PROPOSITION II: THE GROUP CASE, WHERE THE UNIT IS MISSING

$X = H$ under $G = H \times H$, $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$, with H connected reductive over $k = \mathbb{C}$ and Weyl group $W \neq 1$. Then $I\backslash LX = I_H\backslash LH/I_H$, the basic object is the constant sheaf on $I_H\backslash K_H/I_H$, and the Iwahori-generated part is the thick two-sided ideal it generates in the affine Hecke category of H:

$$\mathcal{C}_I = \mathcal{H}_{c,I_H} = \mathcal{D}_c(I_H\backslash LH/I_H), \qquad c := \mathrm{IC}_0 = \underline{k}_{I_H\backslash K_H/I_H},$$
$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathcal{A} * c * \mathcal{B} : \ \mathcal{A}, \mathcal{B} \in \mathcal{H}_{c,I_H}\}.$$

By Proposition I this is the paper's Definition 4.3 and, since (H1) holds for the group, the Iwahori span of the whole spherical Hecke category. Is it all of $\mathcal{C}_I$? No: the unit Δ_e is missing. For functions the witness is the sign character; for sheaves it is a sign-type module functor through an affine Whittaker category, which also identifies what is missing.

Setting

Write $K = K_H$, $I = I_H$, $\widetilde{W}$ the extended affine Weyl group of H, $W \subseteq \widetilde{W}$ the finite Weyl group, $\Omega \subseteq \widetilde{W}$ the elements of length zero, so $\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$ and Ω indexes the connected components of $\mathrm{Fl} = LH/I$. Iwahori decompositions:

$$LH = \bigsqcup_{w \in \widetilde{W}} IwI, \qquad K = \bigsqcup_{w \in W} IwI, \qquad LH = \bigsqcup_{w \in \widetilde{W}} IwI^-,$$

I^- the opposite Iwahori, $I \cap I^- = T(\mathcal{O})$, $I^- = T(\mathcal{O}) \ltimes I_u^-$ with I_u^- pro-unipotent, containing the root subgroup $U_{-\alpha}$ for every positive affine root α. $\Delta_w \in \mathcal{H}_{c,I}$ is the constant sheaf on IwI/I extended by zero; Δ_e is the unit of convolution; $c = \underline{k}_{K/I}$ extended by zero is the basic object; $c_s = \underline{k}_{P_s/I}$ for a simple affine reflection s with parahoric $P_s \supseteq I$, $P_s/I \cong \mathbb{P}^1$.

Parabolic equivariance and the eigen-property. For a simple affine reflection s, left convolution by Δ_s is pull–push along $I\backslash LH/I \leftarrow I\backslash P_s \times^I LH/I \to I\backslash LH/I$, an $\mathbb{A}^1$-bundle followed by a proper map. If $\mathcal{F}$ is left P_s-equivariant (its pullback to $I\backslash LH/I$ comes from $P_s\backslash LH/I$), then

$$\Delta_s * \mathcal{F} \simeq \mathcal{F} \otimes H_c^*(\mathbb{A}^1) = \mathcal{F}[-2].$$

Left K-equivariant objects are left P_s-equivariant for every finite simple $s \in W$; so c, $c * \mathcal{B}$ and every q^*M, $M \in \mathcal{D}_c(K\backslash LH/I)$, satisfy $\Delta_s * (-) = (-)[-2]$ for finite s. This is the categorical form of $T_s \cdot f = q\, f$ for a K-invariant function f.

The affine Whittaker category and the averaging functor. Fix a character $\psi\colon I_u^- \to \mathbb{G}_a$ nontrivial on each simple affine root subgroup $U_{-\alpha_i}$, $i = 0, \dots, r$ (it factors through the abelianization $\prod_i U_{-\alpha_i}$), and the exponential $\mathcal{D}$-module exp on $\mathbb{G}_a$. The affine Whittaker category and the right averaging functor are

$$\mathcal{W} := \mathcal{D}_c\big(I\backslash LH/(I_u^-, \psi)\big), \qquad \Phi := \mathrm{Av}_\psi^r\colon \mathcal{C}_I = \mathcal{D}_c(I\backslash LH/I) \longrightarrow \mathcal{W},$$
$$\Phi(\mathcal{F}) = a_!\big(\mathcal{F} \boxtimes \psi^*\exp\big) \quad \text{along} \quad a\colon I\backslash LH \times I_u^- \to I\backslash LH, \ (x, u) \mapsto xu.$$

