THREE SPANS AND SKYSCRAPERS ON THE LINE

We compare the thick span of Proposition I with the draft's two spans, and show with skyscrapers on $\mathbb{A}^1$ that the double orthogonal of a small category can be strictly larger than the thick span.

1. Three spans

Let $\mathcal{D}$ be a small, idempotent-complete, stable ∞-category over $k = \mathbb{C}$, $\mathcal{S}$ a set of objects of $\mathcal{D}$, and $\widehat{\mathcal{D}} = \operatorname{Ind}\mathcal{D}$. Then $\widehat{\mathcal{D}}$ is compactly generated and its compact objects are exactly $\mathcal{D}$ [HTT, Prop. 5.3.5.5, Lemma 5.4.2.4]. $\underline{\mathrm{Map}}$ is the derived mapping complex, so $\underline{\mathrm{Map}}(A,B) = 0$ iff $\mathrm{Hom}(A, B[n]) = 0$ for all $n \in \mathbb{Z}$.

Definition 1.1. $\mathrm{Thick}_{\mathcal{D}}(\mathcal{S})$ is the smallest full subcategory of $\mathcal{D}$ containing 0 and $\mathcal{S}$ and closed under isomorphisms, shifts, cofibres and retracts; dropping retracts gives the stable span $\mathrm{Stab}_{\mathcal{D}}(\mathcal{S})$.

Definition 1.2. For a stable ∞-category $\mathcal{E}$ with all small colimits, $\langle\mathcal{S}\rangle^{\mathrm{GR}}_{\mathcal{E}}$ is the smallest full stable subcategory of $\mathcal{E}$ containing $\mathcal{S}$ and closed under small colimits; for presentable $\mathcal{E}$, $\langle\mathcal{S}\rangle^{\mathrm{GR}}_{\mathcal{E}} = \mathcal{E}$ is GR's generation [GR17, Vol. I, Ch. 1, §5.4.1, Prop. 5.4.5].

Definition 1.3. $\mathcal{S}^{\perp_{\mathcal{D}}} = \{M \in \mathcal{D} : \underline{\mathrm{Map}}_{\mathcal{D}}(s, M) = 0 \ \forall s \in \mathcal{S}\}$ and $\langle\mathcal{S}\rangle^{\mathrm{M}}_{\mathcal{D}} = {}^{\perp_{\mathcal{D}}}(\mathcal{S}^{\perp_{\mathcal{D}}}) = \{X \in \mathcal{D} : \underline{\mathrm{Map}}_{\mathcal{D}}(X, M) = 0 \ \forall M \in \mathcal{S}^{\perp_{\mathcal{D}}}\}$ [Mez26, Def. 5.1].

In the draft, 1.2 is Def. 3.7 (used in Def. 3.28), 1.3 is Def. 3.32 (used for Prop. 3.47), and Remark 3.29 proposes 1.1.

2. How the spans compare

Lemma 2.1.

$$\mathrm{Thick}_{\mathcal{D}}(\mathcal{S}) \overset{(1)}{=} \mathcal{D} \cap \langle\mathcal{S}\rangle^{\mathrm{GR}}_{\widehat{\mathcal{D}}} \overset{(2)}{=} \mathcal{D} \cap {}^{\perp_{\widehat{\mathcal{D}}}}\left(\mathcal{S}^{\perp_{\widehat{\mathcal{D}}}}\right) \overset{(3)}{\subseteq} \langle\mathcal{S}\rangle^{\mathrm{M}}_{\mathcal{D}}.$$

