PROPOSITION IV: THE CONDITIONS OF PROPOSITION III ON TWO CASES

Proposition III reduces $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$ to three checks at each finite level (N, M) of LX: (I1) finitely many pieces, (I2) connected stabilizers, (I3) each piece reached by a Hecke translate. Proposition IV runs the checks on two X: the group, where (I3) fails, and $\mathbb{G}_m$, the smallest positive case. ($\mathbb{A}^n$ under GL_n, formerly IV.1, is proved directly as Proposition IIa of the Iwahori two-levels run.) Notation as in Propositions I and III: levels (N, M), pieces $\mathcal{P}_{N,M}$, standards Δ_P, $\mathcal{C}_I^{(N,M)} \subseteq \mathcal{C}_I$, and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathcal{G} *_I \mathrm{IC}_0 : \mathcal{G} \in \mathcal{H}_{c,I}\}$.

Proposition IV (Case studies)**.**

IV.2 *$X = H$ under $H \times H$, $W \neq 1$:* (I3) *fails at every closed piece $I\omega I$, $\omega \in \Omega$. Consistently, $\mathcal{C}_I^{\mathrm{Iwa}} \neq \mathcal{C}_I$ (Proposition II).*

IV.3 *$X = \mathrm{GL}_2 / \mathrm{Sp}_2 \simeq \mathbb{G}_m$:* (I1), (I2) *and* (I3) *hold, so $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$.*

Case	Pieces of level (N, M)	(I3)	$\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$
H (IV.2)	the double cosets IwI, for $M \gg 0$	fails at the closed $I\omega I$	no (Proposition II)
$\mathbb{G}_m$ (IV.3)	the orbits $\varpi^m \mathcal{O}^\times$, incomparable	holds	yes

The difference is at the closed pieces, where only $Y = \mathrm{IC}_0$ is allowed in (I3): for $\mathbb{A}^n$ the closed piece $T_M = t^M \mathcal{O}^n$ is itself a translate of IC_0; for the group every translate of IC_0 is killed by the functor Φ of Proposition II, and Δ_ω is not.

IV.2 · The group

$X = H$ under $H \times H$, $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$, with $W \neq 1$ ((S2)). As in Proposition II, the Hecke action is two-sided:

$$\mathcal{C}_I = \mathcal{D}_c(I \backslash LH / I), \qquad c := \mathrm{IC}_0 = \underline{k}_{K/I}, \qquad \mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathcal{A} * c * \mathcal{B} : \ \mathcal{A}, \mathcal{B} \in \mathcal{H}_{c,I}\}.$$

Proposition II proves $\mathcal{C}_I^{\mathrm{Iwa}} \neq \mathcal{C}_I$. Here the failure is located among the conditions of Proposition III: it is (I3), at the closed pieces.

Orbits, closures and closed pieces.

$$LH = \bigsqcup_{w \in \widetilde{W}} IwI, \qquad \overline{IwI} = \bigsqcup_{v \leq w} IvI, \qquad \widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega.$$

$\leq$ is the Bruhat order, so every orbit closure is a finite union of orbits and, for $M \gg 0$, the pieces of a level are the double cosets IwI it contains. The closed ones are $I\omega I = \omega I$, $\omega \in \Omega$ of length zero, one in each connected component of LH.

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The functor Φ of Proposition II. $\Phi = \mathrm{Av}^r_\psi : \mathcal{C}_I \to \mathcal{W} \simeq \mathrm{Vect}^\Omega$, right Whittaker averaging. It is exact, commutes with left convolution, $\Phi(\mathcal{A} * \mathcal{F}) = \mathcal{A} * \Phi(\mathcal{F})$, and

$$\mathcal{C}_I^{\mathrm{Iwa}} \subseteq \ker \Phi, \qquad \Phi(\Delta_e) \neq 0$$

by Proposition II.

