SPHERICAL GENERATION AT THE LEVEL OF FUNCTIONS

The classical analogue of the categorical statement $\mathcal{C}^{\mathrm{Sat}} = \mathcal{C}$: over a non-archimedean local field, the basic function $\mathbf{1}_{X(\mathcal{O})}$ is asked to generate the spherical functions on $X(F)$ over the spherical Hecke algebra. A triangular criterion is the function-level form of Proposition I; Sakellaridis's theorem decides the question under hypotheses that are listed and assumed, not verified; the two GL families are decided through their restriction maps; $\mathbb{A}^n$, M_n and rank one are done by hand. This is the former §3 of the project paper, extended.

1. Setting

F is a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field $\mathbb{F}_q$, with $\operatorname{char} F = 0$ where a cited theorem needs it. $G \supseteq B \supseteq T$ is split connected reductive over $\mathcal{O}$, and X is a spherical G-variety over $\mathcal{O}$. Coefficients are complex.

$$K := G(\mathcal{O}), \qquad \mathcal{S}(X) := C_c^\infty(X(F)), \qquad C_K := \mathcal{S}(X)^K, \qquad \mathcal{H}_K := C_c(K\backslash G(F)/K), \qquad \varphi_0 := \mathbf{1}_{X(\mathcal{O})} \in C_K.$$

With the Haar measure of volume one on K, for $f \in \mathcal{H}_K$, $\varphi \in C_K$ and $\lambda \in \Lambda_T^+$, $t^\lambda := \lambda(\varpi)$,

$$(f *_K \varphi)(x) := \sum_{g \in G(F)/K} f(g)\,\varphi(g^{-1}x), \qquad T_\lambda := \mathbf{1}_{Kt^\lambda K}, \qquad G(F) = \bigsqcup_{\lambda \in \Lambda_T^+} Kt^\lambda K.$$

$\mathcal{H}_K$ is an associative algebra with unit $\mathbf{1}_K$ and basis $\{T_\lambda\}$, C_K is a unital left module, and the Satake transform $\mathsf{S}f(\chi) = \int_{T(F)} \delta_B^{1/2}(t)\chi(t) \int_{U(F)} f(tu)\,du\,dt$ is an isomorphism of algebras $\mathcal{H}_K \simeq \mathbb{C}[A^*]^W$, $A^* := \operatorname{Hom}(\Lambda_T, \mathbb{C}^\times)$ the dual torus, with $\mathsf{S}(T_\lambda) = q^{\langle\rho,\lambda\rangle}x^\lambda +$ lower monomials (Cartier, §1.3–1.4, §4; Macdonald). In particular $\mathcal{H}_K$ is commutative.

Lemma A. (1) Every K-orbit in $X(F)$ is compact, and $\mathbf{1}_{Kx} \in C_K$ when Kx is open. If $X = G/H$ is homogeneous, $X(F)$ is a finite disjoint union of open $G(F)$-orbits, every K-orbit is compact open, and $\{\mathbf{1}_{Kx}\}_{Kx \in K\backslash X(F)}$ is a basis of C_K.

(2) For $\lambda \in \Lambda_T^+$,

$$(T_\lambda *_K \varphi_0)(x) = \#\{gK \subseteq Kt^\lambda K :\ x \in gX(\mathcal{O})\} \in \mathbb{Z}_{\geq 0}, \qquad \operatorname{supp}(T_\lambda *_K \varphi_0) = K\,t^\lambda X(\mathcal{O}),$$

a finite union of K-orbits when $X(\mathcal{O})$ meets finitely many of them, in particular for homogeneous X; and $(T_\lambda *_K \varphi_0)(t^\lambda x_0) \geq 1$ for $x_0 \in X(\mathcal{O})$.

