Contents

Résolution de Demazure affines et formule de Casselman-Shalika

This is a note for the orignal paper of Ngô and Polo. [**14**].

Introduction

Let $G \in \mathrm{AlgGrp}_k^{\text{cn.red.split}}$, $k = \mathbb{F}_q$. For each $\lambda \in X_\bullet(T)_+$, it is possible to construct a projective k-scheme $\bar{\mathrm{Gr}}_\lambda$, whose set of k points is

$$\overline{\mathrm{Gr}^\lambda}(k) := \bigsqcup_{\lambda' \leq \lambda} K\varpi^{\lambda'}K/K$$

of which the group K, viewed as an algebra group over k of infinite dimension, acts through a quotient of finite type. The action induces a stratification of open orbits

$$\overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\lambda' \leq \lambda} \mathrm{Gr}^{\lambda'}$$

The scheme $\overline{\mathrm{Gr}^\lambda}$ is not smooth in general, for a prime $l \neq \mathrm{char}\, k$, it is natural do consider the l-adic IC complex

$$\mathcal{A}_\lambda := \mathrm{IC}(\overline{\mathrm{Gr}^\lambda}, \bar{\mathbb{Q}}_\lambda)$$

which is K-equivariant. The associated function from Frobenius trace

$$A_\lambda(x) := \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_\lambda)_x)$$

defined on the set of k points of $\overline{\mathrm{Gr}^\lambda}$, can be viewed as a function of the unramified Hecke algebra [**8**], of compactly supported functions in $G(F)$ this is biequivariant wrt $G(\mathcal{O})$.
Let $\check{G}$ be the group defined over $\bar{\mathbb{Q}}_l$ whose roots is dual to that of G. In [Sat63], Satake constructed a canonical isomorphic of the hecke algebra $\mathcal{H}$ with the algebra of regular functions on $\check{G}$, which are $\mathrm{Ad}(\check{G})$ equivariant. After Lusztig and Kato, see [**11**], the Satake transform of A_λ is equal to, up to a sign, the character of V_λ, irreducible representation of height weight of λ of $\widehat{G}$. More recently, Ginzburg, [**12**], has proved a Tannakian equivalence between K equivariant perverse on Gr with the convolution structure, and the algebraic representations of $\check{G}$ with the tensor structure.
The constant terms which are the Fourier coefficients of the functions A_λ are remarkably simple. Let $B := TU$ the a subgroup of Borel of G and ρ the half sum of

positive roots of T in Lie(U). After Lustzig and Kato the constant integral term is equal to

$$\int_{U(F)} A_\lambda(x\varpi^\nu)\,dx = (-1)^{2\langle\rho,\nu\rangle} q^{\langle\rho,\nu\rangle} m_\lambda(\nu)$$

where $m_\lambda(\nu)$ is the dimension of the weight space ν in $V(\lambda)$.
Example:
The principle object of this paper is to prove the gometric statement of the above result. For each $\nu \in X_\bullet(T)$ there is a well defined subscheme $S_\nu \subset \mathrm{Gr}$ such that

$$S_\nu(k) := U(F)\varpi^\nu G(\mathcal{O})/G(\mathcal{O})$$

We show that the complex

$$R\Gamma_c(S_\nu \otimes_k \bar{k}, \mathcal{A}_\lambda)$$

is concentrated in degree $2\langle\rho,\nu\rangle$ and that the Frobenius endomoprhism acts on $H^{2\langle\rho,\nu\rangle}$ as multiplication by $q^{2\langle\rho,\nu\rangle}$....
When ν is dominant, we can define a moprhism $h : S_\nu \to \mathbb{G}_a$ such that $\theta(x) = \psi(h(x))$, where $\psi : k \to \bar{\mathbb{Q}}_l^\times$ is a nontrivial additive character on k. We show that the complex

$$R\Gamma_c(S_\nu \otimes_k \bar{k}, \mathcal{A}_\lambda \otimes h^*\mathcal{L}_\psi)$$

Here is the organization of the article. After recalling in section 2, known results on affine Grassmanian, we state the principle theorems in 3.2 and 3.4 in section 3. The proof of the theorem occupies the rest of the article. This is based on the study of the geometry of certain resolutions from the simplest $\overline{\mathrm{Gr}^\lambda}$, which corresponds to when λ is minuscule or quasi-minuscule. This strategy is used in **[13]**, where the conjecture of **[7]** is proved for GL_n.
In section 4 and section 5, we prove geometric properties of the intersection $S_\nu \cap \overline{\mathrm{Gr}^\lambda}$, which were probably well known but cannot be found in the literature. 5.2 allows us to show the statements 3.2, 3.4 in the case ν is conjugated by λ by an element of the Weyl group.
We then study in section 6, the geometry of $\overline{\mathrm{Gr}^\lambda}$ in the most simple case, that is, when λ is minuscule section 6, or when it is quasiminuscule section 7. If λ is minuscule, then $\overline{\mathrm{Gr}^\lambda}$ is equal to Gr^λ and is isomorphic to the scheme G/P of subgroups of G which are conjugate to some parabolic P, further, only the ν which are conjugate to λ are involved, so that 3.2 and 3.4 follows as in the case from 5.2.

1. Notation

Let k be a finite field of q elements of characteristic p, with algebraic closure $\bar{k}$. Let T be split maximal torus of G and B, B^- be the Borel subgroups such that $B \cap B^- = T$. We denote $\langle -, - \rangle$ the natural paring $X, X^\vee := \mathrm{Hom}(\mathbb{G}_m, T)$. Let $R \hookrightarrow X$ be the system of roots associated to (G, T) and R_+ the roots corresponding to B (resp. B^-) and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots. For each $\alpha \in \Phi$, we denote U_α the the root subgroup of G corresponding to α. Let $\Phi^\vee \hookrightarrow X_\bullet$ be the dual roots provided by the bijection

$$\Phi \to \Phi^\vee \quad \alpha \mapsto \alpha^\vee$$

Denote by $\Phi_+^\vee$ the set of positive coroots. Let W be the Weyl group of (G, T). [1] Let

$$\rho := (1/2) \sum_{\alpha \in R_+} \alpha$$

the half sum of positive roots. For each simple root, we have

$$\left\langle \rho, \alpha^\vee \right\rangle = 1$$

We denote $Q^\vee := \mathbb{Z}\Phi^\vee$ (resp. $Q_+^\vee := \mathbb{N}_{\geq 0}\Phi_+^\vee$). We denote by $X_{\bullet,+}$ the cone of dominant cocharacter

$$X_{\bullet,+} := \{\lambda \in X_\bullet \ : \ \langle \alpha, \lambda \rangle \geq 0 \forall \alpha \in \Phi_+\}$$

We consider the partial order on $X_\bullet$ as follows: $\nu \geq \nu'$ if and only if $\nu - \nu' \in Q_+^\vee$. In the case of GL_n, this has a particular simple characterization, see [**13**].

We denote $\check{G}$ the dual group over $\bar{\mathbb{Q}}_l$. It is provided with $\check{T} \hookrightarrow \check{B}$. For each $\lambda \in X_{\bullet,+}$ We denote

$$\Omega(\lambda) := \{\nu \in X_\bullet \ : \ \forall w \in W \quad w\nu \leq \lambda\}$$

This is the set of weight of $\check{T}$ in V_λ, the $\check{G}$-simple $\bar{\mathbb{Q}}_l$ module of highest weight λ. We denote M the set of minimal elements[2] in $X_{\bullet,+} \setminus \{0\}$.

Proposition 1.1. Let $\mu \in M$. We have the following equivalent:

1. If $\langle \alpha, \mu \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Phi$, and μ is a minimal element in $X_{\bullet,+}$, then $\Omega(\mu) = W\mu$. In this case, we say that μ is minuscule cocharacter. [3]
2. Otherwise,[4] there exists a unique root such that $\langle \gamma, \mu \rangle \geq 2$; its a maximal positive root, and we have $\mu = \gamma^\vee$ and $\Omega(\mu) = W\mu \cup \{0\}$. In this case, we say that μ is *quasi-miniscule.*

PROOF. The first [**3**, Chap. VI, Ex. 1.24]. We prove the second. Let $\gamma \in \Phi$ such that $\langle \gamma, \mu \rangle \geq 2$. ☐

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[1] The Weyl group is given by $N_G(T)/Z_G(T)$. Typical example to keep in mind is $s := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, see [**1**, 26]

[2] The condition of being minimal: is that there does not exists such that

[3] Take $\mu = (1, 0)$.

[4] In GL_2 there is only *one* positive root. Thus, this criteria simply says that as long as (a, b) satisfies $a \geq b + 2$, then it is not minuscule.

Example 1.2. Let $G = \mathrm{GL}_n$. Then the set of minimal elements in $X_{\bullet,+}\backslash 0$ are classified by:

- Characters.

  $$(l,\ldots,l) \quad l \in \mathbb{Z}$$

  In the representation theoretic side, but the det map takes diagonal elements

  $$(t_i)_{i=1}^n \mapsto \left(\prod t_i\right) \mapsto \left(\prod t_i\right)^n$$

  for $n \in \mathbb{Z}$.
- Miniscule + twisted by characters.

  $$(l+1,\ldots,l+1,l,\ldots,l)\, l \in \mathbb{Z}$$
- Quasiminuscle.

  $$(1,0,\ldots,0,-1)$$

$$\begin{tikzcd}
\mathrm{GL}_n \arrow[rr] \arrow[dr, "\det"'] & & \mathbb{C}^\times \\
& \mathrm{GL}_n/[GL_n, GL_n] \simeq \mathbb{C}^\times \arrow[ur] &
\end{tikzcd}$$

2. La Grassmannienne affine

Recall the construction, [**10**]. As *loc. cit.* call a k-space, resp. k-group a sheaf of set, resp. of group over the Alg_k with respect to fppf topology. Consider a the k-group LG and the K-subgroup $L^{\geq 0}G$.
It is clear that $L^{\geq 0}G$. is represented by the projective limit of schemes of finite type

$$R \mapsto G(R[[\varpi]]/\varpi^n)$$

Denote by $L^{(N)}G(R)$ the set of $g \in LG(R)$ such that both the order of the poles of $\rho(g)$ and $\rho(g^{-1})$ does not exceed N. After *loc. cit.* $L^{(N)}(G)$ is representable by a scheme and

$$\mathrm{Gr} \simeq \varinjlim \mathrm{Gr}^{(N)}$$

where $\mathrm{Gr}^{(N)} = L^{(N)}G/L^{\geq 0}G$. Denote $L^{\leq 0}G$ the k group $R \mapsto G(R[\varpi^{-1}])$ [5]and let

$$L^{<0}G := \ker(L^{\leq 0}G \xrightarrow{\varpi^{-1}\mapsto 0} G)$$

Example 2.1. $L^{<0}G$ has entries of the form

$$\begin{pmatrix} 1+\frac{1}{t}p(\frac{1}{t}) & \frac{1}{t}p(\frac{1}{t}) \\ \frac{1}{t}p(1/t) & 1+\frac{1}{t}p(\frac{1}{t}) \end{pmatrix} \quad p \in k[x]$$

This is a subgroup of LG.

