PROPOSITION II: THE HYPOTHESES H1–H5 IN THE GROUP CASE

Proposition I deduces $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$ from hypotheses (H1)–(H5). We verify them for $X = (H \times H)/\operatorname{diag} H \cong H$, reducing each to Schubert geometry in Gr_H.

1. Setting

$k = \mathbb{C}$; G connected reductive, θ an involution with G^θ connected, $X = G/G^\theta \ni x_0$. S a maximal θ-split torus, $T \supseteq S$ a θ-stable maximal torus, $B \supseteq T$ a Borel in a minimal θ-split parabolic, Φ^+ its roots;

$$\Lambda_T^+ = \{\lambda \in X_*(T) : \langle \Phi^+, \lambda\rangle \geq 0\}, \quad \Lambda_S^+ = X_*(S) \cap \Lambda_T^+, \quad \nu \leq \mu \iff \mu - \nu \in \textstyle\sum_{\check\Phi^+} \mathbb{Z}_{\geq 0}\check\alpha.$$

With $F = k(\!(t)\!)$, $LX = X(F)$, and $\tau(gG^\theta) = g\theta(g)^{-1}$ [CY, Prop. 17]:

$$LX = \bigsqcup_{\mu \in \Lambda_S^+} \mathcal{O}_\mu, \quad \mathcal{O}_\mu = L^+G \cdot x_\mu, \quad \tau(x_\mu) = t^\mu; \qquad \varphi(\lambda) = \lambda - \theta\lambda, \quad t^\lambda x_0 = x_{\varphi(\lambda)}.$$

$\mathcal{C}$: bounded L^+G-equivariant constructible complexes on LX supported on finitely many orbit closures; $\mathcal{C}_\mu$ likewise on $\mathcal{O}_\mu$; $a_\mu \colon \mathcal{O}_\mu \hookrightarrow LX$; IC_0 = constant sheaf on $\mathcal{O}_0 = L^+X$, extended by zero. $\mathcal{H} = \mathrm{Shv}_{\mathrm{cons}}(L^+G\backslash LG/L^+G)$ acts by

$$\mathcal{A} * \mathcal{F} = m_!(\mathcal{A} \mathbin{\widetilde{\boxtimes}} \mathcal{F}), \quad m \colon LG \times^{L^+G} LX \to LX; \qquad \mathcal{C}^{\mathrm{Sat}} = \mathrm{Thick}_{\mathcal{C}}\{V_\lambda * \mathrm{IC}_0 : \lambda \in \Lambda_T^+\},$$

V_λ the Satake sheaf (IC of $\mathrm{Gr}_G^{\leq\lambda}$). Let $Z \supseteq D$ be closed finite unions of orbits.

(H1) *$\overline{\mathcal{O}_\mu} = \bigsqcup_{\nu\leq\mu} \mathcal{O}_\nu$, $\#\{\nu \leq \mu\} < \infty$; each $M \in \mathcal{C}$ is supported on finitely many orbit closures.*

(H2) *Along $D \hookrightarrow Z \hookrightarrow LX$, $\mathcal{O}_\mu \hookrightarrow Z$: exact $f^*, f_!$ ($f_! = f_*$, right adjoint to f^*, if f closed), compatible with composition, and $j_!j^*M \to M \to i_*i^*M \xrightarrow{+1}$ for $Z = \mathcal{O}_\mu \sqcup D$, $\mathcal{O}_\mu$ open.*

(H3) *$\mathcal{C}_\mu$ has a bounded t-structure with $\mathcal{C}_\mu^\heartsuit \simeq \mathrm{Vect}_k^{\mathrm{fd}}$, $V \mapsto \underline{k}_{\mathcal{O}_\mu} \otimes V$.*

(H4) *$*$ is defined, exact, and lands in $\mathcal{C}$; in particular $V_\lambda * \mathrm{IC}_0 \in \mathcal{C}$.*

(H5) *$\forall\mu\ \exists\lambda$: $\mathrm{supp}(V_\lambda * \mathrm{IC}_0) \subseteq \overline{\mathcal{O}_\mu}$ and $a_\mu^*(V_\lambda * \mathrm{IC}_0) \simeq \bigoplus_{s=1}^N \underline{k}_{\mathcal{O}_\mu}[d_s]$, $0 < N < \infty$.*

Proposition I (this project)**.** *If* (H1)–(H5) *hold and $\mathcal{C}$ is idempotent complete, then $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$, and $\underline{\mathrm{Map}}_{\widehat{\mathcal{C}}}(V_\lambda * \mathrm{IC}_0, F) = 0\ \forall\lambda \Rightarrow F = 0$ in $\widehat{\mathcal{C}} = \mathrm{Ind}\,\mathcal{C}$.*

2. The group case

H connected reductive, $T_H \subseteq B_H$, opposite B_H^-, roots Φ_H^+, simple coroots $\alpha_i^\vee$, ρ_H, w_0; $\Lambda_H^+ \subseteq X_*(T_H)$ dominant, $\eta' \le_H \eta \iff \eta - \eta' \in \sum \mathbb{Z}_{\ge 0}\alpha_i^\vee$.

$$G = H \times H, \qquad \theta(g_1, g_2) = (g_2, g_1), \qquad p\colon LH \to \mathrm{Gr}_H = LH/L^+H.$$

$\mathrm{Gr}_H^\eta = L^+Ht^\eta L^+H/L^+H$ is smooth of dimension $\langle 2\rho_H, \eta\rangle$; $\mathrm{Gr}_H^{\le\eta} = \bigsqcup_{\eta' \le_H \eta} \mathrm{Gr}_H^{\eta'}$ is projective [Zhu, Prop. 2.1.5]; $\mathrm{IC}_\eta^H|_{\mathrm{Gr}_H^\eta} \simeq \underline{k}[\langle 2\rho_H, \eta\rangle]$; $\star$ is convolution, unit IC_0^H [Zhu, §5.1]; $J_nH = H(k[t]/t^n)$, $K_n = \ker(L^+H \to J_nH)$ pro-unipotent.

