PROPOSITION IV: THE AFFINE SPACE $\mathbb{A}^n$ UNDER GL_n

Proposition I deduces $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$ from the hypotheses H1–H5 on the orbits of L^+G on LX. We ask how far its proof reaches for $X = \mathbb{A}^n$ under $G = \mathrm{GL}_n$, which is not a symmetric space and for which H1(a) fails. The answer: the proof is a finite dévissage, and the finiteness it needs is restored one finite level of LX at a time; on the pieces of a level, H2–H5 hold with the central Satake objects alone, and these already generate.

1. Setting

$k = \mathbb{C}$, $F = k(\!(t)\!)$, $\mathcal{O} = k[\![t]\!]$, val the t-adic valuation with $\operatorname{val} 0 = \infty$. $G = \mathrm{GL}_n$ acts on column vectors; T is the diagonal torus and B the upper triangular Borel subgroup. Then

$$LX = F^n, \qquad L^+X = \mathcal{O}^n, \qquad L^+G = \mathrm{GL}_n(\mathcal{O}), \qquad x_0 = e_n, \qquad \operatorname{ord} v = \min_i \operatorname{val} v_i \in \mathbb{Z} \cup \{\infty\}.$$

X is smooth, affine and spherical, with open B-orbit $B \cdot e_n = \{v_n \neq 0\}$. The Hecke labels, the central labels and the *leading exponent* are

$$\Lambda_T^+ = \{\lambda \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n\}, \qquad m \cdot \mathbf{1} = (m, \ldots, m), \qquad \varphi(\lambda) := \lambda_n, \qquad t^\lambda x_0 = t^{\lambda_n} e_n.$$

$\mathcal{C} = \mathrm{Shv}_{\mathrm{cons}}(L^+G \backslash L\mathbb{A}^n)$ is the category of Proposition I for this X; $\mathrm{IC}_0 = \underline{k}_{\mathcal{O}^n}$, the constant sheaf on L^+X; $\mathcal{C}^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}}\{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\}$, with $V_\lambda = \mathrm{IC}(\mathrm{Gr}_G^{\leq \lambda})$ and $\mathcal{A} * \mathcal{F} = m_!(\mathcal{A} \widetilde{\boxtimes} \mathcal{F})$ along $m \colon LG \times^{L^+G} LX \to LX$. Constant objects are unshifted; a fixed shift does not change thick spans.

Lemma 1.1 (Orbits)**.** *The L^+G-orbits on F^n are the shells $X_m = \{\operatorname{ord} v = m\} = t^m(\mathcal{O}^n \backslash t\mathcal{O}^n)$, $m \in \mathbb{Z}$, and the point 0. The closure of a shell is a lattice:*

$$\overline{X_m} = t^m\mathcal{O}^n = \bigsqcup_{m' \geq m} X_{m'} \sqcup \{0\}, \qquad \partial X_m = t^{m+1}\mathcal{O}^n.$$

Proof. If $\operatorname{ord} v = 0$, some $v_i \in \mathcal{O}^\times$, and the matrix with columns $e_1, \ldots, \widehat{e_i}, \ldots, e_n, v$ has determinant $\pm v_i$: it lies in L^+G and sends e_n to v, so $L^+G \cdot e_n = X_0$. A nonzero v is $t^m v_0$ with $m = \operatorname{ord} v$ and v_0 primitive, and L^+G preserves each $t^m\mathcal{O}^n$, hence ord; so $L^+G \cdot v = X_m$. In $t^m\mathcal{O}^n / t^M\mathcal{O}^n$, $M > m$, the shell X_m is the preimage of the complement of the proper linear subspace $t^{m+1}\mathcal{O}^n / t^M\mathcal{O}^n$, dense open; so X_m is open and dense in the closed subset $t^m\mathcal{O}^n$ of LX. □

So H1(a) fails: every orbit closure contains infinitely many orbits and the closed orbit $\{0\}$, which has codimension $(M-m)n \to \infty$ in $t^m\mathcal{O}^n$. Moreover $L^+X = \overline{X_0}$ is not an orbit, so $\mathrm{IC}_0 = \mathrm{IC}(\overline{X_0})$ is not a standard object; and the only closed finite unions of orbits are $\varnothing$ and $\{0\}$, so H2 fails at $\{0\}$: the skyscraper δ_0 is not in $\mathcal{C}$. H1(b), H3 and H4 hold, and H5 holds at every shell with $\lambda = m \cdot \mathbf{1}$ (Steps 3–5 below), but no Satake object is supported on $\{0\}$.