Proof. (1) Kx is the image of the compact group K. For homogeneous X, $X(F)$ is the disjoint union of the $G(F)$-orbits of representatives of $\ker(H^1(F,H) \to H^1(F,G))$, finitely many since $\operatorname{char} F = 0$ (Borel–Serre); each orbit $G(F)x \simeq G(F)/G_x(F)$ is open, the orbit map being a submersion of p-adic manifolds, and Kx is the image of the open set K under the open map $G(F) \to G(F)x$. A K-invariant $\varphi \in \mathcal{S}(X)$ is constant on each orbit and its compact support meets finitely many of the open orbits. (2) $(T_\lambda *_K \varphi_0)(x) = \sum_{gK \subseteq Kt^\lambda K} \mathbf{1}_{X(\mathcal{O})}(g^{-1}x)$, and the term for gK is 1 iff $x \in gX(\mathcal{O})$; the support is $Kt^\lambda K \cdot X(\mathcal{O}) = Kt^\lambda X(\mathcal{O})$ since $X(\mathcal{O})$ is K-stable, and the coset $t^\lambda K$ contributes 1 at $t^\lambda x_0$. □

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2. The statement and a triangular criterion

$$\text{(SG)} \qquad \mathcal{H}_K \cdot \varphi_0 = C_K;$$

for homogeneous X it says that every orbit indicator is a Hecke translate of φ_0.

Proposition B (triangular generation). Let Ω be a set of compact open K-orbits in $X(F)$ whose indicators span C_K, with a partial order in which every lower set $\{\nu \le \mu\}$ is finite. Suppose that for every $\mu \in \Omega$ there is $f_\mu \in \mathcal{H}_K$ with $\operatorname{supp}(f_\mu *_K \varphi_0) \subseteq \bigcup_{\nu \le \mu} O_\nu$ and $f_\mu *_K \varphi_0 \neq 0$ on O_μ. Then (SG) holds. Conversely, under (SG) such f_μ exist for every order.

Proof. Induction on $n(\mu) := |\{\nu \le \mu\}|$. $f_\mu *_K \varphi_0$ is K-invariant, hence $f_\mu *_K \varphi_0 = c_\mu \mathbf{1}_{O_\mu} + \sum_{\nu<\mu} c_\nu \mathbf{1}_{O_\nu}$ with $c_\mu \neq 0$ and finitely many terms; the $\mathbf{1}_{O_\nu}$, $\nu < \mu$, lie in $\mathcal{H}_K \cdot \varphi_0$ by induction, so $\mathbf{1}_{O_\mu}$ does. For the converse take f_μ with $f_\mu *_K \varphi_0 = \mathbf{1}_{O_\mu}$. □

Corollary C (dominant lifts). Let X be homogeneous, with $K\backslash X(F)$ indexed by Λ_X^+, $\mu \mapsto O_\mu$, and ordered with finite lower sets. If every μ has $\lambda \in \Lambda_T^+$ with $t^\lambda x_0 \in O_\mu$ and $Kt^\lambda X(\mathcal{O}) \subseteq \bigcup_{\nu \le \mu} O_\nu$, then (SG) holds, with $f_\mu = T_\lambda$.

Proof. Lemma A(2) and Proposition B. □

Finite lower sets are H1(a); a label λ with $t^\lambda x_0 \in O_\mu$ is a dominant lift of μ, the condition $\varphi(\Lambda_T^+) = \Lambda_S^+$ of Proposition III; the support bound and the nonzero top value are the function-level reading of H5. No implication between the levels is proved.

3. The group

Proposition D. $X = H$ under $G = H \times H$, $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$, $K = K_H \times K_H$. Then $C_K = \mathcal{H}_{K_H}$, $\varphi_0 = \mathbf{1}_{K_H}$ is the unit, $\mathcal{H}_K = \mathcal{H}_{K_H} \otimes \mathcal{H}_{K_H}^{\mathrm{op}}$ acts by $(a \otimes b) \cdot \varphi = a *_K \varphi *_K b$, and (SG) holds.

Proof. $a *_K \mathbf{1}_{K_H} *_K b = a *_K b$ ranges over $\mathcal{H}_{K_H}$. In Corollary C: the orbits are the $K_H t^\lambda K_H$, the lift of λ is $(\lambda, 0)$, and $Kt^{(\lambda,0)}X(\mathcal{O}) = K_H t^\lambda K_H$ is a single orbit. □

4. Sakellaridis's theorem, with its hypotheses

Sakellaridis, *Spherical functions on spherical varieties*, Amer. J. Math. 135 (2013), arXiv:0905.4244v3, works under the following standing assumptions, (A1)–(A5), (§1.7, §7, §8 of loc. cit.), none of which is verified here for any family.

