we get

$$D[\mathcal{L}_{\xi,v}] = (q^{-1}[\mathcal{L}_{s_\alpha}] + q^{-1} - 1)D[\mathcal{L}_{s\xi,v'}]. \tag{33}$$

Using induction hypothesis and (29), we can compute $b_{\eta,u;\xi,v}$ for any $u < v$.

8.5.2. Case 2: $\mathcal{O}_v$ is not of type IIb for any s_α. We adopt the trick in §7.2 of *loc. cit.*. Let J be the set of simple affine roots such that $\mathcal{O}_v$ is of type IIIb or IVb for s_α, $\alpha \in J$. By Lemma 29, $\psi_v\alpha = -\alpha$ for any $\alpha \in J$ and $\psi_v\alpha > 0$ for any $\alpha \in \Delta_{\text{aff}} - J$. Note that the action of ψ_v on affine roots is the composition of θ with w_v, and θ preserves Δ_{aff} by assumption. Then J is a proper subset of simple affine roots, since otherwise w_v would map Δ_{aff} to $-\Delta_{\text{aff}}$, which is impossible. Let $P = P_J$ be the standard parahoric subgroup associated to J, with pro-unipotent radical P^+ and Levi subgroup L_P. Note that P, P^+, L_P are all ψ_v-stable. The set J gives the simple roots of L_P.

Lemma 77. *We have $Px_v = \overline{\mathcal{O}}_v$.*

Proof. On the one hand, for any $\alpha \in J$ since $\mathcal{O}_v$ is of type b for s_α, $\mathcal{O}_v$ is open in $P_\alpha x_v$. Thus $\overline{P_\alpha x_v} = \overline{\mathcal{O}}_v$ is stable under P_α-action. Therefore $\overline{\mathcal{O}}_v$ is stable under P-action. We get $Px_v \subset \overline{\mathcal{O}}_v$.

On the other hand, P is stable under ψ_v. By Lemma 30 and the same proof of Lemma 31 for Iwahori subgroup, $\tau(P) = \tau_v(P)n_v = P^{\text{inv}\circ\psi_v,\circ}n_v$ is closed in P and further closed in LG. Thus $\tau(Px_v)$ is closed in $\tau(LX)$ and Px_v is closed in LX. □

In view of the above, we can define quotient map

$$\pi : \overline{\mathcal{O}}_v = Px_v \simeq P/P^{\psi_v} \to L_P/L_P^{\psi_v}.$$

We can see π is smooth with fibers isomorphic to $P^+/(P^+)^{\psi_v}$. It induces a bijection between I_0-orbits on $\overline{\mathcal{O}}_v$ to the orbits of $B_{L_P} := I_0 \cap L_P$ on $X_{L_P} := L_P/L_P^{\psi_v}$. Observe that X_{L_P} is a finite dimensional symmetric variety, whose set of B_{L_P}-orbits are parametrized by $\{\pi u \mid u \leq v\}$. We denote $\mathcal{O}_{\pi u}$ the B_{L_P}-orbit corresponding to πu and $\delta(\pi u) = \dim \mathcal{O}_{\pi u}$. We have the following reduction to the finite dimensional case:

Lemma 78. (i) $\pi^*\mathcal{L}_{\eta,\pi u} = \mathcal{L}_{\eta,u}$, $u \leq v$, $\eta \in \hat{X}(T_0/T_u)$,
(ii) $\mathrm{IC}_{\xi,v} \cong \pi^*\mathrm{IC}_{\xi,\pi v}[\delta(v) - \delta(\pi v)]$,
(iii) $\mathbb{D}_\delta(j_{v,!}\mathcal{L}_{\xi,v}) \cong \pi^*(\mathbb{D}(j_{\pi v,!}\mathcal{L}_{\xi,\pi v}))\langle 2\delta(v) - 2\delta(\pi v)\rangle$.

Proof. Part (i) follows from $T_u = T_0^{\psi_v} = T_{\pi u}$ and Part (iii) follows from (53). Since we are using the $*$-adapted perverse t-structures on placid schemes (thus, $*$-pull back along smooth maps are t-exact), we have $(j_v)_{!*}\mathcal{L}_{\xi,v} \cong \pi^*((j_{\pi v})_{!*}\mathcal{L}_{\xi,\pi v})$ and it follows that

$$\mathrm{IC}_{\xi,v} = (j_v)_{!*}\mathcal{L}_{\xi,v}[\delta(v)] \cong \pi^*((j_{\pi v})_{!*}\mathcal{L}_{\xi,\pi v}[\delta(\pi v)])[\delta(v) - \delta(\pi v)] = \pi^*\mathrm{IC}_{\xi,\pi v}[\delta(v) - \delta(\pi v)].$$

Part (ii) follows. □

In view of the above, the computation of $b_{\eta,u;\xi,v}, c_{\eta,u;\xi,v,i}$ can be reduced to the finite dimensional symmetric variety X_{L_P}, which has been done in [MS98]. As a corollary of the algorithm we obtain the following property of $b_{\eta,u;\xi,v}$:

Lemma 79. *$q^{\delta(v)}b_{\eta,u;\xi,v}$ is a polynomial in q with coefficients in $\mathbb{Z}$ of degree at most $\delta(v) - \delta(u)$.*

Proof. We do induction on $\delta(v)$. When $u \geq v$, $b_{\eta,u;\xi,v} = 0$. Suppose the statement holds for all $b_{\eta',u';\xi',v'}$ with $\delta(v') < \delta(v)$.

When $\mathcal{O}_v$ is of type IIb for s_α, by Lemma 28, we have $\delta(v) - \delta(v') = \mathrm{codim}_{\mathcal{O}_v/\mathcal{O}_{v'}} = 1$. Then the desired claim follows from (33), the induction procedure, and the observation that all the coefficients appearing in the affine Hecke action as in the list (29) are polynomials in q of degree less or equal to 1.

When $\mathcal{O}_v$ is not of type IIb for any s_α, we have seen before that the problem can be reduced to a symmetric variety of a generalized Levi subgroup. Then the statement in [LV83, Theorem 1.10] implies that $q^{\delta(\pi v)}b_{\eta,\pi u;\xi,\pi v} \in \mathbb{Z}[q]$ is of degree at most $\delta(\pi v) - \delta(\pi u)$. According to A.4 we have

$$\delta(\pi v) - \delta(\pi u) = \mathrm{codim}_{\overline{\mathcal{O}}_{\pi u}/\overline{\mathcal{O}}_{\pi v}} = \mathrm{codim}_{\overline{\mathcal{O}}_u/\overline{\mathcal{O}}_v} = \delta(v) - \delta(u)$$

and together with Lemma 78 (iii) we obtain

$$q^{\delta(v)}b_{\eta,u;\xi,v} = q^{\delta(\pi v)}b_{\eta,\pi u;\xi,\pi v} \in \mathbb{Z}[q]$$

is of degree at most $\delta(v) - \delta(u) = \delta(\pi v) - \delta(\pi u)$.

□

8.5.3. *Applications to spherical orbits.* Note that the above results for Iwahoric orbits also provide an algorithm to compute the multiplicities for L^+G-equivariant IC-complexes. Precisely, let $\mathcal{L}_{\chi,\lambda}$ be a L^+G-equivariant local system on L^+G-orbit LX_λ, where χ is a representation of the component group of stabilizers in L^+G on LX_λ. Consider the associated IC-complex

$$\mathrm{IC}_{\chi,\lambda} := (j_\lambda)_{!*}\mathcal{L}_{\chi,\lambda}[\delta(\lambda)] \in \mathrm{Perv}_\delta(L^+G\backslash LX)$$

We can write down similar formula as (31) in the Grothendieck group of mixed L^+G-constructible sheaves:

$$[\mathscr{H}^i\mathrm{IC}_{\chi,\lambda}] = \delta_{i,-\delta(\lambda)}[\mathcal{L}_{\chi,\lambda}] + \sum_{\mu<\lambda} C_{\chi',\mu;\chi,\lambda,i}[\mathcal{L}_{\chi',\mu}] \tag{34}$$

where $C_{\chi',\mu;\chi,\lambda,i} \in \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$, and each $\mathcal{L}_{\chi',\mu}$ is a L^+G-equivariant local system on LX_μ.

Let h be the map from the Grothendieck group of mixed L^+G-constructible sheaves into that of mixed I_0-constructible sheaves and denote $h[\mathcal{L}_{\chi',\mu}] = \sum_{(\eta,u)\subset(\chi',\mu)}[\mathcal{L}_{\eta,u}]$. Denote by $\mathcal{O}_{v_\lambda}$ be unique open I_0-orbit in LX_λ and let $\mathcal{L}_{\xi_\chi,v_\lambda} = \mathcal{L}_{\chi,\lambda}|_{\mathcal{O}_{v_\lambda}}$. Then

$$\mathrm{IC}_{\chi,\lambda} = \mathrm{IC}_{\xi_\chi,v_\lambda}. \tag{35}$$

By comparing (31) with the image under h of (34), we see the left hand sides are the same. The matching of coefficients on the right hand sides provides relation

$$c_{\eta,u;\xi_\chi,v_\lambda,i} = \begin{cases} \delta_{i,-\delta(\lambda)}, & (\eta,u)\subset(\chi,\lambda), \\ C_{\chi',\mu;\chi,\lambda,i}, & (\eta,u)\subset(\chi',\mu),\ \mu<\lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

From the above, we obtain

Lemma 80. *$c_{\eta,u;\xi_\chi,v_\lambda,i}$ depends only on the L^+G-orbit of u.*

In view of the above, to compute $C_{\chi',\mu;\chi,\lambda,i}$, it suffices to take the open I_0-orbit $I_0v_\lambda \subset LX_\lambda$ and any I_0-orbit $I_0u \subset LX_\mu$, which uniquely determines $(\xi_\chi, v_\lambda) \subset (\chi,\lambda)$ and $(\eta,u)\subset(\chi',\mu)$, then compute $c_{\eta,u;\xi_\chi,v_\lambda,i}$.

9. Parity vanishing and Poincaré polynomials of IC-complexes

In this section, we prove the parity vanishing of equivariant IC-complexes on the spherical and Iwahori orbits. We will first prove for Iwahori orbits following the strategy of Mars-Springer [MS98] in the finite dimensional situation, then deduce the result for spherical orbits. Along the way, we provide an algorithm that computes the Poincaré polynomials of Iwahori-equivariant IC-complexes in an inductive procedure. We keep the same notations and assumptions as in §4.1.

