Let $\mathcal{X}(\mathbf{X})^+$ denote the weights of *regular* Borel eigenfunctions on $\mathbf{X}$; in other words, $\mathcal{X}(\mathbf{X})^+$ is the set of highest weights in the decomposition of $k[\mathbf{X}]$ into a multiplicity-free direct sum of highest weight modules. This monoid induces a partial order $\succeq$ on the set $\mathfrak{a}_X = \mathfrak{a}_X = \mathcal{X}(\mathbf{X})^* \otimes \mathbb{Q}$ by: $\mu \succeq \lambda \iff \langle \mu, \chi \rangle \geq \langle \lambda, \chi \rangle$ for all $\chi \in \mathcal{X}(\mathbf{X})^+$. We will extend this order to $\mathfrak{a} = \mathcal{X}(\mathbf{B})^* \otimes \mathbb{Q}$ by pull-back ($\mu \succeq \lambda \iff \bar{\mu} \succeq \bar{\lambda}$, where $\bar{\mu}, \bar{\lambda}$ denote the images in $\mathfrak{a}_X$), and to the tori A, A_X via the natural maps (which we denote by "log"):

$$\begin{aligned} \log : A &\to \mathcal{X}(\mathbf{B})^* \subset \mathfrak{a}, \\ \log : A_X &\to \mathcal{X}(\mathbf{X})^* \subset \mathfrak{a}_X. \end{aligned} \tag{5.22}$$

Since $\mathcal{X}(\mathbf{X})^+$ is contained in the set of dominant weights of $\mathcal{X}(\mathbf{B})$, the set of elements $\succeq 0$ in $\mathfrak{a}$ contains the coroot cone. Finally, we will use, as we have done so far, the notation $\mathfrak{a}^+, A^+$ (resp. $\mathring{\mathfrak{a}}^+, \mathring{A}^+$ for anti-dominant (resp. strictly anti-dominant) elements, and similarly for $\mathfrak{a}_X$, $\mathfrak{a}_{X,\Theta}$, etc. (with respect to the root system of $\mathbf{X}$). We are still in the wavefront case, so we have a surjection: $\mathfrak{a}^+ \twoheadrightarrow \mathfrak{a}_X^+$.

Choose a splitting of the exact sequence of groups:

$$1 \to k^\times \to k(\mathbf{X})^{(\mathbf{B})} \to \mathcal{X}(\mathbf{X}) \to 1, \tag{5.23}$$

denoted by $\mathcal{X}(\mathbf{X}) \ni \chi \mapsto f_\chi \in k(\mathbf{X})^{(\mathbf{B})}$. (Such a splitting exists because $\mathcal{X}(\mathbf{X})$ is free.)

Having fixed a maximal torus $\mathbf{A} \subset \mathbf{B}$, we choose a special maximal compact subgroup $K \subset G$ stabilizing a special point in the apartment of A in the building of G. Hence, K acts transitively on the points of the flag variety of G and satisfies the Cartan decomposition: $G = K\Lambda^+K$, where Λ^+, denotes the *anti-dominant* elements in $\Lambda := \mathcal{X}(\mathbf{B})^*$, considered as elements of G via any choice of uniformizer in F, which leads to a splitting of the log map (5.22).

Define, for every $\chi \in \mathcal{X}(\mathbf{X})^+$, a K-invariant "norm" on X:

$$\|x\|_\chi := \max_{k \in K} |f_\chi(xk)|.$$

5.5.1. Lemma. *The norms $\|x\|_\chi$ have the following properties:*

(1) $\|x\|_{\chi_1+\chi_2} \leq \|x\|_{\chi_1} \cdot \|x\|_{\chi_2}$;
(2) *For any $a \in A^+$, $\|xa\|_\chi \leq |\chi(a)| \cdot \|x\|_\chi$.*

Proof. The first property is obvious. The second follows by viewing the points of X, via the evaluation maps, as elements in the dual V_χ^* of the highest weight module V_χ spanned by f_χ. Then V_χ^* can be realized as sections of the line bundle over the flag variety of $\mathbf{G}$ induced from the character χ of $\mathbf{B}$, that is: regular functions on $\mathbf{G}$ satisfying $F(bg) = \chi(b)F(g)$ for $b \in \mathbf{B}$, and for such a section F the norm we previously defined is equivalent to the norm: $\|F\| = \max_{k \in K} |F(k)|$. From this realization it is easy to see that $\|a \cdot F\| \leq |\chi(a)|\|F\|$ for $a \in B$ dominant, where by $a \cdot F$