Proposition 2.2. The morphism

$$L^{<0}G \times L^{\geq 0}G \to LG$$

is an open immersion.

We identify $L^{<0}G$ with the open $L^{<0}Ge_0$ where e_0 is a fixed based point of Gr. The Grassmanin Gr is covered by the open tralsates $gL^{<0}Ge_0$. These are easy to study for the local geometry of Gr. For example $L^{<0}G$ is not reduced in general, neither is Gr.
The group $L^{\geq 0}G$ acts naturally on Gr. For all $\lambda \in X_\bullet$ denote e_λ the point $\varpi^\lambda e_0$ of Gr. For $\lambda \in X_{\bullet,+}$ denote Gr^λ the $L^{\geq 0}G$ orbit of e_λ. Denote $\overline{\mathrm{Gr}^\lambda}$ the closure of Gr^λ. Also

$$L^{\geq\lambda}G := \mathrm{ad}\varpi^\lambda L^{\geq 0}G, \quad L^{<\lambda}G := \mathrm{ad}\varpi^\lambda L^{<0}G$$

Example 2.3. $G = \mathrm{GL}_2$, let $\lambda = (a, 0) \in X_{\bullet,+}$ so that $a \in \mathbb{N}_{\geq 0}$. Then

$$L^{\geq\lambda}G" = " \left\{ \begin{pmatrix} \mathcal{O} & t^a\mathcal{O} \\ \frac{1}{t^a}\mathcal{O} & \mathcal{O} \end{pmatrix} \right\}$$

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[5] $L^{\leq 0}G$ is often referred as negative loop group, and is also identified as G^{X-x} where $X = \mathbb{P}^1_k$.

Denote J the prieimage of $U \hookrightarrow B$ under the homomoprhism $L^{\geq 0}G \to G$ deinfed by $\varpi \mapsto 0$. Thus, we have the diagram

$$\begin{tikzcd} J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^{\geq 0}G \arrow[d] \\ U \arrow[r, hook] & G \end{tikzcd}$$

This is a projective limit of unipotent groups. Denote by

$$J^{\geq \lambda} := J \cap L^{\geq \lambda}G$$

$$J^{\lambda} := J \cap L^{<\lambda}G$$

Example 2.4. $G = \mathrm{GL}_2$, then

$$J(k) = \begin{pmatrix} 1 + tk[[t]] & k[[t]] \\ tk[[t]] & 1 + tk[[t]] \end{pmatrix} = \begin{pmatrix} 1 + t\mathcal{O} & \mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix}$$

On the other hand, we see that under the language of Moy-Prasad filtration, $J \simeq \langle T_1(\mathcal{O}), U_{\alpha,1,x} : \alpha \in \Phi \rangle$, can be thought of also as the associated loop group of a parahoric group scheme over $\mathcal{O}$.

- $$J^{(1,0)}(k) = k[\frac{1}{t}] \cap k[[t]] = k$$
- Or in general, $\lambda = (a, 0)$. We have
  $$L^{<\lambda}(k) = \begin{pmatrix} 1 + \frac{1}{t}p(\frac{1}{t}) & t^{a}\frac{1}{t}p(\frac{1}{t}) \\ t^{-a}\frac{1}{t}p(\frac{1}{t}) & 1 + \frac{1}{t}p(\frac{1}{t}) \end{pmatrix}$$
  $$J^{\lambda}(k) = \mathrm{Span}_k \left\{1, \ldots, t^{a-1}\right\}$$
  This is the *finite part* of the decomposition of $L^{<\lambda}G \times L^{\geq\lambda}G \simeq LG$. Don't confuse this with LU! This also coincides with Equation 1.

Let $\alpha \in R$, $i \in \mathbb{Z}$, let $U_{\alpha,i}$ be the image of the homomorphism

$$\mathbb{G}_a \to LG$$

$$x \mapsto U_\alpha(\varpi^i x)$$

The multiplication defines an isomoprhism

$$\prod_{\alpha \in R_+, \langle \alpha, \lambda \rangle > 0} \prod_{i=0}^{\langle \alpha, \lambda \rangle - 1} U_{\alpha,i} \xrightarrow{\simeq} J^{\lambda} \tag{1}$$

where we made a choice of total order on the set of factors. In particular J^{λ} is isomorphic to an affine space of dimension $2\langle \rho, \lambda \rangle$.

Example 2.5. In the context of GL_n: $\Phi_+ := \{e_i - e_j \, : \, i < j\}$. When $\langle \alpha, \lambda \rangle > 0$, where α is the index of root subgroup. So $\alpha = e_i - e_j$, $\lambda \in X_{\bullet,+}$, the condition means that $\lambda_i > \lambda_j$, i.e. $i > j$.
In the case of $n = 2$, we have $\lambda_1 > \lambda_2$. Thus, this counts the difference between $\lambda_1 - \lambda_2 - 1$. This is the same as that in $L^{<\lambda}(k)$.

Proposition 2.6. The natural morphism

$$J^\lambda \to \mathrm{Gr}^\lambda$$

$$j \mapsto je_\lambda$$

is an open immersion.

Proof. It is clear that multiplication induces an isomorphism

$$J^\lambda \times J^{\geq\lambda} \xrightarrow{\simeq} J$$

It is also clear that the multiplication induces an open immersion

$$J \times B^- \to L^{\geq 0}G$$

Moreover, $J^{\geq\lambda}$ and B^- are subgroups of $L^{\geq\lambda}G$ which fixes e_λ. The lemma follows. □

It follows from 2.6 that Gr^λ is smooth irreducible and of dimension $2\langle \rho, \lambda \rangle$. There exists an embedding $\mathrm{Gr}^\lambda \hookrightarrow \mathrm{Gr}^{(N)}$ for N sufficiently large, hence the closure $\overline{\mathrm{Gr}^\lambda}$ is a porjective scheme, irreducible and stable by the action of $L^{\geq 0}G$. It is well known, see [**11**, 11], that $\overline{\mathrm{Gr}^\lambda}$ is the union of orbits $\mathrm{Gr}^{\lambda'}$ such that $\lambda' \leq \lambda$. In particular, if μ is minuscule [6], then Gr^μ is a smooth projective scheme.
Let [7]

$$L^{>0}G := \ker\left(L^{\geq 0}G \to G\right)$$

Example 2.7. $G = \mathrm{GL}_2$, then

$$L^{>0}G = \begin{pmatrix} 1 + t\mathcal{O} & t\mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix}$$

This is a projective limit of unipotent groups. It is clear that for $\lambda \in X_{\bullet,+}$ the moprhism

$$L^{>0}G \cap L^{\geq\lambda}G \times L^{>0}G \cap L^{<\lambda}G \xrightarrow{\simeq} L^{>0}G$$

is an isomorphism and that [8]

$$L^{>0}G \cap L^{<\lambda}G = \prod_{\alpha\in\Phi_+, \langle\alpha,\lambda\rangle>1} \prod_{i=1}^{\langle\alpha,\lambda\rangle-1} U_{\alpha,i}$$

Example 2.8. Let λ be minuscule. Then the intersection is empty.

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[6] don't we only need being minimal in $X_{\bullet,+}$?

[7] Loops with formal series with no constant terms.

[8] Taking $\lambda = (1, 0)$, whose that the only term that matters is in the top right.

Let P_λ be the parabolic subgroup generated by B^- and by the radical subgroups with $\langle \alpha, \lambda \rangle = 0$, this would be equivalent to the one constructed in 2.2. The Weyl group of W is equal to the stabilizer W_λ of λ. We denote N_λ^+ the opposite unipotent radical of parabolic opposite to P_λ. It is clear that

$$P_\lambda \subset L^{\geq \lambda} G$$

and that

$$J^\lambda = N_\lambda^+ \ltimes L^{>0} G \cap L^{<\lambda} G \tag{2}$$

Example 2.9.

Proposition 2.10. We have

$$L^+ G \cap L^{\geq \lambda} G = P_\lambda \ltimes (L^{>0} G \cap L^{\geq \lambda} G)$$

In particular, the group $L^{\geq 0} G \cap L^{\geq \lambda} G$ is geometrically connected and we have $G \cap L^{\geq \lambda} G = P_\lambda$.

Proof. It sutffices to show that the multiplication morphism

$$\left(L^{>0} G \cap L^{\geq \lambda} G \right) \times P_\lambda \to L^{\geq 0} G \cap L^{\geq \lambda} G$$

is an isomorphism. Let $g \in L^{\geq 0} G$, which can be written in the form

$$g = g^+ g^- u w p$$

where $g^+ \in L^{>0} G \cap L^{\geq \lambda} G$, $g^- \in L^{>0} G \cap L^{<\lambda} G$ where $u \in U \cap w U_\lambda^+ w$[9], and $p \in P_\lambda$. ☐

2.1. Product decomposition of parabolics. Before we begin, note that there are bijections

$$\mathrm{Borel}(T) \simeq \mathrm{WeylChambers} \simeq \left\{ \Phi^+ \subset \Phi \right\}$$

(1) The second to third map: pick a Weyl chamber, and any cocharacter λ. Then we can define positive and negative roots via:

$$\Phi^+ := \{ \lambda : \langle \alpha, \lambda \rangle > 0 \}$$

We can further construct a basis of Φ^+ by considering the indecomposable roots [**6**, 10], this are $\Delta \subsetneq \Phi+$, such that cannot be written as the sum $\beta_1 + \beta_2$, of $\beta_1, \beta_2 \in \Phi^+$. [10].

Definition 2.11. The connected components of $\mathbb{R} \otimes X_\bullet \setminus \bigcup H_\alpha$ are the Weyl chambers, where $H_\alpha := \{ \lambda \in X_{\bullet, \mathbb{R}} : \langle \alpha, \lambda \rangle = 0 \}$.

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[9] This is a Bruhat decomposition argument.

[10] This can argued by minimality, choose α which is not in $\Phi^+ \setminus \mathbb{Z}_{\geq 0} \Delta$, which minimizes its pairing with $\langle -, \lambda \rangle$. But $\langle \alpha, \lambda \rangle = \langle \beta_1, \lambda \rangle + \langle \beta_2, \lambda \rangle$, where $\beta_i \in \Phi^+$, so $\langle \beta_1, \lambda \rangle$ contradicts minimality.