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Φ is exact and commutes with left convolution, $\Phi(\mathcal{A}*\mathcal{F}) = \mathcal{A}*\Phi(\mathcal{F})$ for $\mathcal{A} \in \mathcal{H}_{c,I}$, because the left I-action and the right I_u^--action on LH commute. (The averaging is over a pro-unipotent group; on bounded supports it is a finite-dimensional unipotent averaging, and $a_!$ and a_* agree up to a shift that is suppressed.) The only input about exp is

$$H_c^*(\mathbb{A}^1, \exp) = 0.$$

Hypotheses

(S1) *$|W| \in k^\times$; used only through Proposition I(a). The rest is unconditional in characteristic zero.*

(S2) *Nontrivial Weyl group: $W \neq 1$, that is, H is not a torus. For a torus, $K = I$, $c = \Delta_e$ and $\mathcal{C}_I^{\mathrm{Iwa}} = \mathcal{C}_I$; the proof needs one finite simple reflection.*

Proposition II. *Assume* (S1) *and* (S2). *The Iwahori-generated part of the affine Hecke category of H is a proper thick two-sided ideal:*

$$\boxed{\Delta_e \notin \mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathcal{A}*c*\mathcal{B}\}, \qquad \textit{more precisely} \qquad \mathcal{C}_I^{\mathrm{Iwa}} \subseteq \ker\Phi \quad \textit{and} \quad \Phi(\Delta_e) \neq 0.}$$

For functions: $\mathcal{H}_I e_K \mathcal{H}_I \subsetneq \mathcal{H}_I$, because the sign character $\varepsilon(T_w) = (-1)^{\ell(w)}$ kills e_K and not T_e.

Proof

Step 1 is the function-level proof. Step 2 rewrites the ideal through left K-equivariant objects. Steps 3–4 compute $\mathcal{W}$ and the action of Δ_s on it. Step 5 is the shift trick, and step 6 concludes.

Step 1 (functions: the sign character).

$$\varepsilon\colon \mathcal{H}_I \to \mathbb{C},\ T_w \mapsto (-1)^{\ell(w)}, \qquad \varepsilon(\mathbf{1}_K) = \sum_{w\in W}(-1)^{\ell(w)} = 0, \qquad \varepsilon(T_e) = 1,$$
$$\text{so} \qquad \mathcal{H}_I e_K \mathcal{H}_I \subseteq \ker\varepsilon \subsetneq \mathcal{H}_I.$$

$T_s \mapsto -1$, $T_\omega \mapsto 1$ respects the Iwahori–Matsumoto presentation, and $w \mapsto ws$ pairs even with odd lengths in $W \neq 1$ [IM, Bo]. The three function-level spans agree with $\mathcal{H}_I e_K \mathcal{H}_I$; the sheaf statement needs an argument that survives retracts, which K_0-classes do not: steps 2–6.

Step 2 (the ideal is the left span of the K-equivariant objects).

$$\mathcal{C}_I^{\mathrm{Iwa}} = \mathrm{Thick}\{\mathcal{A}*\mathcal{F} :\ \mathcal{A} \in \mathcal{H}_{c,I},\ \mathcal{F} \in q^*\mathcal{D}_c(K\backslash LH/I)\},$$
$$\Delta_s * \mathcal{F} \simeq \mathcal{F}[-2] \ \text{ for such } \mathcal{F} \text{ and finite simple } s.$$

Left convolution with $c = \underline{k}_{K/I}$ is left K-averaging: $c*\mathcal{B} \simeq q^*q_*\mathcal{B}$ (pull–push along $I\backslash LH/I \to K\backslash LH/I$, shifts suppressed). So $\mathcal{A}*c*\mathcal{B} = \mathcal{A}*q^*(q_*\mathcal{B})$ lies in the right-hand side. Conversely, for $M \in \mathcal{D}_c(K\backslash LH/I)$, the key lemma for q makes M a retract of q_*q^*M; applying q^*, q^*M is a retract of $q^*q_*q^*M = c*q^*M$, hence $\mathcal{A}*q^*M$ is a retract of $\mathcal{A}*c*q^*M \in \mathcal{C}_I^{\mathrm{Iwa}}$. Finally q^*M is left K-equivariant, hence left P_s-equivariant for every finite simple s, and the eigen-property gives $\Delta_s * q^*M \simeq q^*M[-2]$.