9.1. **Purity and parity vanishing.** Let $\mathrm{IC}_{\xi,v} := (j_v)_{!*}\mathcal{L}_{\xi,v}[\delta(v)]$ (resp. $\mathrm{IC}_{\chi,\lambda} := (j_\lambda)_{!*}\mathcal{L}_{\chi,\lambda}[\delta(\lambda)]$) be the IC-complex associated to the I_0-equivariant local system $\mathcal{L}_{\chi,v}$ on $\mathcal{O}_v$ (resp. L^+G-equivariant local system $\mathcal{L}_{\chi,\lambda}$ on LX_λ). According to Section 8.2, each $\mathrm{IC}_{\xi,v}$ has a canonical Weil structure $\Phi = \{\Phi_n,\ \text{for for } n \text{ divisible by } n_0\}$, where $\Phi_n : (F^n)^*\mathrm{IC}_{\xi,v} \cong \mathrm{IC}_{\xi,v}$ are systems of isomorphisms satisfying $(\Phi_n)^m = \Phi_{nm}$. The discussion in *loc. cit.* also implies that $\mathrm{IC}_{\chi,\lambda}$ carries a canonical Weil structure.

We recall the notion of pointwise purity (see Definition 109 for a more general discussion). A perverse sheaf $(\mathcal{F},\Phi) \in \mathrm{Perve}_\delta(I_0\backslash LX)^{\mathrm{Weil}}$ is called $*$-pointwise pure (resp. !-pointwise

pure) of weight w if for any I_0-orbit $j_v : \mathcal{O}_v \subset LX$ and a point $x \in (\mathcal{O}_v)^{F^n}$, the eigenvalues of Φ^n on the stalk $\mathscr{H}^m(j_v^*\mathcal{F})_x$ (resp. $\mathscr{H}^m(j_v^!\mathcal{F})_x$) are algebraic numbers in $\overline{\mathbb{Q}}_\ell^\times$ all of whose complex conjugates have absolute value $(q^n)^{\frac{w+m}{2}}$. It is called pointwise pure if it is both $*$ and !-pointwise pure. We have similar notions of pointwise purity for $\mathrm{Perv}_\delta(G(\mathcal{O})\backslash LX)^{\mathrm{Weil}}$.

We have the following key pointwise purity result.

Theorem 81. *The complex* $\mathrm{IC}_{\xi,v}$ *(resp.* $\mathrm{IC}_{\chi,\lambda}$*) is pointwise pure of weight* $\delta(v)$ *(resp.* $\delta(\lambda)$*).*

Proof. Note that the local system $\mathcal{L}_{\xi,v}$ is pointwise pure of weight 0 in the sense of Definition 109 (2) and hence by Example 111 the IC-complex $\mathrm{IC}_{\xi,v} = (j_v)_{!*}\mathcal{L}_{\xi,v}[\delta(v)]$ is pure of weight $\delta(v)$ in the sense of Definition 109 (i). Then the existence of contracting slices in Proposition 47 and Proposition 58 together with the criterion in Proposition 113 imply $\mathrm{IC}_{\xi,v}$ is $*$-pointwise pure of weight $\delta(v)$.

On the other hand, Lemma 110 implies $\mathbb{D}_\delta(\mathrm{IC}_{-\xi,v})$ is !-pointwise pure of weight $-\delta(v)$ and the isomorphism $\mathbb{D}_\delta(\mathrm{IC}_{-\xi,v})(-\delta(v)) \cong \mathrm{IC}_{\xi,v}$ implies $\mathrm{IC}_{\xi,v}$ is !-pointwise pure of $-\delta(v) + 2\delta(v) = \delta(v)$.

The same proof works for the case $\mathrm{IC}_{\chi,\lambda}$. □

We are ready to prove the parity vanishing of IC-complexes.

Theorem 82. *For any* $(\xi, v), (\eta, u) \in \mathscr{D}$*, we have*

(i) $c_{\eta,u;\xi,v,i} \in \mathbb{N}q^{\frac{1}{2}(i+\delta(v))}$*. Moreover,* $c_{\eta,u;\xi,v,i} = 0$ *if* $i + \delta(v)$ *is odd.*
(ii) $\mathscr{H}^i\mathrm{IC}_{\xi,v} = 0$ *if* $i + \delta(v)$ *is odd.*
(iii) $\mathscr{H}^i\mathrm{IC}_{\chi,\lambda} = 0$ *if* $i + \delta(\lambda)$ *is odd.*

Proof. (i): The first claim follows from the $*$-pointwise purity of $\mathrm{IC}_{\xi,v}$ in Theorem 81. The second vanishing property follows from Lemma 76 and Lemma 79

(ii): This follows from the definition (31).

(iii): It follows from (ii) and (35). □

Remark 83. The I_0-orbits on the closed orbit $L^+G/L^+K \subset LX$ correspond one-to-one to the B_0-orbits on $X = G/K$. Thus Theorem 82 is a generalization of results of Lusztig-Vogan [LV83] and Mars-Springer [MS98].

9.2. **Affine Kazhdan-Lusztig-Vogan polynomials.** For I_0-equivariant local systems $\mathcal{L}_{\eta,u}$ and $\mathcal{L}_{\xi,v}$, define Poincaré polynomial

$$P_{\eta,u;\xi,v} := \sum_i [\mathcal{L}_{\eta,u}, \mathscr{H}^i(j_{!*}\mathcal{L}_{\xi,v})|_{\mathcal{O}_{\bar{u}}}]q^{\frac{i}{2}}. \tag{37}$$

By Theorem 82.(i), $[\mathcal{L}_{\eta,u}, \mathscr{H}^i(j_{!*}\mathcal{L}_{\xi,v})|_{\mathcal{O}_{\bar{u}}}]q^{\frac{i}{2}} = c_{\eta,u;\xi,v,i-\delta(v)}$. Thus

$$P_{\eta,u;\xi,v} = \sum_i c_{\eta,u;\xi,v,i-\delta(v)}.$$

Here we take convention that $c_{\xi,v;\xi,v,i} = \delta_{i,-\delta(v)}$, and $c_{\eta,u;\xi,v,i} = 0$ if u is not in the closure of $\mathcal{O}_v$ or u is in the closure of $\mathcal{O}_v$ but $\mathcal{L}_{\eta,u}$ is not a subquotient of $\mathcal{L}_{\xi,v}|_{\mathcal{O}_{\bar{u}}}$. Then

$$h[j_{!*}\mathcal{L}_{\xi,v}] = \sum_{\eta,u} P_{\eta,u;\xi,v}[\mathcal{L}_{\eta,u}].$$

Theorem 84. *For any pair $(\eta,u),(\xi,v)$, $P_{\eta,u;\xi,v}$ is a polynomial in q with non-negative integer coefficients. It is the unique family of polynomials in $q^{\frac{1}{2}}$ satisfying the following conditions:*

(i) $P_{\xi,v;\xi,v}=1$.
(ii) *If* $\mathcal{O}_u\neq\mathcal{O}_v$, $\deg_q(P_{\eta,u;\xi,v})\leq\frac{1}{2}(\delta(v)-\delta(u)-1)$.
(iii) $C_{\xi,v}:=\sum_{\eta,u}P_{\eta,u;\xi,v}[\mathcal{L}_{\eta,u}]$ *satisfies* $D_\delta C_{\xi,v}=q^{-\delta(v)}C_{-\xi,v}$.

Remark 85. Note that $\delta(v)-\delta(u)$ is equal to the codimension of $\overline{\mathcal{O}}_u\subset\overline{\mathcal{O}}_v$ which is independent of the choice of the dimension theory δ.

Proof. By Theorem 82, we know $P_{\eta,u;\xi,v}$ satisfies all the above except (ii). By the property of intermediate extension, $c_{\eta,u;\xi,v,i-\delta(v)}\neq 0$ only if $\mathcal{O}_u\subset\overline{\mathcal{O}}_v$ and $\delta(u)<-(i-\delta(v)$, so that $\frac{1}{2}i\leq\frac{1}{2}(\delta(v)-\delta(u)-1)$. Then (ii) follows from Theorem 82.(i).

Conversely, suppose $P_{\eta,u;\xi,v}$ is a family of polynomials in $q^{\frac{1}{2}}$ satisfying (i)-(iii). Denote by $\overline{P_{\eta,u;\xi,v}}$ the polynomial in $q^{\frac{-1}{2}}$ obtained from $P_{\eta,u;\xi,v}$ via $q^{\frac{1}{2}}\mapsto q^{\frac{-1}{2}}$. By the same discussion as (32), we obtain

$$P_{\eta,u;-\xi,v}-q^{\delta(v)-\delta(u)}\overline{P_{-\eta,u;\xi,v}}=q^{\delta(v)}\sum_{u<z<v}b_{\eta,u;\zeta,z}\overline{P_{\zeta,z;\xi,v}}. \tag{38}$$

By condition (ii), the degrees of q in $P_{\eta,u;-\xi,v}$ range from 0 to $\frac{1}{2}(\delta(v)-\delta(u)-1)$, while the degrees of q in $q^{\delta(v)-\delta(u)}\overline{P_{-\eta,u;\xi,v}}$ range from $\frac{1}{2}(\delta(v)-\delta(u)+1)$ to $\delta(v)-\delta(u)$. Thus (38) determines $P_{\eta,u;\xi,v}$ by doing induction on $\delta(v)-\delta(u)$ with base case given by condition (i). □

9.3. **Relative Kostka-Foulkes polynomials.** We can deduce similar results for L^+G-orbits from the above. We resume the discussion in §8.5.3. For L^+G-equivariant local systems $\mathcal{L}_{\chi,\lambda}$, we defined coefficients $C_{\chi',\mu;\chi,\lambda,i}$ parallel to $c_{\eta,u;\xi,v,i}$. Recall we denoted the restriction of $\mathcal{L}_{\chi,\lambda}$ to the open I_0-orbit in LX_λ by $\mathcal{L}_{\xi_\chi,v_\lambda}$.