Proof. (1) Put $\mathcal{L} = \langle\mathcal{S}\rangle^{\mathrm{GR}}_{\widehat{\mathcal{D}}}$. By [Nee92, Lemma 2.2], applied to the compact objects $s[n]$, $s \in \mathcal{S}$, $n \in \mathbb{Z}$, the objects compact in $\mathcal{L}$ form their thick span; as coproducts in $\mathcal{L}$ are computed in $\widehat{\mathcal{D}}$, these are the objects of $\mathcal{D} \cap \mathcal{L}$. (2) Let $\iota\colon \mathcal{L} \hookrightarrow \widehat{\mathcal{D}}$. The left orthogonal ${}^{\perp}(\mathcal{S}^{\perp})$ contains $\mathcal{S}$ and is closed under colimits, so it contains $\mathcal{L}$. Conversely, $\mathcal{L}$ is the closure of $\{s[n]\}$ under extensions and small colimits (this closure is closed under $[-1]$), hence presentable [HA, Prop. 1.4.4.11(2)], so ι has a right adjoint R [HTT, Cor. 5.5.2.9]. For $X \in \widehat{\mathcal{D}}$ let $\iota RX \to X \to C$ be the counit and its cofibre; $\underline{\mathrm{Map}}(s, \iota RX) \to \underline{\mathrm{Map}}(s, X)$ is an equivalence for $s \in \mathcal{S}$, so $C \in \mathcal{S}^{\perp}$. If $X \in {}^{\perp}(\mathcal{S}^{\perp})$, then $\underline{\mathrm{Map}}(X, C) = 0 = \underline{\mathrm{Map}}(\iota RX, C)$, hence $\underline{\mathrm{Map}}(C, C) = 0$, $C = 0$ and $X \simeq \iota RX \in \mathcal{L}$. (3) $\mathcal{S}^{\perp_{\mathcal{D}}} = \mathcal{S}^{\perp_{\widehat{\mathcal{D}}}} \cap \mathcal{D}$ has fewer test objects, so its left orthogonal in $\mathcal{D}$ is larger. □

Lemma 2.2. *A small stable ∞-category $\mathcal{A}$ with all small colimits is zero. Hence if $\mathcal{S} \not\subseteq \{0\}$, no full stable subcategory of $\mathcal{D}$ with all small colimits contains $\mathcal{S}$, and $\langle\mathcal{S}\rangle^{\mathrm{GR}}_{\mathcal{D}}$ is undefined.*

Proof. Let $X \in \mathcal{A}$ and let I be a set larger than the set of morphisms of $h\mathcal{A}$. Coproducts in $\mathcal{A}$ are coproducts in $h\mathcal{A}$, so $\mathrm{Hom}_{h\mathcal{A}}(\bigoplus_I X, X) \cong \mathrm{End}(X)^I$. If $\mathrm{End}(X)$ had two elements, this set would have at least $2^{|I|} > |I|$ elements. So $\mathrm{End}(X) = \{0\}$ and $X \simeq 0$. This is the dual of Freyd's argument that a small complete category is a preorder [Fre64, Ch. 3, Ex. D]. □

Remark 2.3. Reading "cocomplete" as "closed under finite colimits" gives $\mathrm{Stab}_{\mathcal{D}}(\mathcal{S})$; by [Tho97, Thm. 2.1], $\mathrm{Stab}_{\mathcal{D}}(\mathcal{S}) = \mathcal{D}$ iff $\mathrm{Thick}_{\mathcal{D}}(\mathcal{S}) = \mathcal{D}$ and the $[s]$ generate $K_0(\mathcal{D})$. Proposition I proves only the first: it extracts Δ_μ from $\bigoplus_s \Delta_\mu[d_s]$ as a retract.

3. Skyscrapers on the line

Definition 3.1. $D^b_c(\mathbb{A}^1)$ is the stable ∞-category of bounded complexes M of sheaves of k-vector spaces on $\mathbb{C} = \mathbb{A}^1(\mathbb{C})$ for which some finite $\Sigma_M \subset \mathbb{C}$ has two properties: on $\mathbb{C} \setminus \Sigma_M$ every $\mathcal{H}^n M$ is a local system of finite rank (M is lisse there), and every stalk $(\mathcal{H}^n M)_b$, $b \in \Sigma_M$, is finite-dimensional. For $a \in \mathbb{C}$ with $i_a \colon \{a\} \hookrightarrow \mathbb{A}^1$, put

$$\delta_a = i_{a*}k, \qquad M_a = i_a^* M, \qquad \operatorname{supp} M = \overline{\{x \in \mathbb{C} : M_x \neq 0\}}.$$

Proposition 3.2. *Let $\mathcal{D} = D^b_c(\mathbb{A}^1)$ and $\mathcal{S} = \{\delta_a : a \in \mathbb{C}\}$.*

(a) $\mathcal{S}^{\perp_{\mathcal{D}}} = 0$, so $\langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{D}} = \mathcal{D}$.