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2. The level form of H1–H5

Levels. For $N, M \geq 0$ put

$$Y_N = t^{-N}\mathcal{O}^n = \overline{X_{-N}}, \qquad Y_{N,M} = t^{-N}\mathcal{O}^n/t^M\mathcal{O}^n \cong \mathbb{A}^{(N+M)n}, \qquad \pi_{N,M}\colon Y_N \twoheadrightarrow Y_{N,M},$$
$$G_{N,M} = 1 + t^{N+M}M_n(\mathcal{O}) = \ker\big(L^+G \to J_{N+M}G\big), \qquad J_{N+M}G = \mathrm{GL}_n(\mathcal{O}/t^{N+M}).$$

$G_{N,M}$ acts trivially on $Y_{N,M}$, so the connected algebraic group $J_{N+M}G$ acts on the vector space $Y_{N,M}$ and $\pi_{N,M}$ is equivariant. A *piece* of level (N,M) is the preimage under $\pi_{N,M}$ of a $J_{N+M}G$-orbit; $\mathcal{P}_{N,M}$ is the set of pieces, ordered by $P' \leq P \iff P' \subseteq \overline{P}$; the *tail* is $T_M = t^M\mathcal{O}^n = \pi_{N,M}^{-1}(0)$. Pieces are L^+G-stable, locally closed, unions of orbits; the standard of a piece is $\Delta_P = a_{P!}\underline{k}_P$ for $a_P\colon P \hookrightarrow LX$, and $\Delta_m := \Delta_{X_m}$. The *finite-level category* is

$$\mathcal{C}^{(N,M)} = \big\{r_{N*}\pi_{N,M}^*\bar{M} :\ \bar{M} \in \mathrm{Shv}_{\mathrm{cons}}(J_{N+M}G\backslash Y_{N,M})\big\} \subseteq \mathcal{C}, \qquad \mathcal{C}_W^{(N,M)} = \mathcal{C}^{(N,M)} \cap \mathcal{C}_W,$$

for $r_N\colon Y_N \hookrightarrow LX$ and $W \subseteq LX$ stable and locally closed. The level hypotheses are H1–H5 with "orbit" read as "piece of a level" and "closed finite union of orbits" as "closed union of pieces of one level".

(L1) *(a) For every level, $\#\mathcal{P}_{N,M} < \infty$, each piece is L^+G-stable and locally closed in Y_N, and $\overline{P} \cap Y_N = \bigsqcup_{P' \leq P} P'$. (b) $\mathcal{C} = \bigcup_{N,M} \mathcal{C}^{(N,M)}$.*

(L2) *For closed unions $D \subseteq Z$ of pieces of one level and a piece $P \subseteq Z$: restriction f^* and extension by zero $f_!$ along $D \hookrightarrow Z$, $Z \hookrightarrow LX$, $P \hookrightarrow Z$ and their composites are defined, exact, compatible with composition, and preserve the finite-level categories; $f_! = f_*$ is right adjoint to f^* for closed f; and $j_!j^*M \to M \to i_*i^*M \xrightarrow{+1}$ when P is open in Z with complement D.*

(L3) *For every piece P of level (N,M), $\mathcal{C}_P^{(N,M)}$ has a bounded t-structure with heart $\mathrm{Vect}_k^{\mathrm{fd}}$, $V \mapsto \underline{k}_P \otimes V$.*

(L4) *$*: \mathcal{H} \times \mathcal{C} \to \mathcal{C}$ is defined and $V_\lambda * \mathrm{IC}_0 \in \mathcal{C}$ for all $\lambda \in \Lambda_T^+$.*

(L5) *For every level (N,M) and piece $P \in \mathcal{P}_{N,M}$ there is $\lambda \in \Lambda_T^+$ with $V_\lambda * \mathrm{IC}_0 \in \mathcal{C}^{(N,M)}$, $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{P}$, and $a_P^*(V_\lambda * \mathrm{IC}_0) \simeq \bigoplus_{s=1}^{N_P} \underline{k}_P[d_s]$ in $\mathcal{C}_P^{(N,M)}$, $0 < N_P < \infty$.*

When every orbit closure is a finite union of orbits, the closed finite unions of orbits serve as levels, with the orbits as pieces, and (L1)–(L5) reduce to H1–H5.