(A1): F a p-adic field; G reductive, split, with connected fibres over $\mathcal{O}$ and simply connected derived group.

(A2): X a quasi-affine G-scheme over $\mathcal{O}$ with an open B_F-orbit $\mathring{X}_F$, whose complement of the closure of the complement $\mathring{X}$ is smooth and surjective over $\mathcal{O}$ with special fibre homogeneous under $B_{\mathbb{F}_q}$; X admits a G-eigenmeasure and $\mathring{X}$ a B-invariant measure.

(A3): Axiom 2.2.1 (the rank-one quotients $YP_\alpha/R(P_\alpha)$ of the B-orbits Y of maximal rank are $\mathrm{PGL}_2\backslash\mathrm{PGL}_2$, $SU\backslash\mathrm{PGL}_2$, $T\backslash\mathrm{PGL}_2$ or $N(T)\backslash\mathrm{PGL}_2$ over $\mathcal{O}$); Axiom 2.4.1 (the X-anti-dominant elements A_X^+ represent all K-orbits on $X(F)$, elements with distinct images in Λ_X^+ lying in distinct orbits); Axiom 2.4.2 ($xJ \subseteq xB(\mathcal{O})$ for $x \in A_X^+$, J the Iwahori subgroup). Proved there at almost every place of a global field.

(A4): No spherical root of type N; the Levi L_γ of each simple spherical root γ classical; Statement 6.3.1 for the simple spherical roots; Statement 7.1.5 for the set Θ of virtual weighted colours, conjectured there for affine homogeneous X and checked case by case.

(A5): $X = H\backslash G$ homogeneous and affine, and $\mathring{X}(F)$ a single $B(F)$-orbit; then $K\backslash X(F) \simeq \Lambda_X^+$ and $A_X^* \subseteq A^*$.

Invariants (Sakellaridis–Venkatesh, §2.1, defined for any spherical variety over an algebraically closed field of characteristic 0): $A_X := L(X)/(L(X) \cap H)$ for the Levi $L(X)$ of the parabolic stabilizing $\mathring{X}$; $\Lambda_X := X_*(A_X)$ and Λ_X^+ the X-anti-dominant cocharacters; A_X^* the dual torus, mapped to A^* by $T \twoheadrightarrow A_X$; W_X the little Weyl group; $\delta(X)$ the modular character of $L(X) \cap B$, and $\delta(X)^{1/2} A_X^* \subseteq A^*$ the translate with the transported W_X-action. For symmetric $X = G/G^\theta$: $A_X \simeq S$ via $t \mapsto t\theta(t)^{-1}$, $\Lambda_X^+ = \Lambda_S^+$, and $W_X = N_G(S)/Z_G(S)$.

Theorem E (Sakellaridis, Theorem 8.0.2, Corollary 8.0.4, Example 8.0.1). Assume (A1)–(A5).

(1) There is an isomorphism of $\mathcal{H}_K$-modules $C_K \simeq \mathbb{C}[\delta(X)^{1/2} A_X^*]^{W_X}$ sending φ_0 to 1, where $\mathcal{H}_K \simeq \mathbb{C}[A^*]^W$ acts through the restriction map $r : \mathbb{C}[A^*]^W \to \mathbb{C}[\delta(X)^{1/2} A_X^*]^{W_X}$.

(2) (SG) holds if and only if r is surjective.

(3) For $X = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \backslash \mathrm{Sp}_4 \times \mathrm{Sp}_4$ (the first two factors as Sp_2 in the two copies of Sp_4, the third diagonal), r is not surjective and (SG) fails.