(b) $\mathrm{Thick}_{\mathcal{D}}(\mathcal{S}) = \{M \in \mathcal{D} : \operatorname{supp} M$ finite$\}$; in particular $\underline{k}_{\mathbb{A}^1} \notin \mathrm{Thick}_{\mathcal{D}}(\mathcal{S})$.

Hence $\mathrm{Thick}_{\mathcal{D}}(\mathcal{S}) \subsetneq \langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{D}}$: inclusion (3) of Lemma 2.1 is strict.

Lemma 3.3. *If $M \in D^b_c(\mathbb{A}^1)$ and $a \notin \Sigma_M$, then $i_a^! M \simeq M_a[-2]$.*

Proof. Choose an open disc $B \ni a$ disjoint from Σ_M, and $p \colon B \to \mathrm{pt}$. The counit $p^* p_*(M|_B) \to M|_B$ is an isomorphism: both sides are exact in M, so by induction on the number of nonzero $\mathcal{H}^n M$ it suffices to treat a local system L, which is constant on the contractible B with $H^{>0}(B, L) = 0$. So $M|_B \simeq p^* M_a$. Since $i_a^!$ depends only on $M|_B$ and M_a is a finite sum of shifts of k, it suffices to compute $i_a^! \underline{k}_B$, the local cohomology at a. As $B \simeq \mathrm{pt}$, $B \setminus a \simeq S^1$ and restriction is an isomorphism on H^0,

$$i_a^! \underline{k}_B = \mathrm{fib}\big(R\Gamma(B, k) \to R\Gamma(B \setminus a, k)\big) = \mathrm{fib}\big(k \to k \oplus k[-1]\big) = k[-2]. \qquad \square$$

Proof of Proposition 3.2. Step 1. Since $i_{a*} \dashv i_a^!$, $\underline{\mathrm{Map}}_{\mathcal{D}}(\delta_a, M) \simeq \underline{\mathrm{Map}}_{D(k)}(k, i_a^! M) \simeq i_a^! M$.

Step 2. If $\Sigma \subset \mathbb{C}$ is finite and $M_x = 0$ for $x \notin \Sigma$, then M vanishes on $\mathbb{C} \setminus \Sigma$; the localization triangle and discreteness of Σ give $M \simeq \bigoplus_{b \in \Sigma} i_{b*} M_b$, and $i_a^! i_{b*} V$ is V if $a = b$ and 0 otherwise.

Step 3: (a). Let $M \in \mathcal{S}^{\perp_{\mathcal{D}}}$; by Step 1, $i_a^! M = 0$ for all a. Lemma 3.3 gives $M_a = 0$ for $a \notin \Sigma_M$, and then Step 2 with $\Sigma = \Sigma_M$ gives $M_a \simeq i_a^! M = 0$ for $a \in \Sigma_M$. So $M = 0$ and $\langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{D}} = {}^{\perp_{\mathcal{D}}} 0 = \mathcal{D}$.

Step 4. The stalk functor $M \mapsto M_x = i_x^* M$ is exact and carries retracts to retracts. So for a cofibre sequence $A \to B \to C$ and a retract M of N,

$$\operatorname{supp} M[1] = \operatorname{supp} M, \qquad \operatorname{supp} C \subseteq \operatorname{supp} A \cup \operatorname{supp} B, \qquad \operatorname{supp} M \subseteq \operatorname{supp} N.$$

Hence the objects of finite support form a thick subcategory containing $\mathcal{S}$, and $\mathrm{Thick}_{\mathcal{D}}(\mathcal{S}) \subseteq \{M : \operatorname{supp} M \text{ finite}\}$.

Step 5: (b). Conversely, if $\operatorname{supp} M$ is finite, Step 2 writes M as a finite sum of objects $i_{b*} M_b \simeq \bigoplus_n \delta_b^{\oplus \dim H^n(M_b)}[-n]$, and a finite sum is the cofibre of a zero map. Finally $(\underline{k}_{\mathbb{A}^1})_x = k \neq 0$ for every x, so $\operatorname{supp} \underline{k}_{\mathbb{A}^1} = \mathbb{C}$ is infinite and $\underline{k}_{\mathbb{A}^1} \notin \mathrm{Thick}_{\mathcal{D}}(\mathcal{S})$. □

4. The test object the small category misses

By (2), $\underline{k}_{\mathbb{A}^1} \notin {}^{\perp}(\mathcal{S}^{\perp_{\widehat{\mathcal{D}}}})$, so some nonzero ind-object is orthogonal to every skyscraper. For finite $T \subset \mathbb{C}$ let $U_T = \mathbb{A}^1 \setminus T$ and $j_T \colon U_T \hookrightarrow \mathbb{A}^1$.