Theorem 2.12. Relative Bruhat Decomposition. There is an isomorphism at the level of k points,

$$W := N(k)/Z(k) \xrightarrow{\simeq} P(k)\backslash G(k)/P(p)$$

Following Lusztig, Ginzburg, Mkirkovic and Vilonen, we define the convolution product $\mathcal{A}_{\lambda_1} * \mathcal{A}_{\lambda_2}$ for $\lambda_1, \lambda_2 \in X_{\bullet,+}$. Consider the moprhisms

$$\begin{tikzcd}
& LG \times \mathrm{Gr} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathrm{Gr} \times \mathrm{Gr} & & \mathrm{Gr} \times \mathrm{Gr}
\end{tikzcd}$$

$$\pi_1(g,x) = (ge_0, x) \quad \pi_2(g,x) = (ge_0, gx)$$

The moprhism $\pi - 1$ is the quotient[11] morphism for the action $L^{\geq 0}G$ on $LG \times \mathrm{Gr}$ defined by

$$\alpha_1(h)(g,x) = (gh^{-1}, x)$$

whilst $\pi - 2$ is the quotient morphism of the action of $L^{\geq 0}G$ on $LG \times \mathrm{Gr}$ deinfed by

$$\alpha_2(h)(g,x) = (gh^{-1}, hx)$$

For $\lambda_1, \lambda_2 \in X_{\bullet,+}$ let

$$\overline{\mathrm{Gr}^{\lambda_1}} \bar{\times} \overline{\mathrm{Gr}_2^{\lambda}}$$

be the quotinet of $\pi_1^{-1}(\overline{\mathrm{Gr}^{\lambda_1}} \times \overline{\mathrm{Gr}_2^{\lambda}})$ by $\alpha_2(L^{\geq 0}G)$. The existsence of this question is guaranteed by the local triviality of the moprhism $LG \to \mathrm{Gr}$. More precisely, as the open s of $\overline{\mathrm{Gr}^{\lambda}}$, of the form

$$gL^{<0}Ge_0 \cap \overline{\mathrm{Gr}^{\lambda_1}}$$

the schemes

$$\overline{\mathrm{Gr}^{\lambda_1}} \bar{\times} \overline{\mathrm{Gr}^{\lambda_2}}$$

and

$$\overline{\mathrm{Gr}^{\lambda_1}} \times \overline{\mathrm{Gr}^{\lambda^2}}$$

are isomorphic. Further, these isomoprhisms are clearly compatible with the stratification of $\overline{\mathrm{Gr}_1^{\lambda}} \times \overline{\mathrm{Gr}^{\lambda_2}}$ by the locally closed subsets $\mathrm{Gr}^{\lambda_1'} \times \mathrm{Gr}^{\lambda_2'}$. The projection on second factor defines a moprhism

$$m : \overline{\mathrm{Gr}^{\lambda_1}} \bar{\times} \overline{\mathrm{Gr}^{\lambda_2}} \to \overline{\mathrm{Gr}^{\lambda_1+\lambda_2}}$$

2.1.1. *Some remarks on the twisted products.*

Proposition 2.13. [**16**, 2] $\mathrm{Gr} \tilde{\times} \mathrm{Gr} \cdots \tilde{\times} \mathrm{Gr} \simeq \mathrm{Gr}^n$.

Whenever we have

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[11] The terminology is unclear here. Should edit.

2.2. Examples of parabolics. Let $\lambda = (\lambda_1, \lambda_2)$. Generating from roots. For a root α, we can construct

$$\langle B, M_\alpha \rangle$$

where $M_\alpha := Z(T_\alpha), T_\alpha := \ker(T \xrightarrow{\alpha} \mathbb{G}_m)$.

Example 2.14. $G = \mathrm{GL}_n$. Let $\lambda = \big(\lambda_1 = \cdots \lambda_{m_1} > \cdots > \lambda_{m_{k-1}+1} = \cdots = \lambda_{m_k}\big)$. The parabolic is of the form:

$$P_\lambda := \begin{pmatrix} \boxed{\mathrm{GL}_{m_1}} & * & * \\ & \ddots & * \\ 0 & & \boxed{\mathrm{GL}_{m_k}} \end{pmatrix}$$

Though, later we would consider another way to construct these parabolic from root subgroups, see Sec. 7.
We may consider $\mathrm{ev}_0^{-1}(P_\lambda)$.

Proposition 2.15. [**15**, 2.3.10]

$$\mathrm{ev}_0^{-1}(P_\lambda) \simeq L^{\geq 0}G \cap L^{\geq \lambda}G$$

Proof. Let us consider the $\mathbb{C}$-points. It would be easy to consider the function $\tilde{\lambda}_{(-)} : \{1, \dots, n\} \to \mathbb{Z}$ as a function given by

$$\tilde{\lambda}_x = \lambda_i \text{ if } 1 \leq x \leq \lambda_{m_i}$$

Then

$$L^{\geq 0}G(\mathbb{C}) \cap L^{\geq \lambda}G(\mathbb{C}) = \Big\{ t^{\tilde{\lambda}_i - \tilde{\lambda}_j} a_{ij} \in G(\mathbb{C}[[t]]) \, : \, a_{ij} \in G(\mathbb{C}[t]]) \Big\}$$

□

3. Les énconcés principaux

Recall that U denotes the unipotent radical of B associated to R_+. We define LU,

$$L^{\geq 0}U := LU \cap L^{\geq 0}G, \quad L^{\leq 0}U := LU \cap L^{<0}G$$

For each $\nu \in X_\bullet(T)$ we also denote

$$L^{\geq \nu}U := \varpi^\nu L^{\geq 0}U\varpi^{-\nu}, L^{<\nu}U := \varpi^\nu L^{<0}U\varpi^{-\nu}$$

Example 3.1. $G = \mathrm{GL}_2$. $\lambda := (1,0) \in X_{\bullet,+}$. Then

$$L^{\geq\lambda}U = \begin{pmatrix} 1 & tk[[t]] \\ & 1 \end{pmatrix}, L^{<\lambda}U = \begin{pmatrix} 1 & t(1/t)k[1/t] \\ & 1 \end{pmatrix}$$

In general if $\lambda = (a,b)$, then

$$L^{\geq\lambda}U = \begin{pmatrix} 1 & t^{a-b}k[[t]] \\ & 1 \end{pmatrix}, L^{<\lambda}U = \begin{pmatrix} 1 & t^{a-b}(1/t)k[1/t] \\ & 1 \end{pmatrix}$$

For each $\nu \in X_\bullet$, $L^{<\nu}U$ is a closed subgroup of $L^{<\nu}G$ so we can define $L^{<\nu}Ue_0$ as a closed subset

$$S_\nu \hookrightarrow_{\mathrm{cl}} \varpi^\nu L^{<0}Ge_0 \hookrightarrow \overline{\mathrm{Gr}}_\lambda$$

In particular for all $\lambda \in X_{\bullet,+}$ and $\nu \in X_\bullet$, $S_\nu \cap \overline{\mathrm{Gr}}_\lambda$ is a locally closed subscheme, possibly empty, of $\overline{\mathrm{Gr}}_\lambda$. By the Iwaswa decomposition, this yields a stratification of $\overline{\mathrm{Gr}}_\lambda$. We will give a new proof of the following theorem due to Mirkovic and Vilonen in the case $k = \mathbb{C}$, [**12**].

Theorem 3.2. For each $\lambda \in X_{\bullet,+}$, and $\nu \in X_\bullet$ the complex $R\Gamma_c(S_\nu, \mathcal{A}_\lambda)$ is concentrated in degree $2\langle\rho,\nu\rangle$. Further, the endomoprhism Fr_q acts on $H_c^{2\langle\rho,\nu\rangle}(S_\nu, \mathcal{A}_\lambda)$ as $q^{\langle\rho,\nu\rangle}$.

In the previous statement we wrote $R\Gamma_c(S_\nu, \mathcal{A}_\lambda)$ instead of

$$R\Gamma_c((S_\nu \cap \overline{\mathrm{Gr}^\lambda}) \otimes_k \bar{k}, \mathcal{A}_\lambda)$$

for simplicity. We use this notation systematically in the following and does not cause any ambiguity.

For each $\nu \in X_{\bullet,+}, \nu' \in X_\bullet$, choose a total orer of the positive roots and we have an isomoprhism

$$\prod_{\alpha\in R_+} \prod_{\langle\alpha,\nu'\rangle \leq i < \langle\alpha,\nu\rangle} U_{\alpha,i} = L^{<\nu}U \cap L^{\geq\nu'}U$$

For ν fixed ν' more and more anitdominant , this group forms an inductive system for the limit $L^\nu U$.

Example 3.3. Use $G = \mathrm{GL}_2$, $\nu_1 = (1,0)$. Let $\nu'_n := -(n,-n)$, then

$$L^{\geq\nu'}U = \begin{pmatrix} 1 & t^{-2n}k[[t]] \\ & 1 \end{pmatrix}$$

It is then clear that

$$L^{<\nu} = \varinjlim L^{<\nu}U \cap L^{\geq \nu'_n}U$$

For each simple root $\alpha \in \Delta$, denote $u_{\alpha,i}$ the projection over the factor $U_{\alpha,i}$ and

$$h : L^{<\nu}U \cap L^{\geq \nu'}U \to \mathbb{G}_a$$
$$h(x) := \sum_{\alpha \in \Delta} u_{\alpha,-1}(x)$$

Fix a nontrivial additibev character, $\psi : k \to \bar{\mathbb{Q}}_l^{\times}$, and denote $\mathcal{L}_\psi$ the Artin-Schreier sheaf over $\mathbb{G}_a$ associated to ψ. The character $\theta : U(F) \to \bar{\mathbb{Q}}_l$ considered in introduction is the character $x \mapsto \psi(h(x))$. The following statement was a conjecture of [**7**]

Theorem 3.4. For $\nu \neq \lambda$ in $X_{\bullet,+}$ the complex $R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h^*\mathcal{L}_\psi)$ is zero. For $\nu = \lambda$ the complex is isomorphic to $\bar{\mathbb{Q}}_l$ provided with the action of Frobenius by $q^{\langle \rho, \lambda \rangle}$, at degree $2\langle \rho, \lambda \rangle$.

These results imply the statements about constant terms and Fourier coefficients mentioned in the Grothendiecks' function-sheaf dictionary. We will present the proofs of these two theorems in parallel in the rest of the article.

4. L'action du tore T

The torus T normalizes these subgroups $L^{\geq 0}G, L^{<0}G$, $L^{<\nu}G$, ... of LG so that it acts on all the geometric objects we considered. This action provides a valuable tool to study their geometry. Choose once and for all a strictly dominant cocharacter $\phi : \mathbb{G}_m \to T$. The $\mathbb{G}_m$ action we consider follows from the following compositions

$$\mathbb{G}_m \hookrightarrow L^{\geq 0}\mathbb{G}_m \xrightarrow{L^{\geq 0}\phi} L^{\geq 0}G \circlearrowleft \mathrm{Gr}$$

Proposition 4.1. For all $\nu \in X_\bullet$ the point e_ν is the fixed point of the action $\mathbb{G}_m \circlearrowleft S_\nu$. Furthermore, it is the attractive fixed point.