Proof. (1), (3) are Theorem 8.0.2 and Example 8.0.1 of loc. cit. (2): under (1), $\mathcal{H}_K \cdot \varphi_0$ corresponds to $\operatorname{im} r \cdot 1$; $\Leftarrow$ is Corollary 8.0.4(1). □

Remark 4.1. (a) For the table of examples on p. 76 of loc. cit., Sakellaridis writes that the groups are split and the centres chosen "so that the assumptions of Theorem 7.2.1 are satisfied"; rows 1, 2, 12 are $\mathrm{GL}_m \times \mathrm{GL}_n \backslash \mathrm{GL}_{m+n}$, $\mathrm{Sp}_{2n} \backslash \mathrm{GL}_{2n}$, $H \backslash H \times H$. So (A4) is asserted there for the GL families and the group; the spheres and E_6/F_4 are not in the table. (b) If the generic B-stabilizer B_x is connected, as the paper assumes, $H^1(F, B_x) = 1$ (its maximal torus is split, being a subtorus of a $B(F)$-conjugate of T; its unipotent radical is split), so $\mathring{X}(F) = (B/B_x)(F)$ is one $B(F)$-orbit and (A5) holds. For $\mathrm{SL}_2/\mathrm{SO}_2$ the generic stabilizer is $\{\pm 1\}$ and $\mathring{X}(F)/B(F) \simeq F^\times/F^{\times 2}$: (A5) fails.

Proposition F (the two GL families). Assume (A1)–(A5) for X. Then r is surjective, hence (SG) holds, for

(1) $X = \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$: $A_X^* = \{(y_1, y_1, \ldots, y_n, y_n)\} \subseteq (\mathbb{C}^\times)^{2n}$, $\delta(X)^{1/2}$ moves each pair to $(q^{1/2} y_i, q^{-1/2} y_i)$, $W_X = S_n$ permutes the pairs, and $r(p_{\pm k}) = (q^{k/2} + q^{-k/2})\, p_{\pm k}(y)$ for $p_{\pm k} := \sum_i x_i^{\pm k}$;

(2) $X = \mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$, $p \le q$: $A_X^* = \{(y_1, \ldots, y_p, 1, \ldots, 1, y_p^{-1}, \ldots, y_1^{-1})\}$, $\delta(X)^{1/2}$ moves only the middle block, $W_X = W(C_p)$ acts by signed permutations of the y_i, and $r(p_k) = \sum_i (y_i^k + y_i^{-k}) + c_k$, $c_k \in \mathbb{C}$.

Moreover (SG) for $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ follows from (SG) for $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$.

Proof. Root data. (1) $\theta(x) = (-x_2, -x_1, \ldots, -x_{2n}, -x_{2n-1})$ on Λ_T and $T \twoheadrightarrow S = A_X$ is $(x_i) \mapsto (x_1 x_2, \ldots, x_{2n-1} x_{2n})$, so the characters pulled back from A_X are the $(y_1, y_1, \ldots, y_n, y_n)$; $L(X) = \mathrm{GL}_2^n$, $\delta(X)$ is $|a/b|$ on each block $\mathrm{diag}(a, b)$; the restricted root system is A_{n-1} and $W_X = S_n$ permutes the pairs, so the shift is W_X-invariant and the transported action permutes the y_i. (2) θ permutes coordinates by $i \leftrightarrow n + 1 - i$ $(i \le p)$; the pulled-back characters satisfy

$\chi\circ\theta=\chi^{-1}$; $L(X)=\mathbb{G}_m^{2p}\times\mathrm{GL}_{q-p}$, so $\delta(X)$ is trivial on the y-coordinates and moves the middle 1's to constants $c^{(j)}=q^{(q-p+1)/2-j}$; the restricted root system is BC_p or C_p, $W_X=W(C_p)$.

Restricted power sums. $\mathbb{C}[x^{\pm1}]^{S_N}=\mathbb{C}[x]^{S_N}[e_N^{-1}]$ is generated by the $p_{\pm k}$, $k\ge1$, since $e_N^{-1}=e_N(x^{-1})$ is a polynomial in the p_{-k}. Restricting: (1) $p_{\pm k}\mapsto\sum_i((q^{1/2}y_i)^{\pm k}+(q^{-1/2}y_i)^{\pm k})=(q^{k/2}+q^{-k/2})p_{\pm k}(y)$; (2) $p_k\mapsto\sum_i(y_i^k+y_i^{-k})+\sum_j(c^{(j)})^k$.