we denote the right regular action of a on F: This follows from the Bruhat-Tits comparison ([**HR10**, 10.2.1], cf. [**Car79**, p. 148]) of the Iwasawa and Cartan decompositions: $Ka_\lambda K \subset \cup_{\mu\geq\lambda} Ua_\mu K$, where $\mu \geq \lambda$ means that $\lambda - \mu$ belongs to the coroot cone (and recall that every $\chi \in \mathcal{X}(\mathbf{X})^+$ is ≥ 0 on the coroot cone). An equivalent way to state that is that the norms defined extend to K-invariant norms on V_χ^*, and the spectral norm of $a \in B$ is $|\chi(a)|$. □

For every $\mu \in \mathfrak{a}_X$, let $X_{\succeq\mu}$ denote the set of elements $x \in X$ with $\|x\|_\chi \leq q^{-\langle\chi,\mu\rangle}$ for all $\chi \in \mathcal{X}(\mathbf{X})^+$; obviously, by the first statement of the previous lemma, only a finite number of χ's is needed to define this set. These sets have the following properties:

(1) $X_{\succeq\mu}$ is a set with compact closure in the points of the affine closure $\overline{\mathbf{X}}^{\mathrm{aff}} := \operatorname{spec} k[\mathbf{X}]$; indeed, the K-translates of a generating set of $\mathbf{B}$-eigenfunctions generate the coordinate ring $k[\mathbf{X}]$ (since $BK = G$), and therefore any finite set of generators of $k[\mathbf{X}]$ has bounded evaluations on elements of $X_{\succeq\mu}$.
(2) The sets $X_{\succeq\mu}$ define a filtration of X by K-invariant sets, decreasing with respect to the $\succeq$ ordering on $\mathfrak{a}_X$ (i.e. $X_{\succeq\mu} \subset X_{\succeq\lambda}$ if $\lambda \preceq \mu$).

5.5.2. Lemma. *For any $\lambda \in \Lambda^+$, $\mu \in \mathfrak{a}_X$, we have:*

$$X_{\succeq\mu} \cdot Ka_\lambda K \subset X_{\succeq\mu+\bar{\lambda}},$$

where $\bar{\lambda}$ is the image of λ in $\mathfrak{a}_X$.

Proof. In fact, since the sets are K-invariant, this reduces to the statement: $X_{\succeq\mu} \cdot a_\lambda \subset X_{\succeq\mu+\bar{\lambda}}$. This follows immediately from the second statement of the previous lemma. □

Now we compare these sets for X and X_Θ. First of all, we clarify that *we will use for the definition of the sets $X_{\Theta,\succeq\mu}$ only the set of weights $\mathcal{X}(\mathbf{X})^+$*, despite the fact that X_Θ might have more regular functions; this corresponds to the affine embedding of $\mathbf{X}_\Theta$ obtained from the "affine degeneration" of the affine closure of $\mathbf{X}$, cf. §2.5. This embedding will play the role of $\mathbf{Y}$ in the proof of Proposition 5.4.5. Secondly, we choose a splitting of the sequence analogous to (5.23) for $\mathbf{X}_\Theta$ as follows:

Let $\mathbf{X}^a$ denote the affine closure of $\mathbf{X}$ (i.e. $\mathbf{X}^a = \operatorname{spec} k[\mathbf{X}]$), and consider the affine degeneration $\mathscr{X}^a$ of §2.5 over the base $\overline{\mathbf{A}_{X,ss}}$ (same notation as in §2.5). Notice that $\mathscr{X}^a \times_{\overline{\mathbf{A}_{X,ss}}} \mathbf{A}_X \simeq \mathbf{X}^a \times \mathbf{A}_X$ canonically, so we can define the regular function $F_\chi : (x, a) \mapsto f_\chi(x)\chi(a)$ on it. Choose an affine embedding $\overline{\mathbf{A}_X}$ of $\mathbf{A}_X$ where the kernel $\mathbf{A}_1$ of $\mathbf{A}_X \to \mathbf{A}_{X,ss}$ acts freely, and such that $\overline{\mathbf{A}_X}/\mathbf{A}_1 = \overline{\mathbf{A}_{X,ss}}$. This corresponds to a lifting of cocharacters from $\mathbf{A}_{X,ss}$ to $\mathbf{A}_X$, and gives rise to a base change:

$$\widetilde{\mathscr{X}^a} = \mathscr{X}^a \times_{\overline{\mathbf{A}_{X,ss}}} \overline{\mathbf{A}_X}.$$