# PROPOSITION III: THE HYPOTHESES ON SYMMETRIC SPACES

Proposition I: under H1–H5 below, the Satake objects $V_\lambda * \mathrm{IC}_0$ thickly generate the category $\mathcal{C}$ of $L^+G$-equivariant constructible sheaves on $LX$. We decide H1–H5 for symmetric spaces with connected fixed group, proved up to the transfer to $\mathbb{C}$ of Steps 3 and 5, which are carried out in Chen–Yi's $\ell$-adic theory [CY, App. A.5].

## 1. Setting

Over $\mathbb{C}$: $G$ connected reductive, $\theta$ an involution, $K = G^\theta$ connected, $X = G/K$, $x_0 = eK$, and $\tau\colon X \hookrightarrow G$, $gK \mapsto g\theta(g)^{-1}$, which is equivariant for $g \cdot y = gy\theta(g)^{-1}$ [CY, §2.1]. Fix a maximal $\theta$-split torus $S$, a minimal $\theta$-split parabolic $P$ with $L := Z_G(S) = P \cap \theta(P)$, and $S \subseteq T \subseteq B \subseteq P$. With $\Phi^+ \supseteq \Phi_L^+$ the positive roots of $G \supseteq L$ and $\Lambda_T = X_*(T)$:

$$\Lambda_S = \{\lambda \in \Lambda_T : \theta\lambda = -\lambda\}, \quad \Lambda_S^+ = \Lambda_S \cap \Lambda_T^+, \quad \nu \le \mu \iff \mu - \nu \in \mathbb{Z}_{\ge 0}\check{\Delta}, \quad \varphi(\lambda) = \lambda - \theta\lambda.$$

**Lemma 1.1.** (a) *$\theta\alpha = \alpha$ for $\alpha \in \Phi_L$ and $-\theta\alpha \in \Phi^+$ for $\alpha \in \Phi^+ \setminus \Phi_L$.* (b) *$\varphi(\Lambda_T^+) \subseteq \Lambda_S^+$, and $Z_G(\varphi(\lambda)) = L$ for $\lambda$ regular.*

*Proof.* (a) $\theta = \mathrm{id}$ on $[L, L]$ by maximality of $S$, and $P, \theta(P)$ are opposite with Levi $L$. (b) $\langle \alpha, \varphi\lambda\rangle$ is 0 on $\Phi_L$ and $\langle \alpha, \lambda\rangle + \langle -\theta\alpha, \lambda\rangle$ on $\Phi^+ \setminus \Phi_L$. □

With $t^\lambda = \lambda(t) \in LG$ and $\mathrm{Gr}_{\le\lambda}$ the projective Schubert varieties, convolution [Zhu, (2.1.17)] gives, for $a, b \in \Lambda_T^+$,

$$L^+G\, t^a L^+G \cdot L^+G\, t^b L^+G \subseteq \bigcup\nolimits_{c \le a+b} L^+G\, t^c L^+G. \tag{1}$$

**Theorem 1.2** ([CY, Prop. 17, Thm. 18], [Nad04])**.** *$LX = \bigsqcup_{\mu \in \Lambda_S^+} \mathcal{O}_\mu$, $\mathcal{O}_\mu = L^+G \cdot x_\mu$, $\tau(x_\mu) = t^\mu$, and $\overline{\mathcal{O}_\mu} = \bigsqcup_{\nu \le \mu} \mathcal{O}_\nu$. Moreover $t^\lambda x_0 = x_{\varphi(\lambda)}$ and $L^+X = \mathcal{O}_0$.*

Let $\mathcal{C}_\mu = \mathrm{Shv}_{\mathrm{cons}}(L^+G\backslash\mathcal{O}_\mu)$, $a_\mu\colon \mathcal{O}_\mu \hookrightarrow LX$, $\mathrm{IC}_0 = \underline{k}_{\mathcal{O}_0}$, $V_\lambda = \mathrm{IC}(\mathrm{Gr}_{\le\lambda})$, and $\mathcal{A} * \mathcal{F} = m_!(\mathcal{A}\widetilde{\boxtimes}\mathcal{F})$ for $m\colon LG \times^{L^+G} LX \to LX$. The hypotheses of Proposition I:

