PROPOSITION I: SATAKE OBJECTS GENERATE THE EQUIVARIANT CATEGORY

Setting

$k=\mathbb{C}$; G is connected reductive with an involution θ, G^θ connected, $X=G/G^\theta$, $x_0=eG^\theta$. Fix a maximal θ-split torus S inside a θ-stable maximal torus T, and $B\subseteq P$ with P a minimal θ-split parabolic [CY, §2.1]. Put

$$\Lambda_T^+=X_*(T)^+,\qquad \Lambda_S^+=X_*(S)\cap\Lambda_T^+,\qquad \nu\le\mu \iff \mu-\nu\in\textstyle\sum_{\check\alpha\in\check\Phi^+}\mathbb{Z}_{\ge0}\check\alpha.$$

By [CY, Prop. 17] the L^+G-orbits on the loop space are $LX=\bigsqcup_{\mu\in\Lambda_S^+}\mathcal{O}_\mu$. For L^+G-stable locally closed $W\subseteq LX$, $\mathcal{C}_W$ is the small, idempotent-complete, stable ∞-category of bounded L^+G-equivariant constructible complexes on W; $\mathcal{C}=\mathcal{C}_{LX}$, $\mathcal{C}_\mu=\mathcal{C}_{\mathcal{O}_\mu}$, $\widehat{\mathcal{C}}=\operatorname{Ind}\mathcal{C}$, and $\underline{\mathrm{Map}}$ is the derived mapping complex. With $a_\mu\colon\mathcal{O}_\mu\hookrightarrow LX$, $\partial\mathcal{O}_\mu=\overline{\mathcal{O}_\mu}\setminus\mathcal{O}_\mu$ and $i_\mu\colon\partial\mathcal{O}_\mu\hookrightarrow LX$,

$$\Delta_\mu=a_{\mu!}\underline{k}_{\mathcal{O}_\mu},\qquad \mathrm{IC}_0=\Delta_0,\qquad \mathcal{C}^{\mathrm{Sat}}=\mathrm{Thick}_{\mathcal{C}}\{V_\lambda*\mathrm{IC}_0:\lambda\in\Lambda_T^+\},$$

where $*$ is the spherical Hecke action [CY, §10.2] and $V_\lambda=\mathrm{Sat}(V_\lambda^{\mathrm{rep}})$. $\mathrm{Thick}_{\mathcal{D}}(\mathcal{S})$ is the smallest full subcategory containing 0 and $\mathcal{S}$, closed under isomorphisms, shifts, cofibers and retracts [Stacks, Tag 09SJ]. It satisfies

(T1) two out of three in exact triangles, (T2) $F\big(\mathrm{Thick}_{\mathcal{D}}\,\mathcal{S}\big)\subseteq\mathrm{Thick}_{\mathcal{D}'}\,F(\mathcal{S})$ for exact F.

Hypotheses

A *closed finite union* is a closed L^+G-stable $Z\subseteq LX$ that is a union of finitely many orbits.

(H1) (a) $\overline{\mathcal{O}_\mu}=\bigsqcup_{\nu\le\mu}\mathcal{O}_\nu$ with finitely many $\nu\le\mu$; (b) every $M\in\mathcal{C}$ is supported on a finite union of orbit closures.

(H2) Along embeddings of orbits and closed finite unions, f^* and $f_!$ are exact and compatible with composition, and for closed f, $f_!=f_*$ is right adjoint to f^*. An open orbit j with closed complement i gives $j_!j^*M\to M\to i_*i^*M\xrightarrow{+1}$.

(H3) $\mathcal{C}_\mu$ has a bounded t-structure with heart $\mathrm{Vect}_k^{\mathrm{fd}}$, $V\mapsto\underline{k}_{\mathcal{O}_\mu}\otimes V$.

(H4) $V_\lambda*\mathrm{IC}_0\in\mathcal{C}$ for every $\lambda\in\Lambda_T^+$.

(H5) For every $\mu\in\Lambda_S^+$ some $\lambda\in\Lambda_T^+$ has $\mathrm{supp}(V_\lambda*\mathrm{IC}_0)\subseteq\overline{\mathcal{O}_\mu}$ and $a_\mu^*(V_\lambda*\mathrm{IC}_0)\simeq\bigoplus_{s=1}^{N_\mu}\underline{k}_{\mathcal{O}_\mu}[d_s]$ with $0<N_\mu<\infty$.

(H1)(a) is [CY, Thm. 18]; (H2) is a recollement [Ku, Def. 2.3.4, Lem. 4.3.3]; connected stabilizers give (H3). Proposition II verifies (H1)–(H5) in the group case; Proposition III decides them for symmetric spaces.