Generation. (1) $\mathbb{C}[y^{\pm1}]^{S_n}$ is generated by the $p_{\pm k}(y)$ and $q^{k/2}+q^{-k/2}\neq0$. (2) $\mathbb{C}[y^{\pm1}]^{W(C_p)}=\mathbb{C}[z_1,\dots,z_p]^{S_p}$ with $z_i=y_i+y_i^{-1}$ (a Laurent polynomial invariant under every inversion is a polynomial in the z_i, as $y^k+y^{-k}=z^k+$ lower terms), generated by $P_k=\sum_i z_i^k$; and $r(p_k)-c_k=P_k+(\text{polynomial in }P_1,\dots,P_{k-2})$, so by induction on k every P_k lies in $\operatorname{im} r$.

SL_{2n} *from* GL_{2n}. $X_{\mathrm{GL}}(F)$ is the set of nondegenerate alternating forms on F^{2n} and $X_{\mathrm{SL}}(F)=\{J:\operatorname{Pf}J=1\}$, a single $\mathrm{SL}_{2n}(F)$-orbit since $\operatorname{Pf}(g^{\mathsf T}Jg)=\det g\cdot\operatorname{Pf}J$ (Kneser). The K-orbits on $X_{\mathrm{GL}}(F)$ are the elementary-divisor classes $a=(a_1,a_1,\dots,a_n,a_n)$ with $\operatorname{val}\operatorname{Pf}=\sum a_i$, and two forms of Pfaffian 1 in one class are conjugate under $\mathrm{SL}_{2n}(\mathcal{O})$; so $C_K(X_{\mathrm{SL}})\simeq C_K(X_{\mathrm{GL}})_0$, the span of the classes of Pfaffian valuation 0, with $\varphi_0\mapsto\varphi_0$. Grading $\mathcal{H}_K(\mathrm{GL}_{2n})=\bigoplus_m\mathcal{H}_m$ by $\operatorname{val}\det$, $\mathcal{H}_m\cdot C_K(X_{\mathrm{GL}})_{m'}\subseteq C_K(X_{\mathrm{GL}})_{m+m'}$ and $\mathcal{H}_0\simeq\mathcal{H}_K(\mathrm{SL}_{2n})$ compatibly with the actions, because $Kt^\lambda K\cap\mathrm{SL}_{2n}(F)=K_{\mathrm{SL}}t^\lambda K_{\mathrm{SL}}$ for $\sum\lambda_i=0$. The degree-0 part of $\mathcal{H}_K(\mathrm{GL}_{2n})\cdot\varphi_0=C_K(X_{\mathrm{GL}})$ is (SG) for X_{SL}. □

5. Not homogeneous: $\mathbb{A}^n$ and M_n

Proposition G. $X=\mathbb{A}^n$ under GL_n. The K-orbits on F^n are the shells $X_m=\{v:\operatorname{ord}v=m\}$, $\operatorname{ord}v:=\min_i\operatorname{val}v_i$, and 0; $\mathbf{1}_{\varpi^mK}*_K\varphi_0=\mathbf{1}_{\varpi^m\mathcal{O}^n}$ and $\mathbf{1}_{X_m}=\mathbf{1}_{\varpi^m\mathcal{O}^n}-\mathbf{1}_{\varpi^{m+1}\mathcal{O}^n}$. Hence (SG) holds with the central elements $\mathbf{1}_{\varpi^mK}$ alone.

Proof. K is transitive on the vectors of a given order; ϖ^mK is a single coset, so $(\mathbf{1}_{\varpi^mK}*_K\varphi_0)(v)=\varphi_0(\varpi^{-m}v)$; a K-invariant $\varphi\in\mathcal{S}(X)$ is supported in some $\varpi^{-N}\mathcal{O}^n$ and constant on some $\varpi^N\mathcal{O}^n$, so $\varphi=c\mathbf{1}_{\varpi^N\mathcal{O}^n}+\sum_{-N\le m<N}c_m\mathbf{1}_{X_m}$. The lower sets are infinite and $\{0\}$ is not open: Proposition B does not apply, and differences of central translates replace it (the function-level form of Proposition IV). □

Proposition H. $X=M_n$ under $\mathrm{GL}_n\times\mathrm{GL}_n$, $(g_1,g_2)\cdot x=g_1xg_2^{-1}$, $\varphi_0=\mathbf{1}_{M_n(\mathcal{O})}$; the K-orbits are the elementary-divisor classes $a=(a_1\le\cdots\le a_n)$, $a_i\in\mathbb{Z}\cup\{\infty\}$.