- **(H1)** $\#\{\nu \le \mu\} < \infty$, $\overline{\mathcal{O}_\mu} = \bigsqcup_{\nu \le \mu} \mathcal{O}_\nu$, and each $M \in \mathcal{C}$ is supported on finitely many $\overline{\mathcal{O}_\mu}$.
- **(H2)** $f^*$, $f_!$ along embeddings of orbits and closed finite unions of orbits exist, compose, preserve $\mathcal{C}$, and give $j_!j^*M \to M \to i_*i^*M \xrightarrow{+1}$.
- **(H3)** Each $\mathcal{C}_\mu$ has a bounded t-structure with heart $\mathrm{Vect}^{\mathrm{fd}}$ via $V \mapsto \underline{k}_{\mathcal{O}_\mu} \otimes V$.
- **(H4)** $*$ is defined on $\mathrm{Shv}_{\mathrm{cons}}(L^+G\backslash LG/L^+G) \times \mathcal{C}$ with values in $\mathcal{C}$.
- **(H5)** $\forall \mu \in \Lambda_S^+\ \exists \lambda \in \Lambda_T^+$: $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{\mathcal{O}_\mu}$ and $a_\mu^*(V_\lambda * \mathrm{IC}_0) \simeq \bigoplus_{s=1}^N \underline{k}[d_s]$, $0 < N < \infty$.

## 2. Statement

**Proposition III.** *In the setting of §1, assume the transfer of Steps 3 and 5 to $\mathbb{C}$. Then H1, H2, H4 hold, H3 holds iff $Z_K(S)$ is connected, and under H3, H5 holds iff $\varphi$ is onto. Hence*

$$\boxed{\text{H1–H5} \iff Z_K(S) \text{ connected and } \varphi(\Lambda_T^+) = \Lambda_S^+.}$$

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By direct computation, $(H\times H)/H$, $\mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}$, $\mathrm{GL}_n/(\mathrm{GL}_p\times\mathrm{GL}_q)$ satisfy H1–H5; $E_6/F_4$, $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ ($n\geq 3$), $\mathrm{GL}_{2m}/\mathrm{Sp}_{2m}$ ($m\geq 2$) have $Z_K(S)$ connected but miss $3\check{\omega}_1$, resp. $(1,1,1,1,0,\dots,0,-2,-2)$, $(1,1,1,1,0,\dots,0)$; $\mathrm{SL}_2/\mathrm{SO}_2$ fails both.

## 3. Proof

**Step 1 (H1).** Theorem 1.2; dominant $\nu\leq\mu$ have $|\nu|\leq|\mu|$ for a $W$-invariant norm; supports are finite unions by definition of $\mathcal{C}$.

**Lemma 3.1** (Step 2: support bound)**.** *If $\mathcal{O}_\nu$ meets $L^+G\,t^{\lambda'}\mathcal{O}_{\mu'}$ then $\nu\leq\varphi(\lambda')+\mu'$. Hence $\mathrm{supp}(V_\lambda * \mathrm{IC}_0)\subseteq\overline{\mathcal{O}_{\varphi(\lambda)}}$, and $*$ by $\mathrm{Gr}_{\leq\lambda}$ maps $\bigcup_{\mu\in I}\overline{\mathcal{O}_\mu}$ into $\bigcup_{\mu\in I}\overline{\mathcal{O}_{\varphi(\lambda)+\mu}}$.*

*Proof.* Put $x=-\theta\lambda'$ and $p=\frac{1}{2}(1-\theta)$. As $\tau(kt^{\lambda'}y)=kt^{\lambda'}\tau(y)t^x\theta(k)^{-1}$, (1) gives $D:=\lambda'+\mu'+x^+-\nu\in\mathbb{Z}_{\geq 0}\check{\Phi}^+$. By Lemma 1.1(a), $x$ is $L$-antidominant with $\langle\alpha,x\rangle\geq 0$ for $\alpha\in\Phi^+\setminus\Phi_L$, so $x^+=w_Lx$ and $E:=x^+-x\in\mathbb{Z}_{\geq 0}\check{\Phi}^+_L$. As $p(\lambda')=p(x)$, $p\mu'=\mu'$, $p\nu=\nu$, $pE=0$: $\varphi(\lambda')+\mu'-\nu=D-E=p(D)\in\mathbb{Z}\check{\Phi}\cap\mathbb{Q}_{\geq 0}\check{\Phi}^+=\mathbb{Z}_{\geq 0}\check{\Phi}^+$, since $p$ kills $\check{\Phi}_L$ and sends $\check{\alpha}$ to $\frac{1}{2}(\check{\alpha}+(-\theta\alpha)^\vee)$ for $\alpha\in\Phi^+\setminus\Phi_L$. The same computation gives $\lambda'\leq\lambda\Rightarrow\varphi(\lambda')\leq\varphi(\lambda)$; apply Theorem 1.2. □