Similarly, we define $B_{\chi',\mu;\chi,\lambda}\in\mathbb{Z}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]$ by

$$D_\delta[\mathcal{L}_{\chi,\lambda}]=q^{-\delta(\lambda)}[\mathcal{L}_{-\chi,\lambda}]+\sum_{\mu<\lambda}B_{\chi',\mu;\chi,\lambda}[\mathcal{L}_{\chi',\mu}].$$

We denote by $(\xi,v)\subset(\chi,\lambda)$ if $\mathcal{O}_v\subset LX_\lambda$ and $\mathcal{L}_{\xi,v}=\mathcal{L}_{\chi,\lambda}|_{\mathcal{O}_v}$. By breaking L^+G-equivariant local systems into I_0-equivariant ones in the Grothendieck group and comparing with definition of $b_{\eta,u;\xi,v}$, we obtain relation

$$B_{\chi',\mu;\chi,\lambda}=\sum_{(\eta,u)\leq(\xi,v)\subset(\chi,\mu)}b_{\eta,u;\xi,v},\qquad\forall\ (\eta,u)\subset(\chi',\mu), \tag{39}$$

where we take convention $b_{\xi,v;\xi,v}=q^{-\delta(v)}$ and $B_{\chi,\lambda;\chi,\lambda}=q^{-\delta(\lambda)}$. From this we see $B_{\chi',\mu;\chi,\lambda}$ are integer coefficients Laurent polynomials in q with a pole of order at most $\delta(\lambda)$.

We define $P_{\chi',\mu;\chi,\lambda}$ similarly as $P_{\eta,u;\xi,v}$. Parallel to (38), we have recurrence relation

$$P_{\chi',\mu;-\chi,\lambda}-q^{\delta(\lambda)-\delta(\mu)}\overline{P_{-\chi',\mu;\chi,\lambda}}=q^{\delta(\lambda)}\sum_{\mu<\gamma<\lambda}B_{\chi',\mu;\chi'',\gamma}\overline{P_{\chi'',\gamma;\chi,\lambda}}. \tag{40}$$

By the same argument as the Iwahori case, we obtain:

Theorem 86. *For any pair $(\chi', \mu), (\chi, \lambda)$, $P_{\chi',\mu;\chi,\lambda}$ is a polynomial in q with non-negative integer coefficients. It is the unique family of polynomials in $q^{\frac{1}{2}}$ satisfying the following conditions:*

(i) $P_{\chi,\lambda;\chi,\lambda} = 1$.
(ii) *If* $\mu \neq \lambda$, $\deg_q(P_{\chi',\mu;\chi,\lambda}) \leq \frac{1}{2}(\delta(\lambda) - \delta(\mu) - 1)$.
(iii) $C_{\chi,\lambda} := \sum_{\chi',\mu} P_{\chi',\mu;\chi,\lambda}[\mathcal{L}_{\chi',\mu}]$ *satisfies* $D_\delta C_{\chi,\lambda} = q^{-\delta(\lambda)} C_{-\chi,\lambda}$.

10. Applications

We discuss applications of our main results to relative Langlands duality [BZSV].

10.1. **Semisimplicity criterion.** In the group case, an important application of the parity vanishing of IC-complexes is the semisimplicity of the Satake category $\mathrm{Perv}(L^+G\backslash \mathrm{Gr})$. Using the parity vanishing in Theorem 82 and [CN24] , we prove the following semisimplicity criterion and Langlands dual description for the relative Satake category $\mathrm{Perv}(L^+G\backslash LX)$:

Theorem 87. *Assume the codimensions of L^+G-orbits in the same connected component of LX are even.*

(i) *The relative Satake category* $\mathrm{Perv}_\delta(L^+G\backslash LX)$ *is semisimple.*
(ii) *Assume further that the the L^+G-stabilizers on LX are connected. Then there is an equivalence of abelian categories*

$$\mathrm{Perv}_\delta(L^+G\backslash LX) \cong \mathrm{Rep}(\check{G}_X)$$

where the $\mathrm{Rep}(\check{G}_X)$ *is the category of finite dimensional complex representations of the dual group $\check{G}_X$ of X* [GN10].

Proof. (i) The standard proof in the group case, see, e.g., [Zhu17, Proposition 5.1.1], only uses the parity vanishing of IC-complexes and parity of the dimension of spherical orbits in each connected component, thus is applicable in our setting.

(ii) The assumption implies $\mathrm{Perv}_\delta(L^+G\backslash LX)$ is a semisimple abelian category whose irreducible objects are IC-complexes with trivial local systems. [CN24, Corollary 13.8.] implies that $\mathrm{Perv}(L^+G\backslash LX)$ contains a full subcategory $\mathrm{Rep}(\check{G}_X)$ consisting of IC-complexes with trivial coefficients supported on the orbit closures $\overline{LX}_\lambda$ with $\lambda \in \Lambda_S^+$ of the form $-\theta(\mu) + \mu$, $\mu \in \Lambda_T$. Now the desired claim follows from the Lemma 90 below. □

Example 88. Assume LX is connected (for example, when G is simply connected). According to the codimension formula in Proposition 64, the evenness assumption in the theorem is satisfied if and only if for $LX_\lambda \subset \overline{LX}_\mu$ we have $\mathrm{codim}_{\overline{LX}_\lambda}(\overline{LX}_\mu) = \delta(\mu) - \delta(\lambda) = \langle \rho, \mu - \lambda \rangle \in 2\mathbb{Z}$. Using this formula, one can check that when $X = \mathrm{SL}_{2n}/\mathrm{Sp}_{2n}, \mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}, \mathrm{E}_6/\mathrm{F}_4$ with relative dual groups $\check{G}_X = \mathrm{PGL}_{2n}, \mathrm{PGL}_2, \mathrm{PGL}_3$ (the splitting rank cases), both the evenness and connectedness assumptions are satisfied and we obtain the following new instances of abelian relative Satake equivalence:

(1) $\mathrm{Perv}(L^+\mathrm{SL}_n\backslash L(\mathrm{SL}_n/\mathrm{Sp}_n)) \cong \mathrm{Rep}(\mathrm{PGL}_n)$,
(2) $\mathrm{Perv}(L^+\mathrm{Spin}_{2n}\backslash L(\mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1})) \cong \mathrm{Rep}(\mathrm{PGL}_2)$,
(3) $\mathrm{Perv}(L^+\mathrm{E}_6\backslash L(\mathrm{E}_6/\mathrm{F}_4)) \cong \mathrm{Rep}(\mathrm{PGL}_3)$.

Remark 89. The assumption in the theorem is necessary. For example, consider the complex symmetric variety $X = \mathrm{SL}_2/\mathrm{SO}_2$. Using the codimension formula above one can check

that there are orbits with codimension equal to one. On the other hand, from [BF22, theorem 1.8], the category $\mathrm{Perv}_\delta(L^+\mathrm{SL}_2\backslash LX)$ has a full subcategory equivalent to the abelian category $\wedge^\bullet T^*\mathbb{C}^2$-$\mathrm{mod}^{\mathrm{SL}_2}$ of finite dimensional SL_2-equivariant modules over the exterior algebra $\wedge^\bullet T^*\mathbb{C}^2$. In particular, $\mathrm{Perv}(L^+\mathrm{SL}_2\backslash LX)$ is not semisimple. Another way to see this is to use the real-symmetric equivalence

$$\mathrm{Perv}_\delta(L^+\mathrm{SL}_2\backslash LX) \cong \mathrm{Perv}(L^+\mathrm{SL}_2(\mathbb{R})\backslash \mathrm{Gr}_{\mathrm{SL}_2(\mathbb{R})})$$

in [CN24], where the right hand side is the real Satake category for $\mathrm{SL}_2(\mathbb{R})$. There is a real spherical orbit closure in $\mathrm{Gr}_{\mathrm{SL}_2(\mathbb{R})}$ homeomrphic to the real two dimensional pinched torus and the extension by zero $j_!(\mathbb{C}[1])$ along the open orbit provides a non-semisimple object in $\mathrm{Perv}(L^+\mathrm{SL}_2(\mathbb{R})\backslash \mathrm{Gr}_{\mathrm{SL}_2(\mathbb{R})})$.

Lemma 90. *Assume the codimensions of L^+G-orbits in the same connected component of LX are even. Then every element Λ_S is of the form $-\theta(\mu)+\mu$ for some $\mu \in \Lambda_T$.*

Proof. From diagram (7), we can write $\lambda = (-\theta(\mu_1)+\mu_1)+\lambda'$ where $\lambda' \in \Lambda_S \cap Q$ (recall we assume K is connected). Let Δ_L be the set of simple coroots of the Levi $L \subset G$ that centralizes $2\check{\rho}_M$. According to [Nad05, Lemma 2.1.1], we have $-\theta(\Delta_L) = \Delta_L$ and $\lambda' = \sum_{\alpha\in\Delta_L} n_\alpha \alpha$. Note that we have $-\theta(\alpha) \neq \alpha$ for any $\alpha \in \Delta_L$, otherwise $\alpha \in \Lambda_S^+ \cap Q$ and Proposition 64 would imply

$$\mathrm{codim}_{\overline{LX_\alpha}}(\overline{LX}_0) = \langle \rho, \alpha\rangle = 1$$

contracditing the evenness assumption (note that LX_α and LX_0 are in the same component). Thus we can write $\lambda' = \sum n_\alpha(-\theta(\alpha)+\alpha)$ where α runs through representatives of $(-\theta)$-orbits on Δ_L. The lemma follows. □

Remark 91. The converse of Lemma 90 is not true. For example when $X = \mathrm{GL}_n/\mathrm{GL}_p \times \mathrm{GL}_q$ with $n = p+q$ even, every element Λ_S^+ is of the form $-\theta(\mu)+\mu$ but there are orbits with odd codimension $n-1$.

