PROPOSITION III: SUFFICIENT CONDITIONS FOR IWAHORI GENERATION

For which X is $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$? Proposition II shows that it fails for the group. Proposition III gives a sufficient condition, modelled on the paper's proof of spherical generation but checked on one finite level (N, M) of LX at a time, so that an orbit closure may contain infinitely many I-orbits:

$$\underbrace{\text{(I1) finite levels,} \quad \text{(I2) connected stabilizers,} \quad \text{(I3) triangularity on pieces}}_{\text{at every level } (N,M)} \implies \mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I.$$

Proposition IV checks the conditions on $\mathbb{A}^n$ under GL_n, where they hold although every orbit closure is infinite, on the group, where (I3) fails, and on $\mathbb{G}_m$.

Setting

Notation as in Proposition I: $k = \mathbb{C}$, $G \supseteq B$, $K = L^+G \supseteq I$, X smooth affine spherical, $\mathcal{C}_I = \mathcal{D}_c(I\backslash LX)$, $\mathcal{H}_{c,I} = \mathcal{D}_c(I\backslash LG/I)$ acting by $*_I$, $\mathrm{IC}_0 = \underline{k}_{L^+X}$ extended by zero, $\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathcal{G} *_I \mathrm{IC}_0 : \mathcal{G} \in \mathcal{H}_{c,I}\}$, and (T1), (T2) for thick spans.

Iwahori orbits on LX. $\mathcal{V}$ is the set of I-orbits $O_v \subseteq LX$, ordered by $u \le v \iff O_u \subseteq \overline{O_v}$. Chen–Yi parametrize $\mathcal{V}$ for symmetric X, prove that each $\overline{O_v}$ is placid and a finite union of orbits, and present $LX = \varinjlim_v \overline{O_v}$ as an I-placid ind-scheme [CY]. For $\mathbb{A}^n$ each closure is a lattice and contains infinitely many orbits (Proposition IV.1).

Levels, pieces and their standard objects. A *level presentation* is

$$LX = \varinjlim_N Y_N, \qquad Y_N = \varprojlim_M Y_{N,M}, \qquad \pi_{N,M} \colon Y_N \to Y_{N,M},$$

with Y_N closed, I-stable and placid (for instance an orbit closure), $Y_{N,M}$ of finite type, $\pi_{N,M}$ smooth, affine and surjective, and I acting on $Y_{N,M}$ through a finite-dimensional quotient $I/I_{N,M}$. Then

$$\text{piece of level } (N,M): \quad P = \pi_{N,M}^{-1}(o), \qquad o \subseteq Y_{N,M} \text{ an } I/I_{N,M}\text{-orbit},$$
$$\mathcal{P}_{N,M} := \{\text{pieces of level } (N,M)\}, \qquad P' \le P \iff P' \subseteq \overline{P}, \qquad \Delta_P := j_{P!}\,\underline{k}_P,$$
$$\mathcal{C}_I^{(N,M)} := \{\pi_{N,M}^*\bar{\mathcal{F}} \text{ extended by zero}: \ \bar{\mathcal{F}} \in \mathcal{D}_c(I/I_{N,M}\backslash Y_{N,M})\} \subseteq \mathcal{C}_I.$$

Pieces of one level partition Y_N; each is I-stable, locally closed and a union of I-orbits, and $\overline{P} = \pi_{N,M}^{-1}(\overline{o})$, so $\partial P := \overline{P} \setminus P$ is a union of lower pieces of the same level. An orbit pulled back from a finite level is a piece for $M \gg 0$, with $\Delta_P = i_{v!}\underline{k}$; an orbit of infinite codimension, such as $\{0\} \subseteq L\mathbb{A}^n$, is never a piece and has no standard object in $\mathcal{C}_I$.

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Example 0.1 ($\mathbb{A}^n$ under GL_n)**.** Lattices give the levels, and the finite-dimensional quotient is a congruence quotient of I:

$$Y_N = t^{-N}\mathcal{O}^n, \qquad Y_{N,M} = t^{-N}\mathcal{O}^n/t^M\mathcal{O}^n \simeq \mathbb{A}^{(N+M)n}, \qquad I_{N,M} = 1 + t^{N+M}M_n(\mathcal{O}).$$

The $I/I_{N,M}$-orbits on $Y_{N,M}$ are the images of the orbits $O_{m,j}$ with $-N \le m < M$ and the point 0; so

$$\mathcal{P}_{N,M} = \{O_{m,j} : -N \le m < M,\ 1 \le j \le n\} \sqcup \{T_M\}, \qquad \Delta_{T_M} = \underline{k}_{T_M},$$
$$T_M := \pi_{N,M}^{-1}(0) = t^M\mathcal{O}^n = \{0\} \sqcup \bigsqcup_{m \ge M,\, j} O_{m,j}.$$

The closed piece T_M contains infinitely many orbits; raising M splits off n more of them.