Proposition I. *Assume* (H1)–(H5). *Then*

$$\boxed{\mathcal{C}^{\mathrm{Sat}}=\mathcal{C}}\qquad\text{and}\qquad \underline{\mathrm{Map}}_{\widehat{\mathcal{C}}}(V_\lambda*\mathrm{IC}_0,F)=0\ \ \forall\lambda\in\Lambda_T^+\implies F\simeq0\quad(F\in\widehat{\mathcal{C}}).$$

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Supporting results

Lemma 1 (Orbit combinatorics)**.** *Assume* (H1)*(a). Then* $h(\mu) = \#\{\nu \le \mu\}$ *is finite and strictly increasing along* $<$*;* $\partial\mathcal{O}_\mu = \bigsqcup_{\nu<\mu} \mathcal{O}_\nu$ *is a closed finite union, so* $\mathcal{O}_\mu$ *is open in* $\overline{\mathcal{O}_\mu}$*; and a maximal orbit of a closed finite union* Z *is open in* Z *with closed complement a closed finite union.*

Lemma 2 (Localization)**.** *Assume* (H1)*(a),* (H2)*. Pushforward along* $r_Z\colon Z \hookrightarrow LX$ *sends the standard of* $\mathcal{O}_\nu$ *in* $\mathcal{C}_Z$ *to* Δ_ν*, and every* M *supported on* $\overline{\mathcal{O}_\mu}$ *sits in*

$$a_{\mu!}a_\mu^* M \longrightarrow M \longrightarrow i_{\mu*}i_\mu^* M \xrightarrow{+1} \qquad (\textit{top part} \to M \to \textit{boundary part}).$$

Lemma 3 (Standard generation)**.** *Assume* (H1)–(H3)*. For every closed finite union* Z*,* $\mathcal{C}_Z = \mathrm{Thick}_{\mathcal{C}_Z}\{\Delta_{\nu,Z} : \mathcal{O}_\nu \subseteq Z\}$*, and* $\mathcal{C} = \mathrm{Thick}_{\mathcal{C}}\{\Delta_\mu : \mu \in \Lambda_S^+\}$*.*

Proof. On one orbit, the truncation triangles of the bounded t-structure of (H3) build every object of $\mathcal{C}_\mu$ from $\underline{k}_{\mathcal{O}_\mu}$ by (T1). For Z, induct on the number of orbits: remove a maximal (open) orbit, apply the triangle of (H2), and use (T2) for $j_!$ and i_*. For $\mathcal{C}$, (H1)(b) puts $M \simeq r_{Z*}r_Z^* M$ for some Z; apply (T2) to r_{Z*}. □

Proof of Proposition I

By Lemma 3 and thickness of $\mathcal{C}^{\mathrm{Sat}}$, it suffices to show $\Delta_\mu \in \mathcal{C}^{\mathrm{Sat}}$ for all μ. Induct on $h(\mu)$; fix μ and assume $\Delta_\nu \in \mathcal{C}^{\mathrm{Sat}}$ for all $\nu < \mu$. Take λ from (H5) and $P = V_\lambda * \mathrm{IC}_0 \in \mathcal{C}^{\mathrm{Sat}}$.

(1) *Boundary part.* $i_\mu^* P \in \mathrm{Thick}\{\Delta_{\nu,\partial\mathcal{O}_\mu} : \nu < \mu\}$ by Lemma 3, so by (T2) and the induction hypothesis $i_{\mu*}i_\mu^* P \in \mathrm{Thick}_{\mathcal{C}}\{\Delta_\nu : \nu < \mu\} \subseteq \mathcal{C}^{\mathrm{Sat}}$.

(2) *Top part.* Since $\operatorname{supp} P \subseteq \overline{\mathcal{O}_\mu}$, Lemma 2 gives $a_{\mu!}a_\mu^* P \to P \to i_{\mu*}i_\mu^* P \xrightarrow{+1}$; by (T1), $a_{\mu!}a_\mu^* P \in \mathcal{C}^{\mathrm{Sat}}$.

(3) *Split.* Applying the exact functor $a_{\mu!}$ to (H5),

$$a_{\mu!}a_\mu^* P \simeq \bigoplus_{s=1}^{N_\mu} \Delta_\mu[d_s], \qquad N_\mu \ge 1.$$

(4) *Extract.* $\Delta_\mu[d_1]$ is a retract of an object of $\mathcal{C}^{\mathrm{Sat}}$, so $\Delta_\mu \in \mathcal{C}^{\mathrm{Sat}}$. This is the one place where closure under retracts is needed.

Hence $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$. For detection, $\{A \in \mathcal{C} : \underline{\mathrm{Map}}_{\widehat{\mathcal{C}}}(A, F) = 0\}$ is thick, since $\underline{\mathrm{Map}}(-, F)$ is exact. It contains every $V_\lambda * \mathrm{IC}_0$, hence all of $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$. Writing $F = \operatorname{colim}_i A_i$ with $A_i \in \mathcal{C}$ gives $\underline{\mathrm{Map}}(F, F) = \lim_i \underline{\mathrm{Map}}(A_i, F) = 0$, so $F \simeq 0$. □

Remark. The splitting in (H5) must hold in the equivariant category $\mathcal{C}_\mu$; Proposition III derives it from the decomposition theorem for a dominant lift $\lambda \in \varphi^{-1}(\mu)$, $\varphi(\lambda) = \lambda - \theta\lambda$.

References

[CY] T.-H. Chen and L. Yi, *Singularities of orbit closures in loop spaces of symmetric varieties*, arXiv:2310.20006v4, 2025.

[Ku] J. Kuijper, *An axiomatization of six-functor formalisms*, Forum Math. Sigma **14** (2026), e44.

[Stacks] The Stacks Project Authors, *The Stacks Project*, https://stacks.math.columbia.edu/tag/09SJ.