Step 3 (the affine Whittaker category is Vect^{Ω}).

$$\mathcal{W} \simeq \bigoplus_{\omega\in\Omega} \mathrm{Vect}, \qquad E_\omega := \Phi(\Delta_\omega) \text{ a generator of the } \omega\text{-summand}, \qquad \Phi(\Delta_e) = E_e \neq 0.$$

Orbits. $LH = \bigsqcup_{w\in\widetilde{W}} IwI^-$ and $IwI^- = IwI_u^-$ because $I \supseteq T(\mathcal{O}) = wT(\mathcal{O})w^{-1}$. So the $I \times I_u^-$-orbits on LH are indexed by $\widetilde{W}$, and the stabilizer of the coset Iw in I_u^- is $S_w := I_u^- \cap w^{-1}Iw$, a connected pro-unipotent group.

Which orbits carry ψ-equivariant objects. On the orbit of w, an $(I, (I_u^-, \psi))$-equivariant object is an $(S_w, \psi|_{S_w})$-equivariant object on a point, and these vanish unless $\psi|_{S_w} = 1$: a nonzero equivariant object would trivialize the character on the connected group S_w. Now $U_{-\alpha_i} \subseteq S_w$ iff $w(-\alpha_i)$ is a positive affine root, iff $w(\alpha_i) < 0$. If $w \notin \Omega$, some simple affine root is sent to a negative root (write $w = \omega w'$, $w' \in W_{\mathrm{aff}}$ of positive length, ω permuting the simple affine roots), and ψ is nontrivial on that $U_{-\alpha_i}$: the orbit contributes nothing. If $w = \omega \in \Omega$, ω permutes the positive affine roots, $\psi|_{S_\omega} = 1$, and the category on the orbit is $\mathcal{D}_c(\mathbb{B}S_\omega) \simeq \mathrm{Vect}$.

Gluing. The orbits $I\omega I_u^-$, $\omega \in \Omega$, lie in distinct connected components of LH, and every other orbit carries the zero category; so $\mathcal{W}$ is the direct sum of the Vect summands. $\Phi(\Delta_e)$ restricted to the orbit of e is $H_c^*(S_e, \psi) = H_c^*(\mathrm{pt}) = k$ since $S_e = I_u^- \cap I = 1$; likewise $\Phi(\Delta_\omega)$ is a generator of the ω-summand.

Step 4 (every Δ_s acts on $\mathcal{W}$ by the shift $[-1]$: the categorical sign character).

$$\Phi(c_s) = 0, \qquad \Phi(\Delta_s) \simeq E_e[-1], \qquad \Delta_s * E_\omega \simeq E_\omega[-1] \quad (s \text{ any simple affine reflection, } \omega \in \Omega).$$

$\Phi(c_s)$ restricted to the orbit of e is $H_c^*(I_u^- \cap P_s, \psi)$. The group $I_u^- \cap P_s$ contains $U_{-\alpha_s} \cong \mathbb{A}^1$ as a direct factor on which ψ is nontrivial, so this is $H_c^*(\mathbb{A}^1, \exp) \otimes (\cdots) = 0$; the same on the other orbits. Hence $\Phi(c_s) = 0$. Vanishing: the de Rham complex of the exponential $\mathcal{D}$-module $(\mathcal{O}_{\mathbb{A}^1}, d + dz)$ is $k[z] \to k[z]$, $f \mapsto f' + f$, an isomorphism (triangular for the degree filtration, identity on the graded pieces), so $H_{\mathrm{dR}}^*(\mathbb{A}^1, \exp) = 0$; the same computation with $f' - f$ handles the Verdier dual $\exp(-z)$, hence $H_c^* = 0$. The ℓ-adic counterpart, $R\Gamma_c(\mathbb{A}^1, \mathcal{L}_\psi) = R\Gamma(\mathbb{A}^1, \mathcal{L}_\psi) = 0$ for the Artin–Schreier sheaf, is Laumon (1.1.3.4), attributed to Deligne [La].

c_s restricted to the open cell $IsI/I \cong \mathbb{A}^1$ and to the point e is constant, giving the cofibre sequence $\Delta_s \to c_s \to \Delta_e$. Apply the exact functor Φ: $\Phi(\Delta_s) \to 0 \to E_e$, so $\Phi(\Delta_s) \simeq E_e[-1]$.

Module property: $\Delta_s * E_e = \Delta_s * \Phi(\Delta_e) = \Phi(\Delta_s * \Delta_e) = \Phi(\Delta_s) = E_e[-1]$. For $\omega \in \Omega$, $s\omega = \omega s'$ with $s' = \omega^{-1}s\omega$ again simple, and $\Delta_s * \Delta_\omega = \Delta_{s\omega} = \Delta_\omega * \Delta_{s'}$ (lengths add), so $\Delta_s * E_\omega = \Delta_\omega * \Delta_{s'} * E_e = E_\omega[-1]$.