Proof. For all $x \in L^{<\nu}U(\bar{k})$ is of the form

$$x = \prod_{\alpha\in\Phi_+} \prod_{i<\langle\alpha,\nu\rangle} U_{\alpha,i}(x_{\alpha,i})$$

where $x_{\alpha,i} \in \bar{k}$ are zero for all but a finite number. Thus, for all $z \in \bar{k}^\times$, we have

$$\phi(z)xe_\nu = \prod_{\alpha\in\Phi_+} \prod_{i<\langle\alpha,\nu\rangle} U_{\alpha,i}(z^{\langle\alpha,i\rangle}x_{\alpha,i})e_\nu$$

□

This lemma shows that e_ν are the only fixed points of the action $\mathbb{G}_m \circlearrowleft \mathrm{Gr}$. Further, it implies following statement

Lemma 4.2. If the intersection $S_\nu \cap \overline{\mathrm{Gr}^\lambda}$ is nonempty, ν belongs $\Omega(\lambda)$.

Proof. If a point $x\varpi^\nu$ with $x \in L^\nu U(\bar{k})$ belongs to $\mathrm{Gr}_{\leq\lambda}(\bar{k})$ then the orbit of ... ? □

Proposition 4.3. The Euler-Poincaré characteristic $\chi_c(S_\nu \cap \mathcal{Q}_\lambda)$ is equal to 1 if ν is conjugate to λ by an element of W and 0 otherwise.

This statement can be considered as a geometric interpretation of result of Lusztig, [**11**, 6.1]. Let us use the notation of introduction. Let c_λ be the element of hecke algebra $\mathcal{H}$ defined

$$c_\lambda = (-1)^{2\langle\rho,\lambda\rangle}q^{-\langle\rho,\lambda\rangle}1_\lambda$$

where 1_λ is the characteristic function of $K\varpi^\lambda K$. We know that

$$(c_\lambda) = (K_{\lambda,\mu}(q))^{-1}(A_\lambda)$$

where $K_{\lambda,\mu}(q)$ is the triangular matrices formed the Kazhdan-Lusztig polynomials. The constant terms of the normalizing constants

$$(-1)^{2\langle\rho,\nu\rangle}q^{-\langle\rho,\nu\rangle}\int_{U(F)} c_\lambda(x\varpi^\mu)\,dx$$

5. Les intersections $S_{w\lambda} \cap \overline{\mathrm{Gr}^{\lambda}}$

For all $\lambda \in X_{\bullet,+}$ we considered

$$J^{\lambda} = \prod_{\alpha \in \Phi_+} \prod_{i=0}^{\langle \alpha, \lambda \rangle - 1} U_{\alpha,i}$$

which is clearly a subgroup of $L^{\geq 0}U$. We also prove that the morphism $J^{\lambda} \to \overline{\mathrm{Gr}^{\lambda}}$ is an open immersion. A distinct argument of the content of this section is given in [**4**, 5.2].

Proposition 5.1. Let $\lambda \in X_{\bullet,+}$ induces an isomorphism of J^{λ} with the open subset $\varpi^{\lambda} L^{<0} G e_0 \cap \overline{\mathrm{Gr}^{\lambda}}$ of $\overline{\mathrm{Gr}^{\lambda}}$.

Proof. The image of J^{λ} is contained in $\varpi^{\lambda} L^{<0} G e_0 \cap \overline{\mathrm{Gr}^{\lambda}}$. By 2.6, it is thus a dense open subset of $\varpi^{\lambda} L^{<0} G e_0 \cap \overline{\mathrm{Gr}^{\lambda}}$. □

Now one proves a "loop group" version of the identifying the Schubert cells, as [**2**].

Proposition 5.2. Let $\lambda \in X_{\bullet,+}$ for $w \in W$ the moprhism

$$wJ^{\lambda}w^{-1} \cap LU \xrightarrow{\simeq} S_{w\lambda} \cap \overline{\mathrm{Gr}^{\lambda}}$$

defined by

$$j \mapsto j e_{w\lambda}$$

is an isomorphism. As a consequence $S_{w\lambda} \cap \overline{\mathrm{Gr}^{\lambda}}$ is isomorphic to an affine space of dimension $\langle \rho, \lambda + w\lambda \rangle$

Proof. For $w = 1$, the result follows from the 5.1 due to the following inclusion [12]

$$J^{\lambda} e_{\lambda} \subset S^{\lambda} \cap \overline{\mathrm{Gr}^{\lambda}} \subset \varpi^{\lambda} L^{<0} G e_0 \cap \overline{\mathrm{Gr}^{\lambda}}$$

For $w \in W$, we can reason as follows: as shown,

$$wJ^{\lambda}w^{-1} \to \varpi^{w\lambda} L^{<0} G e_0 \cap \mathrm{Gr}_{\leq \lambda}$$

defined by $j \mapsto j e_{w\lambda}$ is an isomorphism. Thus the multiplication

$$\left(wJ^{\lambda}w^{-1} \cap LU\right) \times \left(wJ^{\lambda}w^{-1} \cap LU^{-}\right) \xrightarrow{\simeq} wJ^{\lambda}w^{-1}$$

Moreover, multiplication induces an isomorphism

$$\prod_{\alpha \in \Phi_+ \cap w^{-1}\Phi_+} \prod_{i=0}^{\langle \alpha, \lambda \rangle - 1} U_{w\alpha,i} \xrightarrow{\simeq} wJ^{\lambda}w^{-1} \cap LU \tag{3}$$

Indeed, J^{λ} consists of elements generated by roots subgroups $U_{\alpha,i}$,

$$\mathrm{ad} w U_{\alpha,i} = U_{w\alpha,i}$$

---

[12] If $\lambda = (1,0)$, $J^{\lambda}(k) = \begin{pmatrix} 1 + t\mathcal{O} & \mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \frac{1}{t}k[1/t] & t \cdot \frac{1}{t}k[1/t] \\ \frac{1}{t} \cdot \frac{1}{t}k[1/t] & k[1/t] \end{pmatrix}$. For sake

Thus, the final terms which contribute to the interesction those $\alpha \in \Phi_+$, which under action of Weyl group α, still remains as a positive root group. Now from the equality,

$$\sum_{\alpha \in \Phi_+ \cap w^{-1}\Phi_+} \alpha = \rho + w^{-1}\rho$$

we obtain the second assertion. □

Note that later on, in section 7, we will study the case when $\lambda = e_1 - e_n$ is quasiminuscule.

Example 5.3. The case of GL_2 is not as interesting. $W \simeq S_2 := \{1, w\}$. Indeed $\lambda + w\lambda = (1, -1) + (-1, 1) = 0$. So the intersections here are always 0-dimensional.

Example 5.4. GL_3. $\lambda = (1, 0, -1)$.

$$\rho = \frac{1}{2}\left((1, -1, 0) + (1, 0, -1) + (0, 1, -1)\right) = e_1 - e_3$$

- $w = (13)$. Again, the intersection is trivial.
- $w = (132)$. $\lambda + w\lambda = (1, 0, -1) + (0, -1, 1) = (1, -1, 0)$. Thus

  $$\langle \rho, \lambda + w\lambda \rangle = 1$$

  The root subgroup $\alpha \in \Phi_+$ where under action of w still remains Φ_+ is that induced from $(0, 1, -1)$.

We can deduce 3.2 in the case $\nu = w\lambda$ and 3.4 in the case $\nu = \lambda$. Indeed the inclusion

$$wJ^\lambda w^{-1} \cap LU \hookrightarrow L^{\geq 0}U \hookrightarrow L^{\geq 0}G$$

implies that $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is contained in the open orbit Gr^λ. Thus the restriction of $\mathcal{A}_\lambda$ to $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is equal to:

$$\mathcal{A}_\lambda\Big|_{S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}} = \bar{\mathbb{Q}}_l[\langle \rho, 2\lambda \rangle](\langle \rho, \lambda \rangle)$$

The statement Thm. 3.2 thus follows. The inclusion $J^\lambda \subset L^{\geq 0}U$ implies that the restriction of h to J^λ is zero. Then 3.4 is true in the case $\nu = \lambda$.
The more general statement below will be needed later. For each $\sigma \in X_{\bullet,+}$ denote

$$h_\sigma : LU \to \mathbb{G}_a \tag{4}$$

the morphism

$$h\mathrm{ad}(\sigma) : x \mapsto h(\varpi^\sigma x \varpi^{-\sigma})$$

and also the induced homomorphism $h_\sigma : S_\lambda \to \mathbb{G}_a$. Since σ is dominant, the restriction of h_σ to $L^{\geq 0}U$, and a fortiori to J^λ is zero. We thus also have the following

Proposition 5.5. For all $\lambda, \sigma \in X_{\bullet,+}$ we have

$$R\Gamma_c(S_\lambda, \mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi) = \bar{\mathbb{Q}}_l[-2\langle \rho, \lambda \rangle](-\langle \rho, \lambda \rangle)$$

Remark 5.6. As we will need later, $R\Gamma_c(S_{w\lambda} \cap \mathrm{Gr}_{\leq \lambda}, \mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi)$. In which we case we also have $S_{w\lambda} \cap \mathrm{Gr}_{\leq \lambda} = S_{w\lambda} \cap \mathrm{Gr}_\lambda$. Then $h_\sigma^{\lambda, w\lambda}$ is trivial on the piece.

6. Minuscules

We utilized the notations fixed in 1. Let μ be nonzero minimal [13] element of $X_{\bullet,+}$. By 1.1, we have the following statement

Proposition 6.1. Let μ be minuscule. We have $\Omega(\mu) = W\mu$. For $\alpha \in R$, we have

$$\langle \alpha, \mu \rangle \in \{0, \pm 1\}$$

For example, in the case of GL_n the minuscule ones are precisely those of the form

$$(l+1, l+1, , \dots, l+1, l, \dots, l) \quad l \in \mathbb{Z}$$

If μ is minuscule, by minimality, this implies the orbit Gr^μ is closed. Since for all elements ν of $\Omega(\mu)$ is conjugate to μ by anaction of W for 3.2, 3.4 it suffices to verify for the case $\lambda = \mu$ and $\nu \in \Omega(\mu)$.

Lemma 6.2. We have a canonical isomorphism $\mathrm{Gr}_\mu \to G/P$ st.

$$S_{w\mu} \cap \mathrm{Gr}_\mu \simeq UwP/P$$

Proof. Given 2.10 and the two assertions of 6.1, we have that $L^{\geq 0}G \cap L^{\geq \mu}G$ is the inverse image of of P under the homomorphism $\mathrm{ev}_0 : L^{\geq 0}G \to G$. For example, see 2.15.

$$\mathrm{Gr}^\mu = L^{\geq 0}G/(L^{\geq 0}G \cap L^{\geq \mu}G) \simeq G/P_\mu$$

Given, again, 6.1 we knw that $J^\mu = U_\mu^+ = \prod_{\langle \alpha, \mu \rangle = 1} U_\alpha$, which is the unipotent subgroup of the opposite parabolic of P. As a consequence

$$wJ^\mu w^{-1} \cap LU = wU_\mu^+ w^{-1} \cap U$$

The second assersion follow from 5.2. $\square$

---

[13] why was this necessary again?