Proposition I′ (level form)**.** *Assume* (L1)–(L5)*. Then $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$, and $\underline{\mathrm{Map}}_{\widehat{\mathcal{C}}}(V_\lambda * \mathrm{IC}_0, F) = 0$ for all λ implies $F = 0$ in $\widehat{\mathcal{C}} = \mathrm{Ind}\,\mathcal{C}$.*

Proof. Fix a level (N,M). (i) *Standard generation at the level*: for every closed union Z of pieces, $\mathcal{C}_Z^{(N,M)} = \mathrm{Thick}\{\Delta_{P,Z} : P \subseteq Z\}$. This is Step 1 of Proposition I with (L1)(a), (L2), (L3) for H1(a), H2, H3: on one piece, truncation in the t-structure of (L3) builds every object from $\underline{k}_P$ by (T1); in Z, a maximal piece is open by (L1)(a), the triangle of (L2) leaves an object on one piece fewer, and (T2) for $j_!$ and i_* closes the induction on $\#\{P \subseteq Z\}$. (ii) *Induction over the pieces*: $\Delta_P \in \mathcal{C}^{\mathrm{Sat}}$ for every P, by induction on $h(P) = \#\{P' \leq P\}$. Take λ from (L5) and $A = V_\lambda * \mathrm{IC}_0$. The boundary part $i_{P*}i_P^*A$ lies in $\mathcal{C}_{\partial P}^{(N,M)}$ by (L2), hence in $\mathrm{Thick}\{\Delta_{P'} : P' < P\} \subseteq \mathcal{C}^{\mathrm{Sat}}$ by (i), (T2) and induction. The top part $a_{P!}a_P^*A$ lies in $\mathcal{C}^{\mathrm{Sat}}$ by (L5)(a), the triangle of (L2) and (T1); by (L5)(b) it is $\bigoplus_s \Delta_P[d_s]$ with $N_P \geq 1$ summands,

so Δ_P is a retract of an object of $\mathcal{C}^{\mathrm{Sat}}$. These are Steps 2–6 of Proposition I. (iii) By (i) and (ii), $\mathcal{C}^{(N,M)} \subseteq \mathcal{C}^{\mathrm{Sat}}$; by (L1)(b), $\mathcal{C} = \bigcup \mathcal{C}^{(N,M)} \subseteq \mathcal{C}^{\mathrm{Sat}}$. Detection is Step 7 of Proposition I, unchanged. □

3. Statement

Proposition IV ($\mathbb{A}^n$ under GL_n)**.** *Let $X = \mathbb{A}^n$ under $G = \mathrm{GL}_n$, $n \geq 1$, with $\mathcal{C}$, IC_0, the shells, levels and pieces as above.*

(a) *H1(a) fails: $\overline{X_m} = \bigsqcup_{m' \geq m} X_{m'} \sqcup \{0\}$. The orbit $\{0\}$ is closed, of infinite codimension, and has no standard object in $\mathcal{C}$; Proposition I does not apply as stated.*

(b) *Given* (L1)*(b), the level hypotheses* (L1)–(L5) *hold, and* (L5) *is witnessed on every piece by a central label: $m \cdot \mathbf{1}$ at X_m and $M \cdot \mathbf{1}$ at T_M.*

(c) *Hence the central Satake objects, which are the constant sheaves on the lattices, already generate, and the Satake objects detect zero in $\widehat{\mathcal{C}}$:*

$$\boxed{\mathcal{C} = \mathrm{Thick}_{\mathcal{C}}\{V_{m\cdot\mathbf{1}} * \mathrm{IC}_0 = \underline{k}_{t^m\mathcal{O}^n} : m \in \mathbb{Z}\} = \mathcal{C}^{\mathrm{Sat}}.}$$

Classical reading. For F non-archimedean, $\mathcal{H}_K \cdot \varphi_0 = C_K$ with the elements $\mathbf{1}_{t^m K}$ alone: $\mathbf{1}_{X_m} = \mathbf{1}_{t^m\mathcal{O}^n} - \mathbf{1}_{t^{m+1}\mathcal{O}^n} = (\mathbf{1}_{t^m K} - \mathbf{1}_{t^{m+1}K}) * \varphi_0$ (F3, Theorem 3, Iwahori workspace).

4. Proof

Step 1 (orbits; part (a)). Lemma 1.1. The skyscraper at 0 is supported on a set that is not a union of pieces of any level, so under (L1)(b) it is not an object of $\mathcal{C}$.