10.2. **Formality of dg extension algebras.** Let $D(L^+G\backslash \mathrm{Gr})$ be the derived Satake category for G and let $D(L^+G\backslash LX)$ be the derived relative Satake category of dg derived category of L^+G-equivariant complexes on LX. Here we use the $*$-sheaves theory, see, Appendix A. We have the Hecke action, denoted by $\star$, of the Satake category $D(L^+G\backslash LG/L^+G)$ on $D(L^+G\backslash LX)$ defined similary as in Section 8.1.1 (replacing I_0 by L^+G). We have the monoidal abelian Satake equivalence

$$\mathrm{Rep}(\check{G}) \cong \mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G): \quad V \to \mathrm{IC}_V.$$

By restricting the action to $\mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G) \cong \mathrm{Rep}(\check{G})$, we obtain a monoidal action of $\mathrm{Rep}(\check{G})$ on $D(L^+G\backslash LX)$. Let $\mathrm{e}_{L^+X} \in D(L^+G\backslash LX)$ be the constant sheaf on the closed L^+G-orbit $LX_0 = L^+X$. Let $\mathrm{IC}_{\mathcal{O}(\check{G})}$ (an ind-object in $\mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G)$) be the image of the regular representation $\mathcal{O}(\check{G})$ under the abelian Satake equivalence. Since $\mathcal{O}(\check{G})$ is a ring object in $\mathrm{Rep}(\check{G})$, the RHom space

$$A_X := R\mathrm{Hom}_{D(L^+G\backslash LX)}(\mathrm{e}_{L^+X}, \mathrm{IC}_{\mathcal{O}(\check{G})} \star \mathrm{e}_{L^+X}) \tag{41}$$

is naturally a dg-algebra, known as the de-equivariantized Ext algebra. The formality conjecture in relative Langlands duality is the following assertion for X a spherical variety(see, e.g., [BZSV, Conjecture 8.1.8]):

Conjecture 92. *The dg-algebra A_X is formal, that is, it is quasi-isomorphic to the graded algebra $A_X \cong \mathrm{Ext}^\bullet_{D(L^+G\backslash LX)}(\mathrm{e}_{L^+X}, \mathrm{IC}_{\mathcal{O}(\check{G})} \star \mathrm{e}_{L^+X})$ with trivial differential.*

Theorem 93. *For X a symmetric variety, Conjecture 92 holds true.*

Proof. The same proof as in Proposition 70 shows that $\mathrm{IC}_{\mathcal{O}(\check{G})} \star \mathrm{e}_{L^+X}$ is a pure complex of weight zero isomorphic to a direct sum of IC-complexes $\mathrm{IC}_{\chi,\lambda}(\frac{\delta(\lambda)}{2})$ shifted by $[2n](n)$. By Theorem 81, the IC-complexes $\mathrm{IC}_{\chi,\lambda}(\frac{\delta(\lambda)}{2})$ are pointwise pure of weight zero and it follows that the Ext group $\mathrm{Ext}^i_{D(L^+G\backslash LX)}(\mathrm{e}_{L^+X}, \mathrm{IC}_{\chi,\lambda}(\frac{\delta(\lambda)}{2})) \cong H^i_{LG^+}(L^+X, j^!\mathrm{IC}_{\chi,\lambda}(\frac{\delta(\lambda)}{2}))$ is pure of weight i (here $j : L^+X \to LX$ is the inclusion). Now the desired formality of A_X follows from the result of [BF08, Section 6.5].

□

11. Disconnected groups

In the proofs from Section 4 to Section 10, the connectedness of K is used only to insure that the points in $LX(k) = X(F)$ land in the image of the natural map $\pi : LG \to LX$ (see Proposition 17 (ii)). In the general case of not necessary connected K, the image $LX_{\mathcal{L}} = \pi(LG) \subset LX$ is an union of components LX, and all the results in the previous sections are valid if we restrict to orbits in $LX_{\mathcal{L}}$. More precisely:

(1) All the results from Section 4 to Section 10 are valid when we consider L^+G or I_0-orbits in $LX_{\mathcal{L}}$.
(2) The formality conjecture Theorem 93 holds true for general K. Indeed, the union of components $LX_{\mathcal{L}} \subset LX$ is LG-invariant and $L^+X \subset LX_{\mathcal{L}}$. It follows that $\mathrm{IC}_V \star \mathrm{e}_{L^+X}$ lands in the full-subcategory $D(L^+G\backslash LX_{\mathcal{L}}) \subset D(L^+G\backslash LX)$ and we have an isomorphism of dg algebras $A_X \cong R\mathrm{Hom}_{D(L^+G\backslash LX_{\mathcal{L}})}(\mathrm{e}_{L^+X}, \mathrm{IC}_{\mathcal{O}(\check{G})} \star \mathrm{e}_{L^+X})$. Now the desired claim follows from part (1) above.

Appendix A. Sheaves on placid schemes

We collect some basic definitions and properties of placid (ind) stacks, dimension theory, and sheaves theory on placid (ind) stacks. The main references for this section are [BKV11, FKV10, Dri06, Ras].

A.1. **Placid schemes and placid stacks.** All schemes are over a fixed algebraically closed field k. When talking about limits and colimits, all the index sets are assumed to be countable.

(1) A placid presentation of a scheme Y is an isomorphism $Y \cong \lim_j Y_j$, where the limit is taken with respect to a filtered set J, each Y_j, $j \in J$ is a scheme of finite type and the transition maps $Y_{j'} \to Y_j$ for $j \to j'$ are affine and smooth. We say that a scheme Y is *placid* if it has an étale covering by schemes admitting placid presentations.
(2) Let $f : X \to Y$ be a finitely presented morphism. Assume $Y \cong \lim_j Y_j$ admits a placid presentation. Then there is an index j_0, and a morphism $f_{j_0} : X_{j_0} \to Y_{j_0}$ of schemes of finite type such that f is the pull-back of f_{j_0}. In particluar, $X \cong \lim_{j \geq j_0} X_{j_0} \times_{Y_{j_0}} Y_j$ is a plaicd presentation of X.
(3) We call a morphism $f : X \to Y$ *pro-smooth* if locally in the étale topology X has a presentation $X = \lim_j X_j$ over Y such that the all the projection $X_j \to Y$ are

smooth and finitely presented and all the transition maps $X_j \to X_{j'}$ are smooth, finitely presented and affine.[8] By definition, a scheme X is placid if and only if there is a pro-smooth morphism $f : X \to Y$ to a finite type scheme Y.

Let $f : X \to Y$ be a pro-smooth morphism. Then Y is placid implies X is placid (see [BKV11, Section 1.1.3]).

(4) An ind-scheme Y is called placid if it can be written as a filtered colimit $Y \cong \operatorname{colim}_i Y^i$ where each Y^i is placid and the transition maps $Y^i \to Y^{i'}$ finitely presented closed embedding.

(5) A morphism $X \to Y$ from a scheme X to an ind-scheme $Y \cong \operatorname{colim}_i Y^i$ is called finitely presented if it factors through a finitely presented morphism $X \to Y^i$. If Y is placid then part (2) implies X is placid.

(6) Let H be placid group scheme acting on a placid ind-scheme Y. We call Y an H-placid ind-scheme if there exists a placid presentation $Y \cong \operatorname{colim}_i Y^i$ such that each Y^i is stable under the H-action.

(7) By a *stack*, we mean a stack in groupoids in the étale topology. A morphism $f : \mathcal{X} \to \mathcal{Y}$ between stacks is called a *covering* if it is a surjective morphism of stacks. A stack $\mathcal{Y}$ is called *placid* if there is a pro-smooth representable covering $Y \to \mathcal{Y}$ from a placid scheme Y. An ind-stack $\mathcal{Y}$ is called placid if there is a presentation $Y \cong \operatorname{colim}_i \mathcal{Y}^i$ where each $\mathcal{Y}^i$ is a placid stack and the transition maps $\mathcal{Y}^i \to \mathcal{Y}^{i'}$ are finitely presented representable closed embeddings.

The quotient stack $\mathcal{Y} \cong H\backslash Y$ of an H-placid ind scheme Y is an example of placid ind-stack.

(8) A morphism $f : \mathcal{X} \to \mathcal{Y}$ between stacks is called pro-smooth if there are coverings $Y \to \mathcal{Y}$ and $X \to \mathcal{X} \times_{\mathcal{Y}} Y$ from schemes X and Y such that the composed map $X \to \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is pro-smooth.

A.2. **Formally smoothness.**

Definition 94. (i) *A morphism $f : X \to Y$ between ind-schemes is called formally smooth if for any k-algebra R and any ideal $I \subset R$ with $I^2 = 0$ and a commutative solid diagram*

$$\begin{array}{ccc} \operatorname{Spec}(R/I) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}(R) & \longrightarrow & Y \end{array}$$

there is a dotted arrow which makes the diagram commute.

(ii) *An ind-scheme X is called formally smooth if the natural map $f : X \to \operatorname{Spec}(k)$ is formally smooth.*

Lemma 95. *Let $f : X \to Y$ be a pro-smooth morphism between schemes. Then f is formally smooth.*

Proof. It follows from the infinitesimal lifting criterion of formally smoothness. □

[8] In [BKV11, Section 1.3.1 (c)] and [FKV10, Section 1.2 (b)], the authors used the term *smooth morphisms* for our pro-smooth morphisms.

Lemma 96. *Let $f : X \to Y$ be a morphism between formally smooth placid ind-schemes. Then f is formally smooth if the differential map $df : T_X \to f^*T_Y$ between tangent sheaves is surjective.*

Proof. In the case when both X and Y are ind-schemes of ind-finite type this is proved in [BD91, Lemma in page 332]. The argument in *loc. cit.* works equally to the placid ind-schemes setting: Let $I \subset R$ be an ideal in a k-algebra R such that $I^2 = 0$. For a morphism $s : S = \mathrm{Spec}(R/I) \to X$, let $E_X(R, I, s)$ (resp. $E_Y(R, I, s)$) be the set of extensions of s to a morphism $\mathrm{Spec}(R) \to X$ (resp. $\mathrm{Spec}(R) \to Y$). Formally smooth of f means the map $f_* : E_X(R, I, s) \to E_Y(R, I, s)$ is surjective for all possible R, I, s as above. Since X (resp. Y) is formally smooth, according to [Gro67, Section 16.5.14], $E_X(R, I, s)$ (resp. $E_Y(R, I, s)$) is a torsor over $\mathrm{Hom}(s^*\Omega^1_X, I)$ (resp. $\mathrm{Hom}(s^*f^*\Omega^1_Y, I)$) where Ω^1_X (resp. Ω^1_Y) is the cotangent sheaf. Since X (resp. Y) is placid, according to [Dri06, Theorem 6.2 (iii)], Ω^1_X (resp. Ω^1_Y) is a Tate sheaf and it follows that there is an isomorphism $\mathrm{Hom}(s^*\Omega^1_X, I) \cong \mathrm{Hom}(I^\vee, s^*T_X)$ (resp. $\mathrm{Hom}(I^\vee, s^*f^*T_Y)$) where $I^\vee = \mathrm{Hom}_{R/I}(I, R/I)$. Moreover, the surjectivity of the differential df induces a surjective map $\mathrm{Hom}(I^\vee, s^*T_X) \to \mathrm{Hom}(I^\vee, s^*f^*T_Y)$ and it follows that the map f_* between the corresponding torsors is also surjective. The lemma follows. □

Lemma 97. *Let $\sigma : X \to X$ be an involution on a formally smooth ind-scheme X. Then the fixed points X^σ is again a formally smooth ind-scheme.*

Proof. Let R, I be as in Lemma 96 and let $s : \mathrm{Spec}(R/I) \to X^\sigma$. The involution σ acts naturally on $\mathrm{Hom}(s^*\Omega^1_X, I)$ and we have $H^1(\mu_2, \mathrm{Hom}(s^*\Omega^1_X, I)) = 0$ since $\mathrm{char}\, k \neq 2$ (here $-1 \in \mu_2$ acts on $\mathrm{Hom}(s^*\Omega^1_X, I)$ by σ). Now we can apply [Gro70, Lemma 9.4] to conclude that s admits a lifting $\tilde{s} : \mathrm{Spec}(R) \to X^\sigma$. The lemma follows. □

A.3. **Placid morphisms.**

Definition 98. (i) *A morphism $f : X \to Y$ between placid schemes is called* placid *if for any étale cover $X' \cong \lim X'_i \to X$ (resp. $Y' \cong \lim Y'_j \to Y$) admitting placid presentation and any map $f' : X' \to Y'$ fitting into the following commutative diagram (not necessarily Cartesian)*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

the composition $X' \to Y' \to Y'_j$ factors as $X' \to X'_i \xrightarrow{f'_{i,j}} Y'_j$ where $f'_{i,j} : X'_i \to Y'_j$ is smooth.