Proposition IV.2. *Assume* (S2). *For every $\omega \in \Omega$, no object of $\mathcal{C}_I^{\mathrm{Iwa}}$ is a nonzero sum of shifts of Δ_ω:*

$$\boxed{A \in \mathcal{C}_I^{\mathrm{Iwa}}, \ \ \mathrm{supp}\, A \subseteq I\omega I, \ \ A|_{I\omega I} \simeq \bigoplus_{s=1}^{r} \underline{k}[d_s] \implies r = 0.}$$

In particular (I3) *fails at every closed piece $I\omega I$.*

Proof. Step 1 (at a closed piece, (I3) *asks for such an A).* No piece lies below a closed piece, so the only admissible Y in (I3) is $\mathrm{IC}_0 = c$, and $A = \mathcal{G} * c$ lies in $\mathcal{C}_I^{\mathrm{Iwa}}$. $\mathrm{supp}\, A \subseteq I\omega I$, which is closed, so $A = i_*(A|_{I\omega I}) \simeq \bigoplus_s \Delta_\omega[d_s]$.

Step 2 (Φ separates Δ_ω from $\mathcal{C}_I^{\mathrm{Iwa}}$). $A \in \mathcal{C}_I^{\mathrm{Iwa}} \subseteq \ker \Phi$ and Φ is exact, so $0 = \Phi(A) \simeq \bigoplus_{s=1}^r \Phi(\Delta_\omega)[d_s]$. Δ_ω is invertible, $\Delta_{\omega^{-1}} * \Delta_\omega = \Delta_e$, and Φ is left $\mathcal{H}_{c,I}$-linear: $\Phi(\Delta_e) = \Delta_{\omega^{-1}} * \Phi(\Delta_\omega)$. Since $\Phi(\Delta_e) \neq 0$, $\Phi(\Delta_\omega) \neq 0$, hence $r = 0$.

Step 3 (functions: the sign character). $\varepsilon(T_w) = (-1)^{\ell(w)}$ and $\ell(\omega) = 0$, so $\varepsilon(T_\omega) = 1$ while $\varepsilon(\mathcal{H}_I e_K \mathcal{H}_I) = 0$: $aT_\omega \in \mathcal{H}_I e_K \mathcal{H}_I$ only for $a = 0$. The class of A is $\big(\sum_s (-1)^{d_s}\big) T_\omega$, which can vanish with $r > 0$ (take $\Delta_\omega \oplus \Delta_\omega[1]$); so the sheaf statement needs Φ, not ε. □

Remarks. Only (I3) fails: orbit closures are finite, so (I1) is automatic; the obstruction is the absence of a Hecke translate of c concentrated on a closed orbit. A support argument is not enough: $\mathcal{A} * c$ is right K-equivariant, but $\mathcal{A} * c * \mathcal{B}$ need not be, so its support need not be a union of cosets IwK. For $W = 1$, $K = I$ and $c = \Delta_e$: every piece is a closed $I\omega I$ and $\Delta_\omega * c = \Delta_\omega$, so (I3) holds, as for $\mathbb{G}_m$ (IV.3).

IV.3 · $\mathrm{GL}_2 / \mathrm{Sp}_2 \simeq \mathbb{G}_m$

Proposition IV.3. *For $X = \mathrm{GL}_2 / \mathrm{Sp}_2 \simeq \mathbb{G}_m$, the smallest positive case,* (I1), (I2) *and* (I3) *hold; hence*

$$\boxed{\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I.}$$

Proof.

$$LX = F^\times, \qquad I\backslash LX = K\backslash LX = F^\times/\mathcal{O}^\times \simeq \mathbb{Z}, \qquad \Delta_{t^\lambda} *_I \mathrm{IC}_0 = \underline{k}_{\varpi^{\langle \det, \lambda\rangle}\mathcal{O}^\times} \text{ (up to shift).}$$

GL_2 acts through det, and $\det(I) = \det(K) = \mathcal{O}^\times$. Every I-orbit $\varpi^m\mathcal{O}^\times$ is closed with connected stabilizer and is a piece at every level $M \geq 1$; (I1) and (I2) hold, and the pieces of a level are pairwise incomparable. (I3): for $\varpi^m\mathcal{O}^\times$ take $\mathcal{G} = \Delta_{t^\lambda}$ with $\langle \det, \lambda\rangle = m$ and $Y = \mathrm{IC}_0$; $\mathcal{G} *_I \mathrm{IC}_0$ is the pushforward of a constant sheaf along $It^\lambda I \times^I \mathcal{O}^\times \to \varpi^m\mathcal{O}^\times$, whose fibres are affine spaces: a nonzero constant sheaf on that single orbit. Every piece is closed and every closed piece is a translate of IC_0: no induction is needed. □

The symmetric families beyond tori are open: deciding (I3) there needs the orbit graph of Chen–Yi's module (Proposition III).