**Step 3 (H2, H4; $\ell$-adic).** Orbit closures are placid with finitely presented embeddings [CY, Thm. 36], so [CY, App. A.5(3)] gives H2 (finite unions of closures are part of the assumed transfer). On $\mathrm{Gr}_{\leq\lambda}\,\widetilde{\times}Z$, $m$ is a closed embedding into $\mathrm{Gr}_{\leq\lambda}\times LX$ followed by projection, hence proper, with image in a closed finite union by Lemma 3.1: this is H4.

**Lemma 3.2** (Step 4: H3)**.** *Let $\mu\in\Lambda_S^+$ and $Z_K(\mu)=K\cap Z_G(\mu)$.* (a) $\pi_0\,\mathrm{Stab}_{L^+G}(x_\mu)\cong\pi_0 Z_K(\mu)$. (b) $Z_K(\mu)=Z_K(\mu)^\circ\cdot Z_K(S)$. (c) *H3 holds at $\mu$ iff $\mathrm{Stab}_{L^+G}(x_\mu)$ is connected.*

(a) The stabilizer is $Z_G(\mu)^\theta\ltimes(\text{pro-unipotent})^\theta$. (b) $K_M=K_M^\circ Z_{K_M}(S)$ for $M=Z_G(\mu)$ (recalled from [KR71, Ri82], not re-checked). (c) The standard heart of $\mathcal{C}_\mu$ is $\mathrm{Rep}^{\mathrm{fd}}(\pi_0\,\mathrm{Stab})$; use Maschke. By Lemma 1.1(b), $Z_K(\varphi\lambda)=Z_K(S)$ for $\lambda$ regular, so H3 $\iff$ all stabilizers connected $\iff$ $Z_K(S)$ connected.

**Lemma 3.3** (Step 5: H5; $\ell$-adic)**.** *Let $M=V_\lambda*\mathrm{IC}_0$, $V_\lambda$ pure of weight $d=\langle 2\rho,\lambda\rangle$.* (a) *$M_{x_{\varphi(\lambda)}}\neq 0$ (without H3).* (b) *$M\simeq\bigoplus_b\mathrm{IC}(\overline{\mathcal{O}_{\nu_b}},\mathcal{L}_b)[s_b]$, a finite sum.*

(a) With $F=\{g\in\mathrm{Gr}_{\leq\lambda}: g^{-1}t^\lambda x_0\in L^+X\}$ and $t^\lambda\in F$, $M_x\simeq R\Gamma(F,V_\lambda)$; since $\mathcal{H}^{-d}V_\lambda\supseteq\underline{k}$ by Deligne's formula, $H^{-d}(M_x)\supseteq H^0(F,k)\neq 0$ [BBD]. (Weights also give $0\neq H_c^{2e}(U)\hookrightarrow H^{2e-d}(M_x)$, with $U=F\cap\mathrm{Gr}_\lambda$ of dimension $e$.) (b) Decomposition theorem [BBD, 5.4.5], as in [CY, proof of Thm. 93].

If $\lambda$ gives H5 at $\mu$: Lemma 3.3(a) gives $\varphi(\lambda)\leq\mu$, and $a_\mu^*M\neq 0$ with Lemma 3.1 gives $\mu\leq\varphi(\lambda)$; so H5 $\Rightarrow$ coverage. Conversely, under H3 and $\varphi(\lambda)=\mu$, H5(a) is Lemma 3.1 and $a_\mu^*M\simeq\bigoplus_{\nu_b=\mu}\mathcal{L}_b[*]$ with $\mathcal{L}_b$ constant (Lemma 3.2(c)), nonzero by Lemma 3.3(a). □

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