Proposition II. *For $G = H \times H$, $\theta(g_1, g_2) = (g_2, g_1)$, the hypotheses* (H1)–(H5) *hold and $\mathcal{C}$ is idempotent complete; hence*

$$\boxed{\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}} \qquad \textit{and the Satake objects detect zero in } \widehat{\mathcal{C}}.$$

Lemma 2.1 (Dictionary)**.** *For $G = H \times H$:*

(a) $(g_1, g_2)G^\theta \mapsto g_1 g_2^{-1}$ *gives* $X \cong H$, $(g_1, g_2)\cdot x = g_1 x g_2^{-1}$, $\tau(x) = (x, x^{-1})$;
(b) $p^*\colon \mathrm{Shv}_{\mathrm{cons}}(L^+H\backslash \mathrm{Gr}_H) \xrightarrow{\sim} \mathcal{C}$;
(c) $B = B_H \times B_H^-$, $S = \{(s, s^{-1})\}$: $\Lambda_T^+ = \Lambda_H^+ \times (-\Lambda_H^+)$, $\Lambda_S^+ = \{(\mu_H, -\mu_H)\}$, $\nu \le \mu \iff \nu_H \le_H \mu_H$;
(d) $x_\mu = t^{\mu_H}$, $\mathcal{O}_\mu = L^+H\, t^{\mu_H} L^+H = p^{-1}(\mathrm{Gr}_H^{\mu_H})$;
(e) $V_{(\lambda_1,\lambda_2)} * \mathrm{IC}_0 \simeq p^*(\mathrm{IC}_{\lambda_1}^H \star \mathrm{IC}_{-\lambda_2}^H)$.

Justification. (b) p is a free right L^+H-quotient [Zhu, A.1.4], [BL, §2.6.2]; (c) $Z_G(S) = T = B \cap \theta B$; (d) Cartan decomposition; (e) $V_\lambda = \mathrm{IC}_{\lambda_1}^H \boxtimes \mathrm{IC}_{w_0\lambda_2}^H$, and inversion sends $\mathrm{IC}_{w_0\lambda_2}^H \mapsto \mathrm{IC}_{-\lambda_2}^H$.

Lemma 2.2 (Finite-type model)**.** *$Z = p^{-1}(Z_H)$, Z_H a finite union of Schubert varieties; for $n \gg 0$, K_n acts trivially on Z_H and $\mathcal{C}_Z \simeq D^b_{J_nH,c}(Z_H)$, independent of n* [BL], [Zhu, §A.1].

Proof of Proposition II. (H1): p is open and surjective, so $\overline{\mathcal{O}_\mu} = p^{-1}(\mathrm{Gr}_H^{\le\mu_H}) = \bigsqcup_{\nu\le\mu} \mathcal{O}_\nu$; and $\mu_H - \nu_H = \sum n_i\alpha_i^\vee$ gives $2\sum n_i = \langle 2\rho_H, \mu_H - \nu_H\rangle \le \langle 2\rho_H, \mu_H\rangle$. Support finiteness is the definition of $\mathcal{C}$; bounded t-structures make $\mathcal{C}$ idempotent complete (Le–Chen 2007).

(H2): by Lemma 2.2 these are the standard operations on $D^b_{J_nH,c}$ along $p(D) \hookrightarrow Z_H$, $\mathrm{Gr}_H^{\mu_H} \hookrightarrow Z_H$ [BL, Part I]; $r_Z^* := i_{Z\subseteq Z'}^*$ is independent of Z' by $i^*i_* \simeq \mathrm{id}$.

(H3): $\mathrm{Stab}(x_\mu) \cong \Gamma_\mu = L^+H \cap t^{\mu_H} L^+H t^{-\mu_H}$, and $1 \to \Gamma_\mu \cap K_1 \to \Gamma_\mu \xrightarrow{\mathrm{ev}_0} P_{\mu_H}^- \to 1$ has connected ends, so $\mathcal{C}_\mu^\heartsuit \simeq \mathrm{Rep}^{\mathrm{fd}}(\pi_0\Gamma_\mu) = \mathrm{Vect}_k^{\mathrm{fd}}$.

(H4): m is proper on $\mathrm{Gr}_G^{\le\lambda} \widetilde{\times} Z$ (closed in $\mathrm{Gr}_G^{\le\lambda} \times LX$), with image p^{-1} of the image of $\mathrm{Gr}_H^{\le\lambda_1} \widetilde{\times} Z_H \widetilde{\times} \mathrm{Gr}_H^{\le -\lambda_2} \to \mathrm{Gr}_H$ [Zhu, (2.1.17)]; so $m_! = m_*$ is exact and lands in $\mathcal{C}$.

(H5): take $\lambda = (\mu_H, 0)$, so $\varphi(\lambda) = \mu$. By Lemma 2.1(e),

$$V_\lambda * \mathrm{IC}_0 \simeq p^*\mathrm{IC}_{\mu_H}^H, \qquad \mathrm{supp} \subseteq p^{-1}(\mathrm{Gr}_H^{\le\mu_H}) = \overline{\mathcal{O}_\mu}, \qquad a_\mu^*(V_\lambda * \mathrm{IC}_0) \simeq \underline{k}_{\mathcal{O}_\mu}[\langle 2\rho_H, \mu_H\rangle],$$

i.e. $N = 1$. Proposition I now gives $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$ and detection in $\widehat{\mathcal{C}}$. □

References

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