7. Quasi-minuscules: étude géométrique

See also exercise of Zhu. Let μ is a quasi-minuscule weight, i.e. a minimal element of $X_{\bullet,+}\setminus\{0\}$, smaller than 0. Recall, that by 1.1 we have

Lemma 7.1. Let μ be quasiminuscule. Then μ is equal to a cocharacter $\gamma^\vee$ associated to a positive maximal root γ.[14] We have $\Omega(\mu) = W\mu \cup \{0\}$. For each root $\alpha \in \Phi\setminus\{\pm\gamma\}$ we have $\langle\alpha,\mu\rangle \in \{0,\pm1\}$.

Example 7.2. Consider the maximal root:

$$e_1 - e_2$$

Then $\langle\mu,\gamma\rangle = 2$, implies that μ is dual coroot. For GL_n us not hard to compute: we can sum up all the positive roots:

$$e_1 - e_n$$

This satisfies that for all *other* roots

$$\langle e_1 - e_n, \alpha\rangle \in \{0,\pm1\}$$

Since 0 is a dominant cocharactere which is smaller smaller than μ,

$$\mathrm{Gr}_{\leq\mu} = \mathrm{Gr}_\mu \cup \mathrm{Gr}_0$$

Denote by P the parabolic subgroup of G generated by T and the subgroup of radical roots U_α such that $\langle\alpha,\gamma^\vee\rangle \leq 0$, see 2.2. Denote

$$V := \mathfrak{h} \oplus \bigoplus_{\alpha\in R\setminus\{\gamma\}} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the Lie algebra of T and where $\mathfrak{g}_\alpha$ are the subspaces of weight α of $\mathfrak{g}$. By the preceding lemma V is the sum of weights ν in $\mathfrak{g}$ such that $\langle\gamma,\nu\rangle \leq 1$. It is a result of the definition of P that V is P -stable.

Example 7.3. For $G = \mathrm{GL}_2$, $\mathfrak{g} = \mathfrak{gl}_2$. Then this is this is the lower Borel, and this is indeed also stable under $P_\gamma = B_-$.

Example 7.4.
- $G = \mathrm{GL}_2$. $\mu = \gamma^\vee = (1,-1)$. We get the lower Borel.
- $G = \mathrm{GL}_4$ this is the first case when we don't get the lower Borel.

$$\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

- $G = \mathrm{GL}_n$, these are those roots $\alpha_{i,j}$ where $1 < i,j < n$.

---

[14] To have an example, consider the root $(1,-1)$.

Identify $\mathfrak{g}_\gamma$ with quotient $\mathfrak{g}/V$ with the structure of P-module, we can thus consider the right fibration

$$\begin{tikzcd}
\mathbb{L}_\gamma := G \times^P \mathfrak{g}_\gamma \arrow[d] \\
G/P
\end{tikzcd}$$

Proposition 7.5. $\mathbb{L}_\gamma \simeq \mathrm{Gr}_\mu$

Proof. The functor $R \mapsto G(R[\varpi]/\varpi^2)$ is $\mathrm{TG} \simeq L^1G$,[15] the tangent bundle. where

$$\mathrm{TG} \simeq G \ltimes \mathfrak{g}$$

from the exact sequence

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^1G \simeq TG \arrow[d] \\
1 \arrow[r] & G
\end{tikzcd}$$

There is a canonical truncation map

$$L^{\geq 0}G \to L^1G = \mathrm{TG} \simeq G \ltimes \mathfrak{g}$$

By 2.10 and the last statement of 7.1, that we have a pullback on the left square hence inducing isomoprhisms on the cokernel.

$$\begin{tikzcd}
L^{\geq 0}G \cap L^{\geq \mu}G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^{\geq 0}G \arrow[r] \arrow[d] & \mathrm{Gr}_\mu \arrow[d, "\simeq"] \\
P \ltimes V \arrow[r] & G \ltimes \mathfrak{g} \arrow[r] & (G \ltimes \mathfrak{g}) \,/\, (P \ltimes V) \simeq G \times^P \mathfrak{g}/V
\end{tikzcd}$$

□

The fiber $\mathbb{L}_\gamma$ compacts in a natural into a straight line fiber of projections. In fact we have

$$\mathbb{L}_\gamma \hookrightarrow \mathrm{Proj}(\mathbb{L}_\gamma \oplus \mathcal{O}_{G/P}) \simeq \mathbb{P}_\gamma$$

we have a natural isomorphism

$$\mathrm{Proj}(\mathbb{L}_\gamma \oplus \mathcal{O}_{G/P}) \simeq \mathrm{Proj}(\mathcal{O}_{G/P} \oplus \mathbb{L}_{-\gamma}) \simeq \mathbb{P}_{-\gamma}$$

we can view $\mathbb{P}\gamma$ as the union of $\mathbb{L}_\gamma$ and $\mathbb{L}_{-\gamma}$. Denote $\epsilon_{\pm\gamma}$ the zero sections of $\phi_{\pm\gamma}$.

$$\begin{tikzcd}
\mathbb{L}_{\pm\gamma} \arrow[d] \\
G/P \arrow[u, bend left, "\epsilon_{\pm\gamma}"]
\end{tikzcd} \tag{5}$$

---

[15] The first jet space

Proposition 7.6. The isomorphism of Lem. 7.5

$$\begin{tikzcd}
\mathbb{L}_\gamma \arrow[r, hook] \arrow[d, "\simeq"] \arrow[dd, bend right=60, "\phi_\gamma"'] & \mathbb{P}_\gamma \arrow[d] & \epsilon_{-\gamma}(G/P) \arrow[l, hook'] \arrow[d] \\
\mathrm{Gr}_\mu \arrow[r, hook] \arrow[d, "p_\mu"] & \mathrm{Gr}_{\leq\mu} & \{e_0\} \arrow[l, hook'] \\
G/P & &
\end{tikzcd}$$

extends and sends $\epsilon_{-\gamma}(G/P)$ to the point ϵ_0. p_μ is projection map as given in Rem. 7.7.

Proof. ☐

Remark 7.7. The argument we are doing is similar to when μ is minuscule [**17**, Cor. 1.24]. Indeed, in this case $\mathrm{Gr}_\mu = \mathrm{Gr}_{\leq\mu}$. Where we have an map

$$\begin{tikzcd}
L^+G/L^+G \cap \mathrm{ad}(\varpi^\mu)L^+G \arrow[r, "\simeq"] \arrow[d, "p_\mu"] & \mathrm{Gr}_\mu \arrow[r, hook] & \mathrm{Gr} \\
G/P_\mu & &
\end{tikzcd}$$

Thus showing that for minuscule pieces $\mathrm{Gr}_{\leq\mu}$ is a smooth projective variety.

We now give an explicit description of $S_{w\mu} \cap \mathrm{Gr}_{\leq\mu}$ using the bundle constructed, $\mathbb{L}_\gamma \simeq \mathrm{Gr}_\mu \xrightarrow{p_\mu=\phi_\gamma} G/P$.

Proposition 7.8. Notation as 5.

$$\begin{tikzcd}
\epsilon_\gamma(UwP/P) \arrow[rr, hook, bend left] \arrow[dr] & \phi_\gamma^{-1}(UwP/P) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_\mu \arrow[d, "p_\mu=\phi_\gamma"] \\
 & UwP/P \arrow[r, hook] \arrow[ul, dashed, bend left, "\epsilon_\gamma"] & G/P
\end{tikzcd}$$

Two cases:

- if $w\gamma \in \Phi_+$ then
$$S_{w\mu} \cap \mathrm{Gr}_{\leq\mu} = S_{w\mu} \cap \mathrm{Gr}_\mu = \phi_\gamma^{-1}(UwP/P)$$
- If $w\gamma \in \Phi^-$ we have
$$S_{w\mu} \cap \mathrm{Gr}_{\leq\mu} = \epsilon_\gamma(UwP/P)$$

Proof. Recall the formula from Theorem 5.2,

$$\prod_{\alpha\in\Phi_+\cap w^{-1}\Phi_+} \prod_{i=0}^{\langle\alpha,\lambda\rangle-1} U_{w\alpha,i} \xrightarrow{\simeq} wJ^\lambda w^{-1} \cap LU \tag{6}$$

As $\langle \alpha, \mu \rangle \leq 1$ for all $\alpha \in \Phi_+ \backslash \{\gamma\}$, by Theorem 7.1, we obtain that this is equal to

$$\begin{cases} U_{w\gamma,1} \prod_{\alpha \in \Phi_+ \cap w^{-1}\Phi_+} U_{w\alpha,0} & w\gamma \in \Phi_+ \\ \prod_{\alpha \in \Phi_+ \cap w^{-1}\Phi_+} U_{w\alpha,0} & w\gamma \in \Phi_- \end{cases}$$

The lemma follows. ☐

Definition 7.9. We denote W_γ the stabilizer of γ in W and Δ_γ the set of simple roots conjugates to γ.

Example 7.10. The Weyl group of GL_n is S_n.

Proposition 7.11. We have a stratification, where $\gamma = \mu^\vee$,

$$S_0 \cap \mathrm{Gr}_{\leq \mu} = \{e_0\} \cup \bigcup_{w \in W/W_\gamma, w\gamma \in \Phi_-} \phi_\gamma^{-1}(UwP/P) \backslash \epsilon_\gamma(UwP/P)$$

In particular, the irreducible components of $S_0 \cap \mathrm{Gr}_{\leq \mu}$ are in bijection with Δ_γ and are all of dimension $\langle \rho, \mu \rangle$. We also have the straitication

(7)

$$\pi_\gamma^{-1}(S_0 \cap \overline{\mathrm{Gr}}_{\leq \mu}) = \bigcup_{w \in W_\gamma, w\gamma \in \Phi_-} \phi_\gamma^{-1}(UwP/P) \cup \bigsqcup_{w \in W/W_\gamma, w\gamma \in \Phi_+} \epsilon_{-\gamma}(UwP/P) \hookrightarrow \mathbb{L}_{-\gamma}$$

Proof. Recall that from 4.2, that the only nonzero intersection of S_λ and $\mathrm{Gr}_{\leq \mu}$ occurs when $\lambda \in \Omega(\mu) = W\mu \cup \{0\}$. We will cover $\mathrm{Gr}_{\leq \mu}$, using the description 7.8. ☐

8. Quasi-minuscules: étude cohomologique

The notation are as the 7. In particular $\mu = \gamma^\vee$ is quasi-minuscule. The resolution

$$\pi_\gamma : \mathbb{P}_\gamma \to \overline{\mathrm{Gr}^\mu}$$

allows us to compute the local intersection cohomology of A_μ at an isolated singularity e_0. The following statement is due to Kazhdan and Lusztig.
Indeed, in the following situation, the hypothesis is much weaker, and their argument applies. We detail the proof for the convenience of the reader.