Step 2 ((L1)). For $-N \leq m < M$, $X_m \subseteq Y_N$ is a union of cosets of $t^{m+1}\mathcal{O}^n \supseteq t^M\mathcal{O}^n$, so $X_m = \pi_{N,M}^{-1}\pi_{N,M}(X_m)$, and $\pi_{N,M}(X_m)$ is one $J_{N+M}G$-orbit, L^+G being transitive on X_m. Every shell with $m \geq M$ and the point 0 map to $0 \in Y_{N,M}$, fixed by $J_{N+M}G$, with preimage T_M. These preimages partition Y_N, and by Lemma 1.1 $\overline{X_m} \cap Y_N = t^m\mathcal{O}^n = \bigsqcup_{m \leq m' < M} X_{m'} \sqcup T_M$:

$$\mathcal{P}_{N,M} = \{X_m : -N \leq m < M\} \sqcup \{T_M\}, \qquad T_M < X_{M-1} < \cdots < X_{-N}, \qquad \#\mathcal{P}_{N,M} = N+M+1.$$

This is (L1)(a). (L1)(b), that every object of $\mathcal{C}$ is pulled back from some $Y_{N,M}$, is the reading of the draft's colimit description of sheaves on the placid ind-scheme $LX = \varinjlim_N Y_N$, $Y_N = \varprojlim_M Y_{N,M}$; it is assumed, as in Proposition IV.1 of the Iwahori workspace.

Step 3 ((L2), (L3)). Pieces of a level are preimages of the orbits of the connected algebraic group $J_{N+M}G$ on the vector space $Y_{N,M}$. The equivariant derived category of $Y_{N,M}$ [BL] has f^*, $f_!$ along stable locally closed embeddings, exact and compatible with composition, with $f_! = f_*$ right adjoint to f^* for closed f and the localization triangle [BBD, §1.4]; pullback along the smooth equivariant surjection $\pi_{N,M}$ commutes with them, so they preserve the finite-level categories. This is (L2). For (L3): the stabilizer of $0 \in Y_{N,M}$ is $J_{N+M}G$, connected. The stabilizer of $t^m e_n \bmod t^M$, $-N \leq m < M$, is $\{g \in \mathrm{GL}_n(\mathcal{O}/t^{N+M}) : ge_n \equiv e_n \bmod t^{M-m}\}$, the preimage under reduction modulo t of $\{\bar{g} \in \mathrm{GL}_n(k) : \bar{g}e_n = e_n\} \cong \mathrm{GL}_{n-1} \ltimes k^{n-1}$ intersected with an affine-linear condition on the last column; the kernel of reduction is connected unipotent, so the stabilizer is connected. The ordinary heart is $\mathrm{Rep}_k^{\mathrm{fd}}(\pi_0 \,\mathrm{Stab}) = \mathrm{Vect}_k^{\mathrm{fd}}$.

Step 4 (central Satake objects). $t^{m\cdot\mathbf{1}} = t^m$ is central in LG, so $\mathrm{Gr}_G^{\leq m\cdot\mathbf{1}} = \{t^m L^+G\}$ is a point and $V_{m\cdot\mathbf{1}}$ its constant sheaf. The convolution space $t^m L^+G \times^{L^+G} \mathcal{O}^n$ maps isomorphically

onto the lattice $t^m\mathcal{O}^n$ by $[t^m g, v] \mapsto t^m gv$, with inverse $w \mapsto [t^m, t^{-m}w]$. Hence

$$V_{m\cdot\mathbf{1}} * \mathrm{IC}_0 \simeq \underline{k}_{t^m\mathcal{O}^n}, \qquad \underline{k}_{t^m\mathcal{O}^n} \in \mathcal{C}^{(N,M)} \ \text{ for } \ -N \le m \le M,$$

the latter because $t^m\mathcal{O}^n/t^M\mathcal{O}^n$ is a $J_{N+M}G$-stable linear subspace of $Y_{N,M}$. This is (L4) for the central labels; for all λ the action is assumed to land in $\mathcal{C}$, with the same status as H4 in Proposition III (see Remark 3 for the support bound).

Step 5 ((L5)). At a shell piece X_m, $-N \le m < M$, take $\lambda = m\cdot\mathbf{1}$: by Step 4, $V_\lambda * \mathrm{IC}_0 = \underline{k}_{t^m\mathcal{O}^n} \in \mathcal{C}^{(N,M)}$ is supported on $\overline{X_m}$ and restricts to $\underline{k}_{X_m}$ on the open subset X_m: one summand, no shift. At the tail, $\lambda = M\cdot\mathbf{1}$ gives $\underline{k}_{T_M}$, supported on T_M with restriction $\underline{k}_{T_M}$. The leading exponent is onto $\mathbb{Z}$ already on the central labels, $\varphi(m\cdot\mathbf{1}) = m$: the coverage condition of Proposition III has a trivial analogue. With Steps 2–4 this is (b).