Lemma 4.1. *Let T run over the finite subsets of $\mathbb{C}$, ordered by inclusion, with the restriction maps $j_{T*}\underline{k}_{U_T} \to j_{T'*}\underline{k}_{U_{T'}}$ for $T \subseteq T'$. In $\widehat{\mathcal{D}} = \operatorname{Ind} D^b_c(\mathbb{A}^1)$, the object $F = \operatorname{colim}_T j_{T*}\underline{k}_{U_T}$ satisfies*

$$\underline{\operatorname{Map}}_{\widehat{\mathcal{D}}}(\delta_a, F) = 0 \quad (a \in \mathbb{C}), \qquad H^0\underline{\operatorname{Map}}_{\widehat{\mathcal{D}}}(\underline{k}_{\mathbb{A}^1}, F) = k.$$

Proof. Objects of $D^b_c(\mathbb{A}^1)$ are compact in $\widehat{\mathcal{D}}$, so both mapping complexes commute with the filtered colimit. The $T \ni a$ are cofinal, and for them $\underline{\operatorname{Map}}(\delta_a, j_{T*}\underline{k}_{U_T}) \simeq \underline{\operatorname{Map}}(j_T^*\delta_a, \underline{k}_{U_T}) = 0$. And $\underline{\operatorname{Map}}(\underline{k}_{\mathbb{A}^1}, j_{T*}\underline{k}_{U_T}) \simeq R\Gamma(U_T, k) \simeq k \oplus k^{|T|}[-1]$, with restriction an isomorphism on H^0. □

Lemma 4.2. *Let $\mathcal{D}' = \operatorname{Thick}_{\widehat{\mathcal{D}}}(D^b_c(\mathbb{A}^1) \cup \{F\})$, a small stable category containing $\mathcal{D} = D^b_c(\mathbb{A}^1)$. Then $\underline{k}_{\mathbb{A}^1} \in \langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{D}}$ but $\underline{k}_{\mathbb{A}^1} \notin \langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{D}'}$: the exact inclusion $\mathcal{D} \hookrightarrow \mathcal{D}'$ does not carry $\langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{D}}$ into $\langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{D}'}$, unlike the thick span.*

Proof. The first is Proposition 3.2(a). By Lemma 4.1, $F \in \mathcal{S}^{\perp_{\mathcal{D}'}}$ and $\underline{\operatorname{Map}}(\underline{k}_{\mathbb{A}^1}, F) \neq 0$. □

5. Consequences for the Satake subcategory

Take $\mathcal{D} = \mathcal{C}$ and $\mathcal{S} = \{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\}$, so $\operatorname{Thick}_{\mathcal{C}}(\mathcal{S}) = \mathcal{C}^{\mathrm{Sat}}$.

- $\operatorname{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G})$ is the thick span of the objects $\mathcal{O} \otimes V$, $V \in \operatorname{Rep}\check{G}$; a fully faithful, exact, Hecke-compatible Φ with $\Phi(\mathcal{O}) = \mathrm{IC}_0$ has essential image $\mathcal{C}^{\mathrm{Sat}}$. With the double orthogonal as definition, the unramified conjecture would also assert equality in (3).
- If $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$, every larger span is $\mathcal{C}$. The draft's proof of Prop. 3.47 tests $F \in \mathcal{C}$, giving only $\langle \mathcal{S} \rangle^{\mathrm{M}}_{\mathcal{C}} = \mathcal{C}$; with $F \in \widehat{\mathcal{C}}$ it gives $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$ by (1)–(2).
- Def. 3.32 as printed uses mapping anima (draft Def. 3.8), testing $\operatorname{Hom}(s[n], M)$ only for $n \geq 0$; [Mez26, Def. 5.1] and our convention test all $n \in \mathbb{Z}$.

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