(ii) *A morphism $f : \mathcal{X} \to \mathcal{Y}$ bewteen placid stacks is called placid if there are pro-smooth representable coverings $Y \to \mathcal{Y}$ and $X \to \mathcal{X} \times_{\mathcal{Y}} Y$ from placid schemes X and Y such that composed map $X \to \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is placid.*

Example 99. Pro-smooth morphisms are placid. However, placid morphisms might not be pro-smooth. More precisely, let $f : X \to Y$ be a placid morphism between schemes admitting placid presentations. For any index j let $f_{i,j} : X_i \to Y_j$ be the smooth morphism as above. We can form the base change $f_j : X_i \times_{Y_j} Y \to Y$ and $f \cong \lim f_j : X \cong \lim_j X_i \times_{Y_j} Y \to Y$

is a presentation over Y where all the projections f_j are smooth but the transition maps $X_i \times_{Y_j} Y \to X_{i'} \times_{Y_{j'}} Y$ might not be smooth.

We have the following useful criterion to recognize placid morphisms.

Lemma 100. *Let $f : X \to Y$ be a formally smooth morphism between placid schemes. Then f is placid.*

Proof. The proof is a variation of the argument in [BKV11, Lemma 1.1.4. (b)].

We can assume $X \cong \lim X_i$ and $Y \cong \lim Y_j$ admit placid presentations. Since Y_j is of finite type, the composition $f_j : X \to Y \to Y_j$ factors as

$$f_j : X \xrightarrow{\pi_i} X_i \to Y_j$$

for some $f_{i,j} : X_i \to Y_j$. Since the projection $X \to X_i$ is pro-smooth the cotangent complex L_{X/X_i} is a flat module concentrated in degree zero. Since both $f : X \to Y$ and the projection $Y \to Y_j$ are formally smooth (Lemma 95), the map f_j is formally smooth and by [dJ, Tag. 0D10] the cotangent complex L_{X/Y_j} satisfies a) $H^{-1}(L_{X/Y_j}) = 0$ and b) $H^0(L_{X/Y_j}) \cong \Omega^1_{X/Y_j}$ is locally projective (in particular, it is flat). Using the distinguished triangle

$$\pi_i^* L_{X_i/Y_j} \to L_{X/Y_j} \to L_{X/X_i}$$

we see that $H^{-1}(\pi_i^* L_{X_i/Y_j}) = 0$ and $H^0(\pi_i^* L_{X_i/Y_j})$ is a flat module. Since π_i is faithfully flat and $f_{i,j} : X_i \to Y_j$ is finitely presented, we conclude that $H^{-1}(L_{X_i/Y_j}) = 0$ and $H^0(L_{X_i/Y_j})$ is projective, and [dJ, Tag. 0D10 and 02GZ] implies $f_{i,j}$ is smooth.

□

It is known that a smooth morphism $f : X \to Y$ is an open morphism, in particular, the image $f(X) \subset Y$ has non-empty intersection $f(X) \cap U \neq \emptyset$ with any open dense subset $U \subset Y$. We have the following generalization to placid morphisms:

Lemma 101. *Let $f : X \to Y$ be a placid morphism. Assume $X \cong \lim_i X_i$ admits a placid presentation with surjective projection maps $X \to X_i$. Then the image of f has non-empty intersection with any open dense finitely presented open embedding $u : U \to Y$.*

Proof. Since the pre-images of open dense subsets under étale morphisms are open dense, we can assume $Y \cong \lim_j Y_j$ admits a placid presentation. As f is placid, the composed map $f_j = \pi_j \circ f : X \to Y \to Y_j$ factors as $X \to X_i \to Y_j$ where $f_{i,j} : X_i \to Y_j$ is smooth. Since $X \to X_i$ is surjective, we see that the image $f_j(X) = f_{i,j}(X_i) \subset Y_j$ is open.

Choose an open subset $U_j \subset Y_j$ such that $U = \pi_j^{-1}(U_j)$. If $f(X) \subset Z = Y \setminus U$ is contained in the closed complement, then $Z_j = \pi_j(Z)$ will contain the open subset $f_j(X) = \pi_j(f(X)) \subset Z_j$. This is impossible since it will imply its pre-image $Z = \pi_j^{-1}(Z_j)$ has non-empty intersection with the open dense subset U. The lemma follows.

□

A.4. **Dimension theory.**

(1) For a scheme Y of finite type over k and $y \in Y$ we denote by $\dim_y(Y)$ the maximum of dimensions of irreducible components of Y containing y. The dimension function $\dim_Y : Y \to \mathbb{Z}$ is defined as $\dim_Y(y) = \dim_y(Y)$. Let $f : X \to Y$ be a finitely presented morphism between finite type schemes. We define $\dim_f = \dim_{X/Y} = \dim_X - f^* \dim_Y : X \to \mathbb{Z}$.

(2) A finite type scheme Y is called *locally equidimensional* if each connected component of Y is equidimentional, equivalently, the dimension function $\dim_Y : Y \to \mathbb{Z}$ is locally constant. A placid presentation $Y \cong \lim_j Y_j$ is called locally equidimentional if each Y_j is locally equidimensional. A placid scheme Y is called locally equidimentional if there exists an étale covering by a scheme with locally equidimensional placid presentation. A placid ind-scheme Y is called locally equidimentional if there is a placid presentation $Y \cong \operatorname{colim} Y^i$ such that each Y^i is locally equidimentional.

(3) Let $f : X \to Y$ be a finitely presented morphism between schemes admitting placid presentations. According to Section 2 of [BKV11, FKV10], there is a well-defined function $\dim_f = \dim_{X/Y} : X \to \mathbb{Z}$ such that $\dim_f = \pi_{j_0}^* \dim_{f_{j_0}}$ where $f_{j_0} : X_{j_0} \to Y_{j_0}$ is as in (A.1)(4) and $\pi_{j_0} : X \to X_{j_0}$ is the projection map.

(4) For every locally finitely presented morphism $f : X \to Y$ of placid scheme, there is a unique function $\dim_f = \dim_{X/Y} : X \to \mathbb{Z}$ such that for every commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle h} & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $f' : X' \to Y'$ is a finitely presented morphism of schemes admitting placid presentations and the vertical maps are étale morphisms, we have $\dim_{f'} = h^* \dim_f$.

(5) For every representable locally finitely presented morphism $f : \mathcal{X} \to \mathcal{Y}$ of placid stacks, there is a unique function $\dim_f = \dim_{\mathcal{X}/\mathcal{Y}} : [\mathcal{X}] \to \mathbb{Z}$, where $[\mathcal{X}]$ is the underlying set of $\mathcal{X}$, such that for every Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle h} & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where the vertical maps are pro-smooth coverings, we have $\dim_{f'} = h^* \dim_f$.

(6) Let $f : \mathcal{X} \to \mathcal{Y}$ be a finitely presented representable locally closed embedding. We define $\operatorname{codim}_f = -\dim_f$ to be called the codimension function of f.

(7) According to [FKV10, Section 2.4], the dimension function is additive, that is, for every pair $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ morphisms as in (5), we have an equality

$$\dim_{g \circ f} = \dim_f + f^* \dim_g .$$

(8) Consider a Cartesian diagram of placid stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle b} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

such that f is locally finitely presented representable and b is placid. According to [BKV11, Lemma 2.3.7], we have

$$h^* \dim_f = \dim_{f'} .$$

(9) Let Y be a placid ind-scheme. A *dimension theory* δ on Y is an assignment to each finitely presented locally equidimensional subscheme $S \subset Y$ a locally constant function $\delta_S : S \to \mathbb{Z}$ such that for any such pair of subschemes $S \subset S' \subset Y$ we have

$$\delta_S - \delta_{S'}|_S = \dim_{S/S'} : S \to \mathbb{Z}.$$

(10) Given two dimension theories δ and δ', their difference $\delta - \delta' : Y \to \mathbb{Z}$ sending $y \to \delta_S(y) - \delta'_S(y)$, where S is any finitely presented subscheme containing y, is a well defined locally constant function on Y. Thus the set of dimension theories on Y is a pseudo torsor over the space of locally constant functions on Y.

(11) Given a locally equidimensional placid ind-scheme $Y \cong \operatorname{colim}_i Y^i$, a dimension theory δ on Y is uniquely characterized by the assignment δ_{Y^i} for each Y^i. Indeed, for any finitely presented locally equidimensional subscheme $S \subset Y$, we have a finitely presented embedding $S \subset Y^i$ for a large enough i and we define $\delta_S = \delta_{Y^i}|_S - \dim_{S/Y^i}$.

Example 102. Assume $Y \cong \operatorname{colim}_i Y^i$ is ind-finite type, that is, each Y^i is locally equidimensional of finite type. Then we have a canonical dimension theory given by the dimension of Y^i, $\delta_{Y^i} = \dim_{Y^i} : Y^i \to \mathbb{Z}$.