(1) Every nonsingular orbit indicator $\mathbf{1}_{Kt^aK}$ lies in $\mathcal{H}_K\cdot\varphi_0$.

(2) For $n=2$, (SG) holds.

(3) For $n\ge3$ the span near the singular locus is not settled here.

Proof. (1) $\mathcal{H}_K=\mathcal{H}_{K_n}\otimes\mathcal{H}_{K_n}^{\mathrm{op}}$ acts by $a*\varphi*b=(a*b)*\varphi$ on the bi-K_n-invariant restriction to $\mathrm{GL}_n(F)$, which determines a locally constant function on $M_n(F)$. Extending $\mathsf S$ termwise to bi-K_n-invariant functions supported in some $\varpi^{-N}M_n(\mathcal{O})$ keeps it multiplicative and injective. For $t=\operatorname{diag}(\varpi^{a_1},\dots,\varpi^{a_n})$, $tu\in M_n(\mathcal{O})$ iff $a\ge0$ and $u_{ij}\in\varpi^{-a_i}\mathcal{O}$ $(i<j)$, of volume $\prod_{i<j}q^{a_i}$, and $\delta_B^{1/2}(t)=q^{-\sum_{i<j}(a_i-a_j)/2}$, so

$$\mathsf S(\varphi_0|_{\mathrm{GL}_n(F)})=\sum_{a\in\mathbb{Z}^n_{\ge0}}q^{\sum_{i<j}(a_i+a_j)/2}x^a=\prod_{i=1}^n\left(1-q^{(n-1)/2}x_i\right)^{-1}$$

(Tamagawa; Godement–Jacquet). With $f:=\mathsf S^{-1}\big(\mathsf S(\mathbf{1}_{Kt^aK})\prod_i(1-q^{(n-1)/2}x_i)\big)\in\mathcal{H}_{K_n}$ we get $f*\varphi_0=\mathbf{1}_{Kt^aK}$ on $\mathrm{GL}_n(F)$, hence on $M_n(F)$. (2) A K-invariant $\varphi\in\mathcal{S}(M_2)$ supported in

$\varpi^{-N}M_2(\mathcal{O})$ and constant on cosets of $\varpi^N M_2(\mathcal{O})$ is, after subtracting the nonsingular orbit indicators with $a_2 < N$, of the form $\sum_{s<N} c_s \mathbf{1}_{\{a_1=s,\, a_2\geq N\}} + c\,\mathbf{1}_{\varpi^N M_2(\mathcal{O})}$, and $\{a_1 = s,\, a_2 \geq N\} = (\varpi^s M_2(\mathcal{O}) \setminus \varpi^{s+1}M_2(\mathcal{O})) \setminus \{a_1 = s,\, a_2 < N\}$ is a difference of central translates minus finitely many nonsingular orbits. □

6. Rank one on the tree, and $\mathbb{G}_m$

Proposition J. $T \subseteq \mathrm{GL}_2$ the diagonal torus.

(1) $X = \mathrm{GL}_2/T$: $K\backslash X(F) = \{d \geq 0\}$, d the distance of a vertex of the tree from the apartment of T, and (SG) holds.
(2) $X = \mathrm{SL}_2/T$, q odd: (SG) fails.
(3) $X = \mathrm{GL}_2/\mathrm{Sp}_2 \simeq \mathbb{G}_m$: $K\backslash X(F) = F^\times/\mathcal{O}^\times$ and (SG) holds.