Proposition 8.1. Let $d = \langle 2\rho, \mu\rangle$ the dimension $\overline{\mathrm{Gr}^\mu}$. For $i \geq 0$, the group $H^i(\mathcal{A}_\mu)_{e_0}$ is trivial. For $i < 0$, we have the short exact sequence

(8)

$$0 \longrightarrow H^{i+d-2}(G/P)(d/2-1) \xrightarrow{(-)\wedge c_{-\gamma}} H^{i+d}(G/P)(d/2) \longrightarrow H^i(\mathcal{A}_\mu)_{e_0} \longrightarrow 0$$

where $c_{-\gamma} \in H^2(X_\gamma)(1)$ is the chern class of $\mathbb{L}_{-\gamma}$.

Proof. Let $\overline{\mathrm{Gr}_\mu}'$ be the open of $\overline{\mathrm{Gr}^\mu}$

$$\overline{\mathrm{Gr}^\mu}' := \mathrm{Gr}_{\leq\mu} \backslash \pi_\gamma \circ \epsilon_\gamma(G/P)$$

thus we have

$$\pi_\gamma^{-1}(\mathrm{Gr}'_{\leq\mu})$$

we have $\pi_\gamma^{-1}(\overline{\mathrm{Gr}^\mu}') = \mathbb{L}_{-\gamma}$. Denote $\mathcal{A}'_\mu$ the restriction of $\mathcal{A}_\mu$ to this open. Denote the inclusion of the closed point $i : \{e_0\} \to \overline{\mathcal{A}}'_\mu$. The natural morphism

$$\mathcal{A}'_\mu \to i_* i^* \mathcal{A}'_\mu$$

induces a restriction of morphism of cohomology (without support)

$$i^* : R\Gamma(\overline{\mathrm{Gr}^\mu}', \mathcal{A}'_\mu) \to (\mathcal{A}'_\mu)_{e_0}$$

We prove that i^* i an isomoprhism. For this we utilize the decomposition of Beilinson, Bernstien, Deligne and Gabber. $\pi_\gamma : \mathbb{P}_\gamma \to \overline{\mathrm{Gr}}_\mu$ is an isomoprhism away from e_0 and we have a decomposition

$$R\pi_{\gamma,*}\bar{\mathbb{Q}}_l[d](d/2) = \mathcal{A}_\mu \oplus \mathcal{C}$$

The zero section $\epsilon_{-\gamma} : G/P \to \mathbb{L}_{-\gamma}$ induces the restriction morphism

$$R\Gamma(\mathbb{L}_{-\gamma}, \bar{\mathbb{Q}}_l) \xrightarrow{\simeq} R\Gamma(G/P, \bar{\mathbb{Q}}_l)$$

which is an isomoprhism since $\mathbb{L}_{-\gamma}$ is a affine fibration. Now this morphism is the direct sum of the identity morphism

$$\mathrm{id} : \mathcal{C} \to \mathcal{C}$$

with the morphism

$$i^* : R\Gamma(\mathrm{Gr}'_{\leq\mu}, \mathcal{A}'_\mu) \to \left(\mathcal{A}'_\mu\right)_{e_0}$$

$\square$

Proposition 8.2. Let $\mathcal{C}$ be the factor supported by e_0 in the decomposition

$$R\pi_{\gamma *}\bar{\mathbb{Q}}_l[d](d/2) = \mathcal{A}_\mu \oplus \mathcal{C}$$

For $i < 0$, we have

$$H^i(\mathcal{C}) = H^{i+d-2}(G/P)(d/2-1)$$

For $i \geq 0$ we have

$$H^i(\mathcal{C}) = H^{i+d}(G/P)(d/2)$$

We can now prove statement 3.2 when case λ is a quasiminuscule cocharacter $\mu = \check{\gamma}$. Consider the discussion after 5.2, it reduces to the case $\nu = 0$.

Proposition 8.3. We have isomorphisms

$$R\Gamma_c(S_0, \mathcal{A}_\mu) \simeq \bar{\mathbb{Q}}_l^{|\Delta_\gamma|}$$

where Δ_γ is the simple roots conjugate to γ.

Proof. By the theorem for base change of proper morphism, we have

$$R\Gamma_c(\pi_\gamma^{-1}(S_0 \cap \overline{\mathrm{Gr}^\mu}, \bar{\mathbb{Q}}_l)[d](d/2) \simeq R\Gamma_c(S_0, \mathcal{A}_\mu) \oplus \mathcal{C} \tag{9}$$

recall that the stratification obtained in 7.8.

$$\pi_\gamma^{-1}(S_0 \cap \bar{\mathrm{G}}\mathrm{r}_\mu) = \bigsqcup_{w\in W/W_\gamma, w\gamma\in\Phi_-} \phi_{-\gamma}^{-1}(UwP/P) \cup \bigsqcup_{w\in W/W_\gamma, w\gamma\in\Phi_+} \epsilon_{-\gamma}(UwP/P)$$

We first compute the dimension of each stratum. We will use the fact that

- If $w\gamma \in \Phi_-$, then $\phi_{-\gamma}^{-1}(UwP/P)$ is an affine space of dimension

  $$\langle \rho, w\mu + \mu\rangle + 1$$

  Indeed, we have an affine bundle of rank 1.

  $$\begin{tikzcd} \phi_\gamma^{-1}(UwP/P) \arrow[d, "\phi_{-\gamma}"] \\ UwP/P \end{tikzcd}$$

  So the dimension of the middle space is $\dim(UwP/P) + 1 = \langle \rho, w\mu + \mu\rangle + 1$, using Lem. 7.8. with

  $$\dim\left(\phi_{-\gamma}^{-1}(UwP/P)\right) \leq d/2$$

  Quillen-Suslin theorem, we even know that this is an affine space, since it is a line bundle over an affine space. This is an equality iff $w\gamma = -l$ for $l \in \Delta$.
- On the other hand if $w\gamma \in \Phi_+$ then the stratum $\epsilon_{-\gamma}(UwP/P)$...

Now we compare the dimensions of the cohomology groups of $R\Gamma_c(\pi_\gamma^{-1}(S_0 \cap \mathrm{Gr}_{\leq\mu}), \bar{\mathbb{Q}}_l)[d]$ and $\mathcal{C}$ in 9, which gives us cohomology of $S_0 \cap \mathrm{Gr}_{\leq\mu}$.

- For $i = 0$. We require $2\langle \rho, \mu\rangle$ to be $\langle \rho, w\mu + \mu\rangle \pm 1$. Equivalently, this is the condition that
  $$\langle \rho, w\mu\rangle = \pm 1$$
  The cardinality of w such that this holds (by splitting to the case positive simple roots) is $2|\Delta_\gamma|$. We also know that $\dim H^0(\mathcal{C}) = |\Delta_\gamma|$.
- For $i > 0$, we require that
  $$\langle \rho, w\mu\rangle > 1$$
  which implies that $w\gamma \in \Phi_+^\vee$. In this case, we compute the cohomology in the second piece in 10. This is precisely the cardinality of the set
  $$|\{w \in W/W_\gamma \,:\, \langle \rho, w\mu + \mu\rangle = (i + d)/2 + 1\}|$$

□

Remark 8.4. The cohomology of such a piece *cannot* be decomposed as

$$\bigoplus_{w\in W/W_\gamma, w\gamma\in\Phi_-} R\Gamma_c\left(\phi_\gamma^{-1}(UwP/P)\right) \oplus \bigoplus_{w\in W/W_\gamma, w\gamma\in\Phi_+} R\Gamma_c\left(\epsilon_{-\gamma}\left(UwP/P\right)\right) \tag{10}$$

Note that $S_0 \cap \mathrm{Gr}_\mu$ is part of a line bundle,

$$\begin{tikzcd}
\mathcal{L}^\times \arrow[d] \\
(G/P)_-
\end{tikzcd}$$

Of which we also have an open closed decomposition

$$\pi^{-1}(\mathrm{Gr}_0) \hookrightarrow \pi^{-1}(S_0 \cap \mathrm{Gr}_{\leq\mu} \hookleftarrow \pi^{-1}(S_0 \cap \mathrm{Gr}_\mu)$$

Let us now prove statemet 3.4 in the case $\nu = 0$ and $\lambda = \mu$ quasi-minuscule. We actually prove something more general. Recall that for each $\sigma \in X_\bullet$, we defined a morphism $h_\sigma : S_0 \to \mathbb{G}_a$ see Eq. 4.

Proposition 8.5. For each $\sigma \in X_{\bullet,+}$ we have the isomoprhism

$$R\Gamma_c(S_0, \mathcal{A}_\mu \otimes h_\sigma^*\mathcal{L}_\psi) = \bar{\mathbb{Q}}_l^{|\Delta_\gamma^\sigma|}$$

where Δ_γ^σ is the set of $\alpha \in \Delta_\gamma$ such that $\langle \alpha, \sigma\rangle > 0$.

Example 8.6. In GL_n, let γ be the quasi-minuscule coroot, $e_i^\vee - e_n^\vee$. $\Delta_\gamma = \Delta$, then $\Delta_\gamma^\sigma = \{\alpha : \langle \alpha, \sigma\rangle > 0\}$ Thus, this counts precisely the number of strictly positive jumps.

The proof of 8.5 is the same as 8.3. We explain it here. The cohomology we are to compute is the sum of the following three pieces:
which is , a particular case of 8.5. It suffices to prove the following geometric statement.

Lemma 8.7. (1) The restrictions $h_\sigma \circ \pi_\gamma$ on each stratum $\epsilon_{-\gamma}(UwP/P)$.

(2) the restrictions to stratum $\phi_{-\gamma}^{-1}(UwP/P)$

(3) The restriction on the latter are linear when restricted to the right bundle $\mathbb{L}_{-\gamma}$.

8.1. Recollection of the work of Kazhdan Lusztig.

[to be added]

9. Convolution

The goal of this section is to prove the following diagram

$$\{\mu_\bullet\text{-dominant paths from 0 to }\nu\} \xrightarrow{\simeq} \mathrm{Irr}\left(\pi^{-1}(S_\nu \cap \mathrm{Gr}_\nu)\right)$$

A better reference is [**17**, 2.1.4].
Let us first recall the construction of twisted product

$$\mathrm{Gr}$$

Recall that M is the minimal cocahracters in $X_{\bullet,+}$. For each $\mu_\bullet = (\mu_1, \dots, \mu_n)$ of elements in M, we can construct the projective subscheme

$$\overline{\mathrm{Gr}^{\mu_\bullet}} = \overline{\mathrm{Gr}^{\mu_1}}\tilde{\times}\cdots\tilde{\times}\overline{\mathrm{Gr}^{\mu_n}} \hookrightarrow_{\mathrm{cl}} \mathrm{Gr}^n$$

The projection of the lass factors of Gr^n defines a proper morphism

$$\overline{\mathrm{Gr}^{\mu_\bullet}} \xrightarrow{m_{\mu_\bullet}} \overline{\mathrm{Gr}^{|\mu_\bullet|}}$$

where $|\mu_\bullet| = \sum_{i=1}^n \mu_i$. Let $\nu_\bullet$ be collection of elements in $X_\bullet$. For $i = 1, \dots, n$, denote $\sigma_i := \nu_1 + \cdots + \nu_i$, we denote

$$S_{\nu_\bullet} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet} := (S_{\sigma_1} \times \cdots \times S_{\sigma_n}) \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet}$$

in Gr^n. It is clear that $S_{\nu_\bullet}$.