A.5. **Sheaf theories.**

(1) We will be working with $\overline{\mathbb{Q}}_\ell$-linear dg-categories. Unless specified otherwise, all dg-categories will be assumed cocomplete, i.e., containing all small colimits, and all functors between dg-categories will be assumed continuous, i.e., preserving all small colimits. For any finite type scheme Y we denote by $D(Y) = \operatorname{Ind}(D^b_c(Y))$ the ind-completion of the bounded dg derived category $D^b_c(Y)$ of ℓ-adic constructible complexes on Y. For any $\mathcal{F} \in D(Y)$ and any integer $n \in \mathbb{Z}$ we will write $\mathcal{F}\langle n\rangle = \mathcal{F}[n](n/2)$. More generally, for any locally constant function $f : Y \to \mathbb{Z}$, we define $\mathcal{F}\langle f\rangle$ such that for any connected component $Y_\beta \subset Y$ we have $\mathcal{F}\langle f\rangle|_{Y_\beta} = \mathcal{F}\langle f(Y_\beta)\rangle|_{Y_\beta}$. We have the Verdier duality fucntor $\mathbb{D}_Y : D(Y) \cong D(Y)^{op}$ satisfying $f^!\mathbb{D}_Y \cong \mathbb{D}_X f^*$. Given a smooth map $f : X \to Y$ between schemes of finite types, we have $f^! \cong f^*\langle 2\dim_f\rangle$ where $\dim_f$ is the relative dimension f.

(2) For a scheme Y admitting a placid presentation $Y \cong \lim_j Y_j$, we define the $*$-version of dg derived category of constructible complexes on Y as the colimit $D(Y) \cong \operatorname{colim}^*_j D(Y_j)$. For any placid stack $\mathcal{Y}$ we define $D(\mathcal{Y}) \cong \lim^*_{[n]\in\Delta^{op}_s} D(X^{[n]})$ where $\{X^{[n]}\}_{[n]\in\Delta^{op}_s}$ is the Čech nerve of a smooth covering $X \to \mathcal{Y}$ admitting a placid presentation (note that by (A.1)(4) each $X^{[n]}$ admits a placid presentation). For any placid ind-stack $\mathcal{Y} \cong \operatorname{colim}_i \mathcal{Y}^i$, we define the $*$-version of dg derived category of constructible complexes on $\mathcal{Y}$ as $D(\mathcal{Y}) = \operatorname{colim}_{i,!} D(\mathcal{Y}^i)$ where the first colimit is with respect to the !-pushforward along the closed embedding $\mathcal{Y}^i \to \mathcal{Y}^{i'}$.

We can also introduce !-version of $D(\mathcal{Y})$ denoted by $D_!(\mathcal{Y})$ using the !-pullbacks in the above definitions.

(3) Let $f : \mathcal{X} \to \mathcal{Y}$ be a finitely presented representable morphism between placid stacks. Then we have (derived) functors $f_*, f_!, f^*, f^!$ between $D(\mathcal{X})$ and $D(\mathcal{Y})$ (resp. $D_!(\mathcal{X})$ and $D_!(\mathcal{Y})$).

A.6. **Verdier dualities.**

Lemma 103. (i) *Let Y be an H-placid ind-scheme. There is a Verdier duality functor*

$$\mathbb{D}'_Y : D(H\backslash Y) \cong D_!(H\backslash Y)^{op} \tag{42}$$

satisfying $(\mathbb{D}'_Y)^2 \cong \mathrm{Id}$.

(ii) *Let Y is locally equidimensional H-placid scheme. There is a canonical equivalence*

$$\Phi_Y : D_!(H\backslash Y) \cong D(H\backslash Y).$$

(iii) *Let $Y \cong \mathrm{colim}_i Y^i$ be a locally equidimensional H-placid ind-scheme. Then a dimension theory δ on Y induces equivalences*

$$\Phi_\delta \cong \mathrm{colim}_i \Phi_{Y^i}\langle 2\delta_{Y^i}\rangle : D_!(H\backslash Y) \cong D(H\backslash Y).$$

Proof. Proof of (ii). Let $\{X^{[n]}\}_{[n]\in\Delta_s^{op}}$ be the Čech nerve of a smooth covering $X \to H\backslash Y$ admitting a placid presentation. The assignment sending $\{\mathcal{F}_j^{[n]}\} \in D_!(X^{[n]}) \cong \mathrm{colim}_j^! D(X_j^{[n]})$, to $\{\mathcal{F}_j^{[n]}\langle -2\dim(X_j^{[n]})\rangle\} \in D(X^{i,[n]}) \cong \mathrm{colim}_j^* D(X_j^{i,[n]})$ induces an equivalence.

Proof of (i). Pick an H-placid presentation $Y \cong \mathrm{colim}_i Y^i$. Let $\{X^{i,[n]}\}_{[n]\in\Delta_s^{op}}$ be the Čech nerve of a smooth covering $X^i \to H\backslash Y^i$ admitting a placid presentation. The isomorphism $f^!\mathbb{D}_Y \cong \mathbb{D}_X f^*$ for maps between finite type schemes implies that the assignment sending $\{\mathcal{F}_j^{i,[n]}\} \in D(X^{i,[n]}) \cong \mathrm{colim}_j^* D(X_j^{i,[n]})$ to $\{\mathbb{D}\mathcal{F}_j^{i,[n]}\} \in D_!(X^{i,[n]}) \cong \mathrm{colim}_j^! D(X_j^{i,[n]})$ induces a well-defined functor

$$\mathbb{D}'_Y : D(H\backslash Y) \cong D_!(H\backslash Y)^{op} \tag{43}$$

satisfying $(\mathbb{D}'_Y)^2 \cong \mathrm{Id}$.

Proof of (iii). The assignment sending $\{\mathcal{F}_j^{i,[n]}\} \in D_!(X^{i,[n]}) \cong \mathrm{colim}_j^! D(X_j^{i,[n]})$ to $\{\mathcal{F}_j^{i,[n]}\langle -2\dim(X_j^{i,[n]}) + 2\delta_{Y^i}\rangle\} \in D(X^{i,[n]}) \cong \mathrm{colim}_j^* D(X_j^{i,[n]})$ induces an equivalence. □

The lemma above implies the following.

Proposition 104. (i) *Let Y be a locally equidimensional H-placid scheme. There is an equivalence*

$$\mathbb{D}_Y := \Phi_Y \circ \mathbb{D}'_Y : D(H\backslash Y) \cong D(H\backslash Y)^{op}$$

satisfying $(\mathbb{D}_Y)^2 \cong \mathrm{id}$.

(ii) *Let $Y \cong \mathrm{colim}_i Y^i$ be a locally equidimensional H-placid ind-scheme. A dimension theory δ on Y induces an equivalence*

$$\mathbb{D}_\delta := \Phi_\delta \circ \mathbb{D}'_Y \cong \mathrm{colim}_i \mathbb{D}_{Y^i}\langle 2\delta_{Y^i}\rangle : D(H\backslash Y) \cong D(H\backslash Y)^{op}$$

satisfying $(\mathbb{D}_\delta)^2 \cong \mathrm{id}$.

Example 105. Assume $Y \cong \mathrm{colim}_i Y^i$ is of ind-finite type. Then, by Example 102, there is a canonical dimension theory $\delta_{Y^i} = \dim_{Y^i}$ for Y and one can check that $D(H\backslash Y) \cong D_!(H\backslash Y)$, $\Phi_\delta \cong \mathrm{id}$, and $\mathbb{D}_\delta \cong \mathbb{D}'_Y$ is the usual Verdier duality.

We will need the following properties of Verdier dualities later.

Lemma 106. *Let $f : S \to Y$ be a finitely presented morphism between locally equidimensional H-placid schemes. Then there are canonical isomorphisms*

(i) $f^!\mathbb{D}_Y \cong \mathbb{D}_S f^*\langle 2\dim_f\rangle$,

(ii) $\mathbb{D}_Y f_! \cong f_*\mathbb{D}_S\langle 2\dim_f\rangle$.

Proof. We give a proof of (i). The case of (ii) is similar. Pick a smooth covering $X \to H\backslash Y$ admitting a placid presentation and let $V = (H\backslash S) \times_{H\backslash Y} X$ be the base change. The map f induces a map $f^{[n]} \cong \lim_j f_j^{[n]} : V^{[n]} \cong \lim_j V_j^{[n]} \to X^{[n]} \cong \lim_j X_j^{[n]}$ between the Čech nerves. Note that we have

$$2\dim_f = 2\dim_{f_j^{[n]}} = -2\dim_{X_j^{[n]}} + 2\dim_{V_j^{[n]}} \tag{44}$$

and it follows that

$$\begin{aligned} f^!\mathbb{D}_Y(\mathcal{F}) &\cong f^!(\lim^*_{[n]} \operatorname{colim}^*_j(\mathbb{D}_{X_j^{[n]}}(\mathcal{F}_j^{[n]})\langle -2\dim_{X_j^{[n]}}\rangle)) \\ &\cong \lim^*_{[n]} \operatorname{colim}^*_j((f_j^{[n]})^!\mathbb{D}_{X_j^{[n]}}(\mathcal{F}_j^{[n]})\langle -2\dim_{X_j^{[n]}}\rangle) \\ &\overset{(44)}{\cong} \lim^*_{[n]} \operatorname{colim}^*_j(\mathbb{D}_{V_j^{[n]}}(f_j^{[n]})^*(\mathcal{F}_j^{[n]})\langle -2\dim_{V_j^{[n]}} + 2\dim_f\rangle) \\ &\cong \mathbb{D}_V(\lim^*_{[n]} \operatorname{colim}^*_j((f_j^{[n]})^*(\mathcal{F}_j^{[n]})))\langle 2\dim_f\rangle \cong \mathbb{D}_V f^*\mathcal{F}\langle 2\dim_f\rangle. \end{aligned}$$

□

A.7. **t-structures and perverse sheaves.** Let Y be a placid scheme. Since $*$-pullback is t-exact with respect to the standard t-structures on finite type schemes, the category $D(Y)$ carries a canonical standard t-structure $(D^{\leq 0}(Y), D^{\geq 0}(Y))$ and we denote by $\mathrm{Shv}(Y)$ the corresponding abelian category of constructible sheaves on Y. Assume Y is an H-placid scheme where H is a placid affine group scheme. We can pull back the standard t-structure on $D(Y)$ along the forgetful map $D(H\backslash Y) \to D(Y)$ and obtain a t-structure $(D^{\leq 0}(H\backslash Y), D^{\geq 0}(H\backslash Y))$ on $D(H\backslash Y)$ with abelian heart

$$\mathrm{Shv}(H\backslash Y) = D^{\leq 0}(H\backslash Y) \cap D^{\geq 0}(H\backslash Y)$$

of H-equivariant constructible sheaves on Y. We denote by

$$\mathcal{H}^0 : D(H\backslash Y) \to \mathrm{Shv}(H\backslash Y)$$

the corresponding cohomological functor.