Proof. (1) $X(F) = \mathrm{GL}_2(F)/T(F)$ by Hilbert 90, and $K\backslash X(F) = T(F)\backslash \mathrm{GL}_2(F)/K$ is the set of $T(F)$-orbits on the lattices in F^2, i.e. of $\overline{T}(F)$-orbits on the vertices of the tree of $\mathrm{PGL}_2(F)$. $\overline{T}(F)$ translates the apartment A_T, and $T(\mathcal{O})$ fixes it pointwise and is transitive on the vertices at distance $d \geq 1$ above a given vertex of A_T (the $[x:1]$, $x \in (\mathcal{O}/\varpi^d)^\times$, by $x \mapsto u_1 u_2^{-1} x$). So $K\backslash X(F) = \{d \geq 0\}$ and $\varphi_0 = \mathbf{1}_{\{d=0\}}$. $T_1 := \mathbf{1}_{K\mathrm{diag}(\varpi,1)K}$ is the adjacency operator of the tree, so

$$T_1\mathbf{1}_{\{0\}} = 2\,\mathbf{1}_{\{0\}} + \mathbf{1}_{\{1\}}, \qquad T_1\mathbf{1}_{\{1\}} = (q-1)\,\mathbf{1}_{\{0\}} + \mathbf{1}_{\{2\}}, \qquad T_1\mathbf{1}_{\{d\}} = q\,\mathbf{1}_{\{d-1\}} + \mathbf{1}_{\{d+1\}}\ (d \geq 2),$$

and by induction $\mathbf{1}_{\{d\}} \in \mathbb{C}[T_1]\cdot\varphi_0$; the central $\mathbf{1}_{\varpi K}$ acts trivially. This is the case $p = q = 1$ of Proposition F.

(2) $X(F) = \mathrm{SL}_2(F)/T(F)$, $X(\mathcal{O}) = K\cdot x_0$, $x_0 = eT$. Let $u \in \mathcal{O}^\times$ be a non-square unit, $s = \mathrm{diag}(u,1)$, and $\sigma : gT \mapsto sgs^{-1}T$. Since s normalizes K, T and every $Kt^\lambda K$ and fixes x_0, every element of $\mathcal{H}_K\cdot\varphi_0$ is σ-invariant. For $\mathrm{val}\,x = -1$, the K-orbits of $y = \left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)T$ and $\sigma(y) = \left(\begin{smallmatrix}1 & ux\\ 0 & 1\end{smallmatrix}\right)T$ are distinct: $k\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\mathrm{diag}(a,a^{-1}) = \left(\begin{smallmatrix}1 & ux\\ 0 & 1\end{smallmatrix}\right)$ forces $k = \begin{pmatrix} a^{-1} & xa^{-1}(ua^2-1) \\ 0 & a\end{pmatrix}$, in K only if $a \in \mathcal{O}^\times$ and $ua^2 \equiv 1 \bmod \varpi$. So $\mathbf{1}_{Ky} \notin \mathcal{H}_K\cdot\varphi_0$. Here (A5) fails.

(3) $\det K = \mathcal{O}^\times$, and $T_\lambda *_K \varphi_0 = |Kt^\lambda K/K|\cdot\mathbf{1}_{\varpi^{\lambda_1+\lambda_2}\mathcal{O}^\times}$, which span $C_c(\mathbb{Z})$. □

7. Status

X	(SG)	reason
H under $H \times H$	holds	φ_0 is the unit (D)
$\mathbb{A}^n$ under GL_n	holds	central translates (G)
M_n under $\mathrm{GL}_n \times \mathrm{GL}_n$	$n = 2$ holds; $n \geq 3$ open	L-factor, central translates (H)
$\mathrm{GL}_2/\mathrm{Sp}_2 \simeq \mathbb{G}_m$	holds	$K\backslash X(F) = F^\times/\mathcal{O}^\times$ (J)
GL_2/T, PGL_2/T	holds	adjacency operator of the tree (J)
SL_2/T, q odd	fails	square classes; (A5) fails (J)
$\mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$	holds under (A1)–(A5)	r onto (F)
$\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$, $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$	holds under (A1)–(A5) for $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$	r onto; degree-0 part (F)
$\mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}$	$n = 2$ holds; $n = 3$ under (A1)–(A5); $n \geq 4$ open	SL_2; $\mathrm{SL}_4/\mathrm{Sp}_4$; r not computed
E_6/F_4	open	not in Sakellaridis's table
$\mathrm{GL}_n/\mathrm{O}_n$, $\mathrm{PGL}_2/\mathrm{PO}_2$	fails	several $G(F)$-orbits on $X(F)$
$\mathrm{SL}_2^3\backslash\mathrm{Sp}_4^2$	fails	r not onto (E(3))