Step 5 (the shift trick: Φ kills every left K-equivariant object, hence the ideal).

$$\mathcal{F} \text{ left } K\text{-equivariant} \implies \Phi(\mathcal{F})[-2] \simeq \Phi(\Delta_s * \mathcal{F}) \simeq \Phi(\mathcal{F})[-1] \implies \Phi(\mathcal{F}) = 0; \qquad \mathcal{C}_I^{\mathrm{Iwa}} \subseteq \ker \Phi.$$

Take a finite simple reflection $s \in W$ ((S2)). Step 2 gives $\Delta_s * \mathcal{F} \simeq \mathcal{F}[-2]$; step 4 and $\mathcal{H}_{c,I}$-linearity give $\Phi(\Delta_s * \mathcal{F}) = \Delta_s * \Phi(\mathcal{F}) \simeq \Phi(\mathcal{F})[-1]$, because $\Phi(\mathcal{F})$ is a finite direct sum of shifts of the E_ω. So $\Phi(\mathcal{F}) \simeq \Phi(\mathcal{F})[1]$: a bounded complex with finite-dimensional cohomology isomorphic to its own shift is zero. Then $\Phi(\mathcal{A} * \mathcal{F}) = \mathcal{A} * \Phi(\mathcal{F}) = 0$ for every $\mathcal{A}$. The kernel of an exact functor is closed under shifts, cofibres and retracts, so it contains the thick span of these objects, which is $\mathcal{C}_I^{\mathrm{Iwa}}$ by step 2.

Step 6 (Δ_e is not Iwahori-generated; what is missing is the sign part).

$$\Phi(\Delta_e) = E_e \neq 0 \implies \Delta_e \notin \mathcal{C}_I^{\mathrm{Iwa}}; \qquad \Phi \text{ factors through } \mathcal{C}_I/\mathcal{C}_I^{\mathrm{Iwa}} \twoheadrightarrow \mathrm{Vect}^{\Omega}.$$

Steps 3 and 5. The quotient $\mathcal{C}_I/\mathcal{C}_I^{\mathrm{Iwa}}$ (Verdier quotient by a thick subcategory) receives Φ, which is essentially surjective; so the Iwahori-generated part misses at least the sign-type module Vect^{Ω}, on which every Δ_s acts by $[-1]$. Dictionary: Φ is the categorical form of the sign character ε of step 1 ($T_s \mapsto -1$ becomes $\Delta_s \mapsto [-1]$, $T_\omega \mapsto 1$ becomes a permutation of the components), that is, of the Iwahori-fixed line of the Steinberg representation; $\ker \Phi$ is the categorical form of $\ker \varepsilon$. For functions, $\ker \varepsilon$ is strictly larger than the ideal when there are other Iwahori-spherical representations without K-fixed vectors; the same is expected for $\ker \Phi$, and nothing here identifies $\mathcal{C}_I^{\mathrm{Iwa}}$ with $\ker \Phi$. □

Remarks

(1) *Where Proposition II sits.* The paper states its conjecture for $\mathcal{D}_c(I\backslash LX)^{\mathrm{Sat}}$, and Devalapurkar's Conjecture 3.4.14 for the subcategory generated by IC_0 under the affine Hecke category. Proposition II is the reason: for the group the Iwahori-generated part is a proper two-sided ideal, and the quotient sees the sign character.

(2) *Whittaker models.* Φ is Iwahori–Whittaker averaging for a character nondegenerate on all affine simple roots. With the finite simple roots only, one gets the Iwahori–Whittaker category of Fl [AB], which categorifies the antispherical module rather than the sign character, since Δ_{s_0} does not act by $[-1]$; on Gr, the Iwahori–Whittaker model of the Satake category [BGMRR]. The fully affine character lies outside these statements ([BY], Lemma 4.2.1, assumes a finite-type set of simple reflections), so the orbit lemma and the shift computation are proved here.

(3) *Why not K_0.* ε kills the class of $\mathcal{A}*c*\mathcal{B}$, but a direct summand of an object with vanishing class need not have vanishing class, and thick spans contain summands. An exact functor whose kernel contains the generators has no such gap: that is the role of Φ.

(4) *Beyond the group.* A general X has no second side to average over. For sheaves, Proposition III gives sufficient conditions and Proposition IV.2 locates the group's failure at the closed orbits.

References

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