Let Y be a finite type scheme. Following [BKV11, Section 6], we can introduce the *$*$-adapted perverse t-structure* $({}^pD^{\leq 0}(Y), {}^pD^{\geq 0}(Y))$ on $D(Y)$ such that for any smooth morphism $f : X \to Y$, the pullback f^* is t-exact.[9] If Y is locally equidimensional, we have

$${}^pD^{\leq 0}(Y) = {}^{p_{cl}}D^{\leq 0}(Y)[-\dim_Y] \tag{45}$$

where ${}^{p_{cl}}D^{\leq 0}(Y)$ is the classical perverse t-structure and $\dim_Y : Y \to \mathbb{Z}$ the dimension function (which is locally constant). In the general case, we consider the canonical equidimensional stratification $\{Y_\beta\}$ of Y and define ${}^pD^{\leq 0}(Y)$ as the gluing of ${}^pD^{\leq 0}(Y_\beta)$, see [BKV11, Section 6.2.3] for more details.

Let Y be a placid scheme. Then we can consider the *perverse t-structure* $({}^pD^{\leq 0}(Y), {}^pD^{\geq 0}(Y))$ on $D(Y)$ given by

$${}^pD^{\leq 0}(Y) = \lim^*_{[n]} \operatorname{colim}^*_j({}^pD^{\leq 0}(X_j^{[n]})) \tag{46}$$

[9] In *loc. cit.*, the authors considered the !-adapted t-structure instead of the $*$-adapted one.

where the colimit is taken with respect to $*$-pullback (here $\{X^{[n]} \cong \lim_j X_j^{[n]}\}_{[n]\in\Delta_s^{op}}$ is the Čech nerve for an étale cover $X \to Y$ admitting a placid presentation). Assume Y is H-placid then, similarly, we can pull back the perverse t-structure on $D(Y)$ along the forgetful map $D(H\backslash Y) \to D(Y)$ and obtain a perverse t-structure $({}^pD^{\leq 0}(H\backslash Y), {}^pD^{\geq 0}(H\backslash Y))$ with abelian heart of H-equivariant perverse sheaves on Y:

$$\mathrm{Perv}(H\backslash Y) = {}^pD^{\leq 0}(H\backslash Y) \cap {}^pD^{\geq 0}(H\backslash Y).$$

We denote by

$${}^p\mathcal{H}^0 : D(H\backslash Y) \to \mathrm{Perv}(H\backslash Y)$$

the corresponding cohomological functor.

Let $Y \cong \mathrm{colim}_i Y^i$ be a locally equidimensional H-placid ind-scheme equipped with a dimension theory $\delta_{Y^i} : Y^i \to \mathbb{Z}$. It follows from $\delta_{Y^i} = \delta_{Y^{i'}}|_{Y^i} + \dim_{Y^i/Y^{i'}}$ and (45) that the functor $(\iota_{i,i'})_! : D(H\backslash Y^i) \to D(H\backslash Y^{i'})$ (where $\iota_{i,i'} : Y^i \to Y^{i'}$ is the transition map) satisfies

$$(\iota_{i,i'})_!({}^pD^{\leq 0}(H\backslash Y^i)[\delta_{Y^i}]) \subset {}^pD^{\leq 0}(H\backslash Y^{i'})[\delta_{Y^{i'}}],$$

$$(\iota_{i,i'})_!({}^pD^{\geq 0}(H\backslash Y^i)[\delta_{Y^i}]) \subset {}^pD^{\geq 0}(H\backslash Y^{i'})[\delta_{Y^{i'}}].$$

Hence there is a perverse t-structure $({}^pD^{\leq 0}(H\backslash Y)[\delta], {}^pD^{\geq 0}(H\backslash Y)[\delta])$ on the colimit $D(H\backslash Y) \cong \mathrm{colim}_{i,!}(D(H\backslash Y^i))$ characterized by

$${}^pD^{\leq 0}(H\backslash Y)[\delta] = \mathrm{colim}_{i,!}({}^pD^{\leq 0}(H\backslash Y^i)[\delta_{Y^i}]), \tag{47}$$

with the abelian heart of H-equivariant perverse sheaves on Y

$$\mathrm{Perv}_\delta(H\backslash Y) = \mathrm{colim}_{i,!}\,\mathrm{Perv}(H\backslash Y^i)[\delta_{Y^i}] = {}^pD^{\leq 0}(H\backslash Y)[\delta] \cap {}^pD^{\geq 0}(H\backslash Y)[\delta].$$

We denote by

$${}^p\mathcal{H}^0_\delta : D(H\backslash Y) \to \mathrm{Perv}_\delta(H\backslash Y)$$

the corresponding cohomological functor.

Lemma 107. *Let $Y \cong \mathrm{colim}_i Y^i$ be a locally equidimensional H-placid ind-scheme equipped with a dimension theory δ.*

(i) *The Verdier duality $\mathbb{D}_\delta : D(H\backslash Y) \cong D(H\backslash Y)^{op}$ (43) is t-exact with respect to the perverse t-structure (47) and induces an equivalence $\mathbb{D}_\delta : \mathrm{Perv}_\delta(H\backslash Y) \cong \mathrm{Perv}_\delta(H\backslash Y)^{op}$.*

(ii) *Assume H is irreducible and admits a placid presentation $H \cong \lim_j H_j$ where the kernel $H \to H_j$ is pro-unipotent. Then the forgetful functor $\mathrm{For} : \mathrm{Perv}_\delta(H\backslash Y) \to \mathrm{Perv}_\delta(Y)$ is fully-faithful.*

Proof. Proof of (i). It follows from

$$\begin{aligned}
&\mathbb{D}_\delta({}^pD^{\leq 0}(H\backslash Y^i)[\delta_{Y^i}]) \cong \Phi_\delta \circ \mathbb{D}'_Y(\lim^*_{[n]} \mathrm{colim}^{*\,p,cl}_j D^{\leq 0}(X_j^{i,[n]})[-\dim_{X_j^{i,[n]}} + \delta_{Y^i}])\\
\cong &\Phi_\delta(\lim^!_{[n]} \mathrm{colim}^{!\,p,cl}_j D^{\leq 0}(X_j^{i,[n]})[\dim_{X_j^{i,[n]}} - \delta_{Y^i}])\\
\cong &\lim^*_{[n]} \mathrm{colim}^{*\,p,cl}_j D^{\leq 0}(X_j^{i,[n]})[\dim_{X_j^{i,[n]}} - \delta_{Y^i}]\langle -2\dim_{X_j^{i,[n]}} + 2\delta_{Y^i}\rangle\\
\cong &\lim^*_{[n]} \mathrm{colim}^{*\,p,cl}_j D^{\leq 0}(X_j^{i,[n]})[-\dim_{X_j^{i,[n]}} + \delta_{Y^i}] \cong {}^pD^{\leq 0}(H\backslash Y^i)[\delta_{Y^i}].
\end{aligned}$$

Proof of (ii). The proof is similar to the case of schemes of finite types : consider the following diagram of functors

$$\begin{array}{ccccc} \mathrm{Perv}_\delta(H\backslash Y) & \xrightarrow{a^*} & \mathrm{Perv}_\delta(H\backslash(H\times Y)) & \xleftarrow{\simeq} & \mathrm{Perv}_\delta(Y) \\ \downarrow \mathrm{For} & & \downarrow \mathrm{For} & \swarrow_{p^*} & \\ \mathrm{Perv}_\delta(Y) & \xrightarrow{p^*} & \mathrm{Perv}_\delta(H\times Y) & & \end{array}$$

where H acts on $H\times Y$ on the first factor and $a, p : H\times Y \to Y$ are the action and projection maps respectively. The assumption implies that both a, p are strongly pro-smooth with all the projection and transition maps being smooth with connected fibers. It follows that a^*, p^* are fully-faithful [BBDG18]. Thus the vertical forgetful maps are fully-faithful. □

Example 108. (IC-complexes) We preserve the setup of Lemma 107. Let $\mathcal{O} = H\cdot y$ be the H-orbit through $y \in Y^i$ and let $S = H_y$ be the stabilizer of y. Choose a placid presentation $H \cong \lim_j H_j$ and let $S_j = \mathrm{Im}(S \subset H \to H_j)$ be the image of S along the projection $H \to H_j$. Then the natural projection map $H_{j'}/S_{j'} \to H_j/S_j$ is smooth affine and unipotent and we get a placid presentation $\mathcal{O} \cong H/S \cong \lim H_j/S_j$. In particular, $\mathcal{O}$ is irreducible placid scheme.

Let $\pi_0(S_j)$ be the group of connected components of S_j and for any representation $\xi \in \mathrm{Rep}(\pi_0(S_j))$ of $\pi_0(S_j)$ we denote by $\mathcal{L}_\xi$ the corresponding H_j-equivariant local system on H_j/S_j. There is an equivalence

$$\mathrm{Rep}(\pi_0(S_j)) \cong \mathrm{Perv}(H\backslash\mathcal{O}) \cong \mathrm{Shv}(H\backslash\mathcal{O}), \quad \xi \to \mathcal{L}_{\xi,\mathcal{O}} := \pi_j^*\mathcal{L}_\xi.$$

Assume the inclusion $j : \mathcal{O} \to Y^i$ is a finitely presented open embedding. Then we have the *standard sheaf* (resp. *co-standard complex*)

$$j_!\mathcal{L}_{\xi,\mathcal{O}} \in \mathrm{Shv}(H\backslash Y^i) \quad (resp.\ \ j_*\mathcal{L}_{\xi,\mathcal{O}} \in D(H\backslash Y^i)) \tag{48}$$

and the intermediate extension

$$j_{!*}\mathcal{L}_{\xi,\mathcal{O}} := \mathrm{Im}({}^p\mathcal{H}^0(f_!(\mathcal{F})) \to {}^p\mathcal{H}^0(f_*(\mathcal{F}))) \in \mathrm{Perv}(H\backslash Y^i). \tag{49}$$

According to the definition of perverse t-structure (47), we have

$$\mathrm{IC}_{\xi,\mathcal{O}} := j_{!*}\mathcal{L}_{\xi,\mathcal{O}}[\delta_\mathcal{O}] \in \mathrm{Perv}_\delta(H\backslash Y) = \mathrm{colim}_{i,!}\,\mathrm{Perv}(H\backslash Y^i)[\delta_{Y^i}]. \tag{50}$$