Proposition 9.1. We have a canonical isomorphism

$$S_{\nu_\bullet} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet} \xleftarrow{\simeq} (S_{\nu_1} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_1}) \times \cdots \times \left(S_{\nu_n} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_n}\right)$$

Proof. One can show easily by recurrence that each point

$$(y_1, \dots, y_n) \in S_{\nu_\bullet} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet}$$

can be uniquely written as

$$\begin{aligned} y_1 &= x_1 \varpi^{\nu_1} e_0 \\ &\cdots \\ y_n &= x_1 \varpi^{n_1} \cdots x_n \varpi^{\nu_n} e_0 \end{aligned}$$

☐

Example 9.2. The decomposition of $y_1, y_2, \dots, y_n$ is an inductive application of the decomposition

$$L^{<\nu_i}N \times L^{\geq \nu_i}N \simeq LN$$

for $i = 1, \dots, n$. In the case of $y_1 \in S_{\nu_1}$, we have

$$\begin{aligned} y_1 &= x\varpi^{\nu_1} \\ &= x_{<\nu_1}\varpi^{\nu_1}x_+ \\ &= x_1\varpi^{\nu_1} \end{aligned}$$

where

$$x = x_{<\nu_1} x_{\geq \nu_1} \in LN, \quad x_{<\nu_1} \in L^{<\nu_1}N, x_{\geq\nu_1} \in L^{\geq \nu_1}N$$
$$x_{\geq \nu_1} = \varpi^{\nu_1} x_+ \varpi^{-\nu_1}, x_+ \in L^{\geq 0}N, \quad x_1 := x_{<\nu_1}$$

and equality is taken as coset class.

$$\begin{aligned} y_2 &= x' \varpi^{\sigma_2} \\ &= (x_1 \varpi^{\nu_1})(x_1\varpi^{\nu_1})^{-1} x' \varpi^{\nu_1} \varpi^{\nu_2} \\ &= (x_1 \varpi^{\nu_1})(\mathrm{ad}((\varpi^{\nu_1})^{-1})(x_1^{-1}x'))\varpi^{\nu_2} \end{aligned}$$

where

$$x' \in LN$$

Corollary 9.3. Let $\mu_1, \dots, \mu_n$ be elements of M. For all $\nu_\bullet$ with $\nu_i \in \Omega(\mu_i)$, all the components of $S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet}$ are o dimension $\langle \rho, |\nu_\bullet| + |\mu_\bullet| \rangle$.

Proof. By 5.2 and 7.11, each $S_{\nu_i} \cap \mathrm{Gr}_{\leq \mu_i}$ has dimension $\langle \rho, \nu_i + \mu_i \rangle$. The corollary thus follows from previous lemmas. □

In fact for arbitrary $\mu \in X_{\bullet,+}$ and $\nu \in \Omega(\mu)$, $S_\nu \cap \mathrm{Gr}_{\leq \mu}$, is pure of dimension $\langle \rho, \nu + \mu \rangle$. This result is stated with not many proof. We were able to prove this using affine lie algebras. Let us put out that we can deduce this dimension formula, without the assertion of pure dimension from 3.2.

Proposition 9.4. The convolution product $\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$ is a perverse sheaf. It decomposes as a direct sum

$$\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} \simeq \bigoplus_{\lambda \leq |\mu_\bullet|} \mathcal{A}_\lambda \otimes V^\lambda_{\mu_\bullet}$$

where the $V^\lambda_{\mu_\bullet}$ is the $\bar{\mathbb{Q}}_l$ vector space whose dimension is the number of irreducible components of $m^{-1}_{\mu_\bullet}(S_\lambda \cap \mathrm{Gr}_{\leq |\mu_\bullet|}$ which are entirely contained in $m^{-1}_{\mu_\bullet}(S_\lambda \cap \mathrm{Gr}_{\leq \lambda})$.

Definition 9.5. Let $\mu_\bullet$ denote a sequence of elements in M. Following [**9**], we call a $\mu_\bullet$ *-path* the following combinatorial data:

- A sequences of vertices in $X_\bullet$ such that for all $i = 1, \dots, n$ we have $\nu_i = \sigma_i - \sigma_{i-1} \in \Omega(\mu_i)$.
- the maps
$$p_i : [0,1] \to X_\bullet \otimes_{\mathbb{Z}} \mathbb{R}$$
satisfying :
  (1) if $\sigma_{i-1} \neq \sigma_i$ we have
  $$p_i(t) = (1-t)\sigma_{i-1} + t\sigma_i$$
  (2) if $\sigma_{i-1} = \sigma_i$ then
  $$p_i(t) = \begin{cases} \sigma_{i-1} - t\alpha_i^\vee & 0 \leq t \leq 1/2 \\ \sigma_{i-1} + (t-1)\alpha_i^\vee & 1/2 \leq t \leq 1 \end{cases}$$
  where $\alpha_i^\vee \in \Delta^\vee_{\mu_i}$, i.e. $\alpha_i^\vee$ is simple coroot conjugate to μ_i.

By putting the images of p_is at the end points, we get a path in $X_\bullet \otimes_{\mathbb{Z}} \mathbb{R}$ going from 0 to σ_n, with vertices $0, \sigma_1, \ldots, \sigma_n$.

Remark 9.6. This is later used in 11.1. For a fix $\mu_\bullet$ and sequence $\nu_\bullet$ induced from the vertices σ_i, how many little man paths are there? Indeed, this should be given by the product $|\Delta_{\mu_i}|$ for i such that $\sigma_i = \sigma_{i-1}$. In the set up of 11.1, this is precisely the points where $\nu_i = 0$, μ_i is quasimuscule.

The $\mu_\bullet$-path is called *dominant*, if the entire image is contained in the dominant chamber, $(X_\bullet \otimes_{\mathbb{Z}} \mathbb{R})_+$.

After 5.2, each $S_{w\mu_i} \cap \mathrm{Gr}_{\leq \mu_i}$ is irreducible. Thus by 7.11, if $\mu_i = \gamma_i^\vee$ is quasiminuscule , and if $\nu = 0$, then we have a bijection

$$\mathrm{Irr}(S_0 \cap \mathrm{Gr}_{\leq \mu_i}) \simeq \Delta_{\mu_i^\vee}$$

Proposition 9.7. for all $\nu \in \Omega(|\mu_\bullet|)$ the set of irreducible components of $\pi^{-1}(S_\nu \cap \bar{\mathrm{Gr}}_{\leq |\mu_\bullet|})$ is in canonical bijection with the $\mu_\bullet$ paths χ from 0 to ν.

Proof. Consider 9.1, we know the set of such components are the irreducible components of $S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet}$ for $|\nu_\bullet| = \nu$. These are counted by considering the number of irreducible components of each $S_{\nu_i} \times \mathrm{Gr}_{\leq \mu_i}$. Result follows then from observation in previous paragraph. ☐

Definition 9.8. Let C_χ denote the component corresponding to χ.

Proposition 9.9. For $\nu \in \Omega(|\mu_\bullet|)$ dominant and χ is a $\mu_\bullet$ dominant path starting from 0 to ν, then the component C_χ is contained in $\pi^{-1}(S_\nu \cap \overline{\mathrm{Gr}^\nu})$. [16]

Proof. Denote $I(\chi)$ the set of indices $i = 1, \ldots, n$ such that $\sigma_{i-1} = \sigma_i$.

- If $i \notin I(\chi)$, ν_i is nonzero and is thus conjugate to μ_i.
- If $i \in I(\chi)$ and μ_i is quasiminsucule, thus $\mu_i = \gamma_i^\vee$, and the hypothesis that χ is dominant implies that $\langle \alpha_i, \sigma_{i-1} \rangle \geq 1$. In fact, the conditions are equivalent. Indeed:

  $$\left\langle \sigma_{i-1} - t\alpha_i^\vee, \beta \right\rangle \geq 0 \quad \beta \in \Delta_s, 0 \leq t \leq \frac{1}{2}$$

  This is equivalent to

  $$\langle \sigma_{i-1}, \beta \rangle \geq t \left\langle \alpha_i^\vee, \beta \right\rangle \quad 0 \leq t \leq \frac{1}{2}$$

  If $\beta = \alpha_i$, this is equivalent to the condition

  $$\langle \sigma_{i-1}, \alpha_i \rangle \geq 1$$

  For other β, the other condition is vacuous: since for any nonequal simple roots, α, β, we have that $\langle \beta^\vee, \alpha \rangle \leq 0$, [**5**, Ch. 6.3]

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[16] How do we think of this π^{-1} what are we supposed to show here?

By 7.11, the irreducible component of $S_0 \cap \mathrm{Gr}_{\leq \gamma_i^\vee}$ corresponding to $\alpha_i = w\gamma_i$ is contained in the trivial $\mathbb{G}_m$-torsor,

$$\phi_{\gamma_i}^{-1}(Uw_iP_i/P_i) \backslash \epsilon_{\gamma_i}^{-1}(Uw_iP_i/P_i)$$

By the proof 8.7, for each $i \in I(\chi)$, each point

$$p_i \in \phi_{\gamma_i}^{-1}(Uw_iP_i/P_i) \backslash \epsilon_{\gamma_i}^{-1}(Uw_iP_i/P_i)$$

can be written uniquely in the form

$$uU_{\alpha_i,-1}(x)e_0 \quad u \in U \cap w^{-1}U_{\gamma_i}^{+}w \quad x \in \mathbb{G}_m$$

□

Example 9.10. Path of 2 terms. $G = \mathrm{GL}_4$, here $\gamma^\vee = e_1^\vee - e_4^\vee$. We consider the simple Weyl conjugate $\alpha^\vee = e_1^\vee - e_2^\vee$.

$$0 \to \gamma^\vee \to \gamma^\vee + \gamma^\vee$$

So our condition requires that

$$\left\langle \gamma^\vee - t\alpha^\vee, \beta \right\rangle \geq 0 \quad 0 \leq t \leq \frac{1}{2}, \beta \in \Delta_s$$

hence

$$\left\langle \gamma^\vee, \beta \right\rangle \geq t \left\langle \alpha^\vee, \beta \right\rangle \quad 0 \leq t \leq \frac{1}{2}\beta \in \Delta_s$$

If $\beta = \alpha$, this shows that $\langle \gamma^\vee, \alpha \rangle \geq 1$. However, if $\beta = e_2 - e_3$, the inequality does not yield any conditions.