Example 0.2 (the group $X = H$ under $H \times H$)**.** Bruhat closures give the levels, and the quotient is by the right action of a congruence subgroup $I_M = \ker(I \to H(\mathcal{O}/t^M))$:

$$Y_N = \bigcup_{\ell(w) \le N} \overline{IwI}, \qquad Y_{N,M} = Y_N/I_M,$$
$$I_{N,M} = I_{M'} \ \text{ with } M' \ge M \text{ such that } w^{-1}I_{M'}w \subseteq I_M \text{ for all } \ell(w) \le N.$$

Each IvI is right I_M-stable, so it is the preimage of its image, and the pieces are the orbits at every level:

$$\mathcal{P}_{N,M} = \{IvI :\ v \le w,\ \ell(w) \le N\}, \qquad \Delta_P = \Delta_v = j_{v!}\,\underline{k}_{IvI},$$
$$|\mathcal{P}_{N,M}| = \#\{v \in \widetilde{W} : v \le w \text{ for some } \ell(w) \le N\} < \infty.$$

Every piece is an orbit, so here a level presentation is the Bruhat stratification.

The affine Lusztig–Vogan module and the orbit graph. Chen–Yi's module M is the free $\mathbb{Z}[q^{\pm 1/2}]$-module on the pairs (ξ, v), ξ an I-equivariant local system on O_v, with the action of the affine Hecke algebra $\mathcal{H}$ of G given by their formula (29): T_s acts on $e_{\xi,v}$ by a combination of $e_{\xi,v}$ and of $e_{\xi',v'}$ for the orbits v' of $P_s \cdot O_v$, according to the type of v for s (the affine form of the Lusztig–Vogan and Mars–Springer tables). The class of $\mathcal{F} \in \mathcal{C}_I$ is $[\mathcal{F}] = \sum_{v,\xi,i}(-1)^i \operatorname{rk} H^i(i_v^*\mathcal{F})_\xi\, e_{\xi,v}$ at $q = 1$, and $[\Delta_s * \mathcal{F}] = T_s \cdot [\mathcal{F}]$. The *orbit graph* has vertex set $\mathcal{V}$ and an edge $v \to v'$ for each simple affine s and each orbit $v' \ne v$ in $P_s \cdot O_v$ on which $T_s e_v$ has a nonzero coefficient. It is the tool for checking (I3) on a symmetric X, where that hypothesis is open.

Hypotheses

(I1) *Finite levels. LX has a level presentation such that*

$$\mathcal{C}_I = \bigcup_{N,M} \mathcal{C}_I^{(N,M)}, \qquad |\mathcal{P}_{N,M}| < \infty \ \text{ for all } N, M.$$

The first part says that every coherent I-equivariant object is supported on some Y_N and pulled back from a finite level; it is the reading, for coherent equivariant objects, of the paper's colimit description of sheaves on a placid scheme, $\mathcal{D}^*(Y_N) = \varinjlim_M \mathcal{D}(Y_{N,M})$ along $*$-pullbacks, and is assumed, not proved, here. The second part replaces "every orbit closure is a finite union of I-orbits" (Chen–Yi for symmetric X; the Iwahori form of the paper's finiteness hypothesis on

orbit closures): when that holds, the pieces of $Y_N = \overline{O_v}$ at large level are its orbits. Function analogue: compact support and local constancy.

(I2) *Connected stabilizers at each level: every stabilizer of $I/I_{N,M}$ on $Y_{N,M}$ is connected.*

On one finite-level orbit $o = H/S$, S connected, the equivariant constructible objects are $\mathrm{Thick}\{\underline{k}_o\}$: their cohomology sheaves are representations of $\pi_0 S = 1$. With disconnected stabilizers replace Δ_P by the $\Delta_{P,\xi}$ for all local systems ξ, in (I3) as well.

(I3) *Triangularity on pieces. For every level (N, M) and every $P \in \mathcal{P}_{N,M}$ there are $\mathcal{G} \in \mathcal{H}_{c,I}$ and $Y \in \{\mathrm{IC}_0\} \cup \{\Delta_{P'} : P' \in \mathcal{P}_{N,M},\ P' < P\}$ such that $A := \mathcal{G} *_I Y \in \mathcal{C}_I^{(N,M)}$ and*

$$\operatorname{supp} A \subseteq \overline{P}, \qquad A|_P \simeq \bigoplus_{s=1}^{r} \underline{k}_P[d_s], \qquad 0 < r < \infty.$$

The Iwahori form of the paper's triangularity hypotheses, with pieces in place of orbits: a Hecke translate of something already generated reaches P with nonzero constant restriction and does not leak above P. For a closed piece only $Y = \mathrm{IC}_0$ is available. For $\mathbb{A}^n$ a single lattice translate of IC_0 works for each piece (Proposition IV.1); for the group it fails at the closed orbits (Proposition IV.2).

Proposition III. *Assume* (I1), (I2) *and* (I3). *Then*

$$\boxed{\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I.}$$

Proof

Step 1 generates one level by standards; step 2 reaches each standard by induction inside the level; step 3 concludes.