Finally, it follows from Lemma 106 that

$$\mathbb{D}_{Y^i}(j_{!*}\mathcal{L}_{\xi,\mathcal{O}}) \cong j_{!*}\mathcal{L}_{-\xi,\mathcal{O}}, \qquad \mathbb{D}_{Y^i}(j_!\mathcal{L}_{\xi,\mathcal{O}}) \cong j_*\mathcal{L}_{-\xi,\mathcal{O}}, \tag{51}$$

and hence from Proposition 104

$$\mathbb{D}_\delta(j_!\mathcal{L}_{\xi,\mathcal{O}}) \cong \mathbb{D}_{Y^i}(j_!\mathcal{L}_{\xi,\mathcal{O}})\langle 2\delta_\mathcal{O}\rangle \cong j_*\mathcal{L}_{-\xi,\mathcal{O}}\langle 2\delta_\mathcal{O}\rangle, \tag{52}$$

$$\mathbb{D}_\delta(\mathrm{IC}_{\xi,\mathcal{O}}) \cong \mathbb{D}_{Y^i}(j_{!*}\mathcal{L}_{\xi,\mathcal{O}}[\delta_\mathcal{O}])\langle 2\delta_\mathcal{O}\rangle \cong (j_{!*}\mathcal{L}_{-\xi,\mathcal{O}}[-\delta_\mathcal{O}])\langle 2\delta_\mathcal{O}\rangle \cong \mathrm{IC}_{-\xi,\mathcal{O}}(\delta_\mathcal{O}). \tag{53}$$

A.8. **Pointwise purity.** Let Y be a locally equidimensional H-placid scheme. In this section we assume the following:

(1) H is irreducible and there is a placid presentation $H \cong \lim_j H_j$ where the kernels $H \to H_j$ are pro-unipotent.
(2) Y is a union of finitely many H-orbits and for any orbit $\mathcal{O} \subset Y$ the inclusion $j : \mathcal{O} \to Y$ is finitely presented.
(3) Assume H and Y are defined over a finite field $\mathbb{F}_q$.

We write $F : Y \to Y$ for the Frobenius morphism and we denote by $D(H\backslash Y)^F$ the corresponding category of F-equivariant complexes whose objects consist of pairs $(\mathcal{F}, \Phi)$ where $\mathcal{F} \in D(H\backslash Y)$ and $\Phi : F^*\mathcal{F} \cong \mathcal{F}$. Similarly, we denote by $\mathrm{Shv}(H\backslash Y)^F$, $\mathrm{Perv}(H\backslash Y)^F$, etc, the corresponding categories of F-equivariant objects.

Definition 109. (i) *A complex $(\mathcal{F}, \Phi) \in D(H\backslash Y)^F$ is called pure of weight w if there exists a smooth cover $\pi : X \to H\backslash Y$ defined over $\mathbb{F}_q$ admitting a placid presentation $X \cong \lim_j X_j$ and a pure complex $\mathcal{F}_{j_0} \in D(X_{j_0})^F$ of weight w in the sense of* [BBDG18] *such that $\pi^*\mathcal{F} \cong \pi_{j_0}^*\mathcal{F}_{j_0}$. Here $\pi_{j_0} : X \to X_{j_0}$ is the transition map.*

(ii) *An F-equivariant local system $(\mathcal{L}, \Phi) \in \mathrm{Shv}(H\backslash\mathcal{O})^F$ is called pointwise pure of weight w if for any positive integer n and any $x \in \mathcal{O}^{F^n}$, the eigenvalues of Φ^n on the stalk $\mathcal{L}_x$ are algebraic numbers in $\overline{\mathbb{Q}}_\ell^\times$ all of whose complex conjugates have absolute value $(q^n)^{\frac{w}{2}}$. If $\mathcal{L}$ is pointwise pure of weight w, then we say $\mathcal{L}[m]$ is pointwise pure of weight $w + m$.*

(iii) *A complex $(\mathcal{F}, \Phi) \in D(H\backslash Y)^F$ is said to be $*$-pointwise pure (resp. $!$-pointwise pure) of weight w if for any H-orbit $j : \mathcal{O} \to Y$ and integer m, the local system $\mathscr{H}^m(j^*\mathcal{F})$ (resp. $\mathscr{H}^m(j^!\mathcal{F})$) is pointwise pure of weight $w + m$ (Note that the assumption (ii) above implies that the functor $j^!$ is well-defined). It is said to be pointwise pure if it is both $*$-pointwise pure and $!$-pointwise pure.*

Lemma 110. *If $\mathcal{F} \in D(H\backslash Y)^F$ is $*$-pointwise pure of weight w, then its Verdier dual $\mathbb{D}_Y(\mathcal{F})$ is $!$-pointwise pure of weight $-w$.*

Proof. It follows from Lemma 106. □

Example 111. Let $j : \mathcal{O} \to Y$ be an open orbit and let $j_{!*}(\mathcal{L}_{\xi,\mathcal{O}})$ be the intermediate extension in Example 108. If $\mathcal{L}_{\xi,\mathcal{O}}$ is a pointwise pure local system of weight w, then $j_{!*}(\mathcal{L}_{\xi,\mathcal{O}})$ is pure of weight w.

We have the following criterion for $*$-pointwise purity generalizing [MS98, Proposition 2.3.3] to the setting of H-placid schemes. To this end, we first recall the notion of contracting slices.

Definition 112. *Let $\mathcal{O} = H \cdot x \subset Y$ be the H-orbit through a point $x \in Y$. A contracting slice at x with respect to $\mathcal{O}$ is a locally closed finite type subscheme $S \subset Y$ defined over $\mathbb{F}_q$ such that*

(i) *S contains x,*
(ii) *the action map $a : H \times S \to Y$ is formally smooth,*
(iii) *there is a homomorphism $\lambda : \mathbb{G}_m \to H$ such that S is preserved by $\lambda(\mathbb{G}_m)$ and the resulting $\mathbb{G}_m$-action on S is attracting with x as its unique fixed point.*

Proposition 113. *Let $(\mathcal{F}, \Phi) \in D(H\backslash Y)^F$ be a pure complex of weight w. Assume there is a contracting slice at each point $x \in Y$ with respect to the orbit $\mathcal{O} = H \cdot x$. Then $\mathcal{F}$ is $*$-pointwise pure of weight w.*

Proof. Let $i : S \subset Y$ be a contracting slice at $x \in Y^{F^n}$. We claim that $\mathcal{F}|_S = i^*\mathcal{F} \in D(S)^F$ is pure of weight w in the sense of [BBDG18]. Assumption (iii) in Definition 112 implies that $\mathcal{F}|_S$ is equivariant with respect to the contracting $\mathbb{G}_m$-action on S and we can apply [MS98, Proposition 2.3.3] to conclude that the eigenvalues of Φ^n on the stalk $\mathscr{H}^m(\mathcal{F}|_x) \cong$

$\mathscr{H}^m((\mathcal{F}|_S)_x)$ are algebraic numbers in $\overline{\mathbb{Q}}_\ell^\times$ all of whose complex conjugates have absolute value $(q^n)^{\frac{w}{2}}$. The proposition follows.

Proof of the claim. Consider the action map $a : H \times S \to Y$. Let $X \cong \lim_j X_j \to Y$ be an étale cover admitting a placid presentation and let $V = (H \times S) \times_Y X$ be the base change. Since the projection $f : V \to H \times S$ is an étale cover, in particular finitely presented, by Section A.1 (4), there is an étale cover $f_{j_0} : V_{j_0} \to H_{j_0} \times S$ such that f is the base change of f_{j_0} and $V \cong \lim_{j \geq j_0} V_j = V_{j_0} \times_{(H_{j_0} \times S)} (H \times S)$ is a placid presentation of V. Moreover, since the projection map $b : V \to X$ is formally smooth (being the base change of the formally smooth map a), it follows from Lemma 100 that the composed map $V \to X \to X_j$ factors as $V \to V_{j'} \xrightarrow{c} X_j$ with c a smooth map. We summarize the above discussion by the following commutative diagram

(54)
$$\begin{array}{ccccc} H_{j'} \times S & \xleftarrow{q} & H \times S & \xrightarrow{a} & Y \\ \uparrow{\scriptstyle f_{j'}} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle h} \\ V_{j'} & \xleftarrow{g} & V & \xrightarrow{b} & X \\ & \searrow^{c} & & & \downarrow{\scriptstyle p} \\ & & & & X_j \end{array}$$

where the upper squares are Cartesian diagrams, the maps $h, f, f_{j'}$ are étale, and the map c is smooth. From the definition of pure complexes in Definition 109, there is a large enough j such that

$$h^*\mathcal{F} \cong p^*\mathcal{F}_j$$

where $\mathcal{F} \in D(X_j)^F$ is pure of weight w. Note that there is an isomorphism

$$(f_{j'})^*(\overline{\mathbb{Q}}_{\ell,H_{j'}} \boxtimes \mathcal{F}|_S) \cong c^*\mathcal{F}_j. \tag{55}$$

Indeed, since g is pro-smooth with contractible fibers it suffices to check (55) after applying g^*, but since $a^*(\mathcal{F}) \cong \overline{\mathbb{Q}}_{\ell,H} \boxtimes \mathcal{F}|_S$ by the H-equivariance of $\mathcal{F}$, we have

$$g^*(f_{j'})^*(\overline{\mathbb{Q}}_{\ell,H_{j'}} \boxtimes \mathcal{F}|_S) \cong f^*((\overline{\mathbb{Q}}_{\ell,H} \boxtimes \mathcal{F}|_S) \cong f^*a^*(\mathcal{F}) \cong b^*h^*\mathcal{F} \cong b^*p^*\mathcal{F}_j \cong g^*c^*\mathcal{F}_j.$$

Since c is a smooth map (between finite type schemes) the complex $c^*\mathcal{F}_j$ is pure of weight w and the isomorphism (55) together with the fact that $f_{j'}$ is étale imply that $\overline{\mathbb{Q}}_{\ell,H_{j'}} \boxtimes \mathcal{F}|_S$ is pure of weight w. Finally, since $e^* \cong e^!\langle 2\dim(H_{j'})\rangle$ on $\overline{\mathbb{Q}}_{\ell,H_{j'}} \boxtimes \mathcal{F}|_S$, where $e : S \to H_{j'} \times S$ is the embedding $e(x) = (1, x)$ ($1 \in H_{j'}$ is the unit), we conclude that $\mathcal{F}|_S \cong e^*(\overline{\mathbb{Q}}_{\ell,H_{j'}} \boxtimes \mathcal{F}|_S)$ is pure of weight w. The claim follows.

□

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