It is not difficult to prove conversely that if the $\mu_\bullet$ path χ is not dominant then $C_\chi \not\subseteq \pi^{-1}(S_\nu \cap \mathrm{Gr}^\mu)$. We leave this to the reader because it is not logically necessary for the rest of the paper. It will only be necessary for us to know that the multiplicity of $\mathcal{A}_\nu$ in $\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$, satisfies

$$\dim(V_{|\mu_\bullet|}) \leq |\mu_\bullet\text{-path } \chi \text{starting from 0 to } \nu|$$

Proposition 9.11. For all $\lambda \in X_{\bullet,+}$, $\mathcal{A}_\lambda$ is a director factor of a convolution product of the form

$$\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$$

with $\nu_1, \ldots, \mu_n \in M$.

Taken into account 9.4 and 9.9 it suffices to show that there exists a dominant $\mu_\bullet$ path from 0 to ν. We prove this combinatorial statement in 10.

Corollary 9.12. Let $\lambda, \lambda' \in X_{*,+}$, the product $\mathcal{A}_\lambda * \mathcal{A}_{\lambda'}$ is perverse.

Proof. $\mathcal{A}_\lambda$ and $\mathcal{A}_{\lambda'}$ are direct factors of $\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$ and $\mathcal{A}_{\mu_1'} * \cdots * \mathcal{A}_{\mu_n'}$. Then $\mathcal{A}_\lambda * \mathcal{A}_{\lambda'}$ is a direct summand of

$$\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} * \mathcal{A}_{\mu_1'} * \cdots * \mathcal{A}_{\mu_n'}$$

□

10. Combinatoire

We omit this section.

11. Fin des démonstrations

We use the notation of Sec. 9. In particular let $\lambda \in X_{\bullet,+}$ and $\mu_\bullet = (\mu_1, \dots, \mu_n)$ elements of M such that $\mathcal{A}_\lambda$ is a direct factor of $\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$, see 9.11.
Proof: consider the ... it suffices to show that the complex

$$R\Gamma_c(S_\nu, \mathcal{A}_1 * \cdots * \mathcal{A}_{\mu_n}) \simeq R\Gamma_c\left(m_{\mu_\bullet}^{-1}\left(S_\nu \cap \mathrm{Gr}^{\leq \mu_\bullet}\right), \mathrm{IC}(\mathrm{Gr}^{\leq \mu_\bullet})\right)$$

Recall that we have the stratification

$$m_{\mu_\bullet}^{-1}\left(S_\nu \cap \mathrm{Gr}^{\leq \mu_\bullet}\right) = \bigcup_{|\nu_\bullet| = \nu} S_\nu \cap \mathrm{Gr}^{\leq \mu_\bullet}$$

and, after Lemma 9.1, we have an isomorphism

$$S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet} \simeq S_{\nu_1} \cap \mathrm{Gr}_{\leq \mu_1} \times \cdots \times (S_{\nu_n} \times \mathrm{Gr}_{\leq \mu_n}) \tag{11}$$

Further this isomorphism induced from the isomorphism of local triviality

$$\varpi^{\mu_1} L^{<0} G e_0 \cap \mathrm{Gr}_{\leq \mu_1}$$

$$R\Gamma_c(S_{\nu_\bullet} \cap \mathrm{Gr}^{\mu_\bullet}, \mathrm{IC}(\mathrm{Gr}^{\leq \mu_\bullet})) \simeq \bigotimes_{i=1}^{n} R\Gamma_c\left(S_{\nu_i} \cap \mathrm{Gr}^{\leq \mu_i}, \mathcal{A}_{\mu_i}\right)$$

Then result follows from Lem. 5.2 and Lem. 8.5.
Proof of theorem Thm. 3.4 Recall that the easy case when $\nu = \lambda$ was discussed after Lem. 5.2. We now prove the more difficult case $\nu \neq \lambda$.
The sequence $\mu_\bullet$, was chosen so that the multiplicity

$$V_{\mu_\bullet}^{\lambda}$$

of $\mathcal{A}_\lambda$ in the decomposition 9.4,

$$\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} \simeq \bigoplus_{\xi \leq |\mu_\bullet|, \xi \in X_{\bullet,+}} \mathcal{A}_\xi \otimes V_{\mu_\bullet}^{\xi}$$

We deduce the decomposition equality $V_{\mu_\bullet}^{\lambda} \neq 0$ and that $\lambda \neq \nu$ to show that

$$R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h^* \mathcal{L}_\psi)$$

it suffices to show that the canonical map

$$R\Gamma_c(S_\nu, \mathcal{A}_\nu \otimes h^* \mathcal{L}_\psi) \otimes V_{\mu_\bullet}^{\nu} \xrightarrow{\simeq} R\Gamma_c(S_\nu, \mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} \otimes h^* \mathcal{L}_\psi)$$

which is a quasi isomorphism. Now from the discussion following lemma, 5.2, Combining this with the trivial case we have just proven in Thm 3.2,

$$R\Gamma_c(S_{\nu_\bullet} \cap \mathrm{Gr}_{\mu_\bullet})$$

Recall that in the stratification

$$m_\bullet^{-1} = \bigcup_{|\nu_\bullet|} S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet}$$

each point $(y_1, \dots, y_n) \in S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet}$ can be written in the unique form, see 9.1,

$$\begin{aligned} y_1 &= x_1 \varpi^{\nu_1} e_0 \\ &\cdots \\ y_n &= x_1 \varpi^{\nu_1} \cdots x_n \varpi^{\nu_n} e_0 \end{aligned}$$

For each $\sigma \in X_\bullet$, we denote h_σ as the composition $LU \xrightarrow{\mathrm{ad}(\sigma)} LU \xrightarrow{h} \mathbb{G}_a$, so that $x \mapsto h(\mathrm{ad}(\sigma)x)$. It is clear that

$$h(y_n) = h(x_1) + h_{\sigma_1}(x_2) + \cdots + h_{\sigma_{n-1}}(x_n)$$

which uses the decompostion

$$y_n = x_1 \mathrm{ad}(\varpi^{\sigma_1}) x_2 \cdots \mathrm{ad}(\varpi^{\sigma_{n-1}}) x_n \varpi^{\sigma_n}$$

Lemma 11.1. If $\sigma \notin X_{\bullet,+}$ we have that

$$R\Gamma_c(S_{\nu'}, \mathcal{A}_{\lambda'} \otimes h^* \mathcal{L}_\psi) = 0$$

PROOF. Observe that the $\mathbb{G}_a$ action on S_ν is induced from the constant embedding

$$\mathbb{G}_a \hookrightarrow LN \circlearrowright LN$$

Let $\alpha \in \Phi$ be a simple root such that $\langle \alpha, \sigma \rangle$ is strictly negative. [17] The subgroups

$$\mathbb{G}_a := U_{\alpha, -(\alpha,\sigma)-1}$$

is contained in $L^{\geq 0}U$ thus act equivaraintly on $(S_\nu, \mathcal{A}_\lambda)$. Thus the restriction of h_σ to the subgroup induces the identity on $\mathbb{G}_a$.
This is equivalent to stating that the existence of commutative diagram.

$$\begin{tikzcd}
\mathbb{G}_a \times S_\nu \arrow[r, hook] \arrow[d, "\mathrm{id} \times h_\sigma"] & LU \times S_\nu \arrow[r, "a"] & S_\nu \arrow[d, "h_\sigma"] \\
\mathbb{G}_a \times \mathbb{G}_a \arrow[rr, "a"] & & \mathbb{G}_a
\end{tikzcd}$$

Via identifying S_ν as the orbit of $LN \circlearrowright \mathrm{Gr}_G$, this square is equivalent to

$$\begin{tikzcd}
\mathbb{G}_a \times LN \arrow[r, hook] \arrow[d, "\mathrm{id} \times h_\sigma"] & LN \times S_\nu \arrow[r, "a"] & LN \arrow[d, "h_\sigma"] \\
\mathbb{G}_a \times \mathbb{G}_a \arrow[rr, "a"] & & \mathbb{G}_a
\end{tikzcd}$$

where the bottom map is the additive map, and the upper map is the natural LN action on itself. This diagram implies

$$\mathrm{act}^* h_\sigma^* \mathcal{L}_\psi \simeq h_\sigma^* \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi$$

---

[17] This is the part where we needed σ to be nondominant, this guarantees the embedded copy of $\mathbb{G}_a$ is in the strict upper borel.

Thus by monoidality of act^*,

$$\begin{aligned}\mathrm{act}^*\left(\mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi\right) &\simeq \mathrm{act}^* \mathcal{A}_\lambda \otimes \mathrm{act}^* h_\sigma^* \mathcal{L}_\psi \\ &\simeq \mathrm{act}^* \mathcal{A}_\lambda \otimes (\mathrm{id} \times h_\sigma)^* a^* \mathcal{L}_\psi \\ &\simeq \mathrm{act}^* \mathcal{A}_\lambda \otimes (h_\sigma^* \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi)\end{aligned}$$

Now recall that the box tensor product satisfies

$$(A \otimes B) \boxtimes (C \otimes D) \simeq (A \boxtimes C) \otimes (B \boxtimes D)$$

It suffices to apply [**13**, Lemme 3.3]. ☐

We deduce the vanishing

$$R\Gamma_c((S_{\nu_\bullet} \cap \mathrm{Gr}_{\mu_\bullet}), \mathrm{IC}_{\mathrm{Gr}_{\leq \mu_\bullet}} \otimes h^* \mathcal{L}_\psi) = 0$$

for the case when $\nu_\bullet$ of which at least one of the partial sums σ_i are non dominant. Let us suppose now $\nu_\bullet$ where each $\nu_i \in \Omega(\mu_i)$ are such that the partial sums are dominant. We say a $\mu_\bullet$ path is of type $\nu_\bullet$ if it has vertices $0, \sigma_1, \ldots, \sigma_n$. Let us observe that the condition $\langle \alpha, \sigma \rangle \geq 1$ in 8.5 is equivalent to the condition $\alpha^\vee/2 + \sigma$ is dominant, i.e. see 9.9.

Putting together Lem. 5.5 and Lem. 8.5 we arrive the following: for $i \neq \langle 2\rho, \nu \rangle$, we have

$$H_c^i(S_{\nu_\bullet} \cap \bar{\mathrm{Gr}}_{\mu_\bullet}, \mathrm{IC}(\bar{\mathrm{Gr}}_{\mu_\bullet}) \otimes h^* \mathcal{L}_\psi) = 0$$

and for $i = 2\langle \rho, \nu \rangle$ we have

$$\dim(V_{\mu_\bullet}^\nu) \geq \dim H_c^i(S_\nu, \mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} \otimes h^* \mathcal{L}_\psi)$$

Result follows.

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