Step 1 (dévissage at one level).

$$Z \subseteq Y_N \text{ a closed union of pieces of level } (N, M)$$
$$\Longrightarrow \quad \{\mathcal{F} \in \mathcal{C}_I^{(N,M)} : \operatorname{supp} \mathcal{F} \subseteq Z\} = \mathrm{Thick}\{\Delta_P :\ P \in \mathcal{P}_{N,M},\ P \subseteq Z\}.$$

$\pi_{N,M}^*$ is exact and commutes with $j_!$ and i_* along preimages of locally closed subsets (smooth base change), and $\pi_{N,M}^* j_{o!} \underline{k}_o = \Delta_P$ for $P = \pi_{N,M}^{-1}(o)$. So it suffices to prove the statement on $Y_{N,M}$, which has finitely many $I/I_{N,M}$-orbits ((I1)). Induct on the number of orbits in $\bar{Z} := \pi_{N,M}(Z)$. Choose an orbit o open in $\bar{Z}$, $\bar{Z}' = \bar{Z} \setminus o$ closed. For $\bar{\mathcal{F}}$ supported on $\bar{Z}$: $j_! j^* \bar{\mathcal{F}} \to \bar{\mathcal{F}} \to i_* i^* \bar{\mathcal{F}}$; $j^* \bar{\mathcal{F}} \in \mathrm{Thick}\{\underline{k}_o\}$ by (I2), so $j_! j^* \bar{\mathcal{F}} \in \mathrm{Thick}\{j_{o!} \underline{k}_o\}$, and $i_* i^* \bar{\mathcal{F}}$ is supported on $\bar{Z}'$. Conclude by (T1).

Step 2 (induction inside one level: every standard is Iwahori-generated).

$$\Delta_{P'} \in \mathcal{C}_I^{\mathrm{Iwa}} \text{ for all } P' < P \text{ in } \mathcal{P}_{N,M} \implies \Delta_P \in \mathcal{C}_I^{\mathrm{Iwa}}.$$

$\mathcal{P}_{N,M}$ is finite ((I1)), so the induction is well founded even when orbit closures are infinite. Take $\mathcal{G}$, Y, A from (I3). $Y \in \mathcal{C}_I^{\mathrm{Iwa}}$ and $\mathcal{C}_I^{\mathrm{Iwa}}$ is an $\mathcal{H}_{c,I}$-submodule (associativity and (T2)), so $A \in \mathcal{C}_I^{\mathrm{Iwa}}$. With $j\colon P \hookrightarrow \overline{P}$, $i\colon \partial P \hookrightarrow \overline{P}$: in $j_! j^* A \to A \to i_* i^* A$ the third term lies in $\mathcal{C}_I^{(N,M)}$ and is supported on ∂P, a closed union of pieces $P' < P$; by step 1 and induction it lies in $\mathcal{C}_I^{\mathrm{Iwa}}$. By (T1), $j_! j^* A \in \mathcal{C}_I^{\mathrm{Iwa}}$. By (I3), $j_! j^* A \simeq \bigoplus_s \Delta_P[d_s]$ with at least one summand, so Δ_P

is a retract of an object of $\mathcal{C}_I^{\mathrm{Iwa}}$, hence in it. (This is the paper's argument for Proposition 4.30, with $\mathcal{H}_{c,I}$ and pieces.)

Step 3 (the claim).

$$\mathcal{C}_I = \bigcup_{N,M} \mathcal{C}_I^{(N,M)} = \bigcup_{N,M} \mathrm{Thick}\{\Delta_P : P \in \mathcal{P}_{N,M}\} \subseteq \mathcal{C}_I^{\mathrm{Iwa}} \subseteq \mathcal{C}_I.$$

(I1), then step 1 with $Z = Y_N$, then step 2 for every piece. □

Remarks

(1) *Template.* Proposition III is the paper's proof of spherical generation (its orbit hypotheses and Proposition 4.30), with the pieces of a level in place of orbits; the Spherical generation workspace records that proof. Working level by level removes the paper's finiteness hypothesis on orbit closures, which fails for $\mathbb{A}^n$.

(2) *Orbits.* Chen–Yi [CY]: the I_0-orbits on LX and the closed ones (Lemmas 23, 31), placidity of closures (§5), dimensions (Proposition 63), the affine Hecke module M and formula (29) (Theorem 1, Proposition 74, Lemma 75), the action (21) on $D(I_0\backslash LX)$. Finite-field model: Mars–Springer [MS], Definition 3.2.7 and table 4.3.1.

(3) *Not claimed.* The finite-level presentation in (I1) is assumed, not derived from the paper's definition of coherent $\mathcal{D}$-modules, so Proposition IV.1 is conditional on it. No implication between the sheaf statement and its function-level shadow is proved: thick spans contain retracts, functions do not see them. (I3) for a symmetric X other than a torus is open.

References

[P] The project draft (`mainv2.tex`): the orbit hypotheses, Proposition 4.30, the placid presentation of LX and the colimit description of sheaves on a placid scheme.

[CY] T.-H. Chen and L. Yi, *Singularities of orbit closures in loop spaces of symmetric varieties*, arXiv:2310.20006v4, 2025.

[MS] J. G. M. Mars and T. A. Springer, *Hecke algebra representations related to spherical varieties*, Represent. Theory 2 (1998), 33–69.