Step 6 (conclusion; part (c)). Proposition I′ gives $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$ and detection. Explicitly, X_m is open in $t^m\mathcal{O}^n$ with closed complement $t^{m+1}\mathcal{O}^n$, and the localization triangle of (L2) for $\underline{k}_{t^m\mathcal{O}^n}$ reads

$$\Delta_m \longrightarrow \underline{k}_{t^m\mathcal{O}^n} \longrightarrow \underline{k}_{t^{m+1}\mathcal{O}^n} \xrightarrow{+1}, \qquad \text{so} \qquad \Delta_m \in \mathrm{Thick}\{V_{m\cdot\mathbf{1}} * \mathrm{IC}_0,\ V_{(m+1)\cdot\mathbf{1}} * \mathrm{IC}_0\}$$

by (T1); the tail standard $\underline{k}_{T_M}$ is itself a central Satake object. No retract is needed: the top part is one copy of Δ_m. Every piece standard of every level lies in $\mathrm{Thick}\{\underline{k}_{t^m\mathcal{O}^n} : m \in \mathbb{Z}\}$, so by (i) of Proposition I′ and (L1)(b) this thick span is $\mathcal{C}$. □

5. Remarks

1. *The point* 0. At level (N, M) it lies in the tail T_M, whose standard is the central Satake object $\underline{k}_{T_M}$. Its skyscraper δ_0 is not pulled back from any level, so it is not an object of $\mathcal{C}$: the orbit-by-orbit conclusion of Proposition I has no content at 0, and the level form replaces it. For functions, local constancy at 0 plays the same role.
2. IC_0 *is not a standard.* $L^+X = \overline{X_0}$ is not an orbit, and $\Delta_0 = \mathrm{fib}(\mathrm{IC}_0 \to V_{\mathbf{1}} * \mathrm{IC}_0)$. For a symmetric space $L^+X = \mathcal{O}_0$ is a closed orbit and $\mathrm{IC}_0 = \Delta_0$.
3. *Non-central labels.* Every lattice $L = g\mathcal{O}^n$ with $g \in L^+G\,t^{\lambda'}L^+G$, $\lambda' \le \lambda$, satisfies $t^{\lambda_1}\mathcal{O}^n \subseteq L \subseteq t^{\lambda_n}\mathcal{O}^n$. Proper base change gives $(V_\lambda * \mathrm{IC}_0)_v \simeq R\Gamma(\{L \in \mathrm{Gr}_G^{\le\lambda} : v \in L\}, V_\lambda)$, hence $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq t^{\lambda_n}\mathcal{O}^n = \overline{X_{\varphi(\lambda)}}$, with stalk $IH^*(\mathrm{Gr}_G^{\le\lambda})$ at every $v \in t^{\lambda_1}\mathcal{O}^n$. For $n = 2$ and $V_\lambda = \underline{k}_{\mathbb{P}^1}[1]$: $\lambda = (0,-1)$ has support $t^{-1}\mathcal{O}^2$, stalk $k[1]$ on X_{-1} and $H^*(\mathbb{P}^1)[1]$ on $\mathcal{O}^2$; $\lambda = (1,0)$ has support $\mathcal{O}^2$, stalk $k[1]$ on X_0 and $H^*(\mathbb{P}^1)[1]$ on $t\mathcal{O}^2$. On $X_{\varphi(\lambda)}$ the fibre is nonempty and the argument of Proposition III, Step 6, applies verbatim to give a nonzero stalk; this is not used above.
4. *No involution.* The two conditions of Proposition III, $Z_{G^\theta}(S)$ connected and $\varphi(\Lambda_T^+) = \Lambda_S^+$, are replaced by connected stabilizers (Step 3) and the surjectivity of the leading exponent on the centre (Step 5), both immediate; the criterion itself, the Cartan embedding and the support bound on Gr_G do not transfer.
5. *Iwahori level and functions.* At Iwahori level the same levels have $(N+M)n+1$ pieces, the rays $O_{m,j}$, and the translates needed are all I-stable lattices $\Lambda_{m,j}$ (Proposition IV.1 of the Iwahori workspace); at spherical level only the lattices $t^m\mathcal{O}^n$ occur. For functions both statements are F3, Theorem 3.
6. *Scope.* Transferred: the dévissage of Proposition I, (T1), (T2), detection. Replaced: orbits by pieces of a level, closed finite unions of orbits by closed unions of pieces, H1 by (L1), the

induction on $h(\mu)$ by the chain of a level and the union over levels. Lost: the statement at $\{0\}$, $\mathrm{IC}_0 = \Delta_0$, the criterion of Proposition III. Gained: generation by the centre t^m, $m \in \mathbb{Z}$, alone.

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