Integral Aspects of Fourier Duality

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- History: Work of Beauville which studied Chow groups of abelian varieties via Fourier transforms.

- Overview:

  i) Abelian varieties.

  ii) Chow groups

  iii) Fourier transforms.

  iv) what we have done.

- For this talk, fix base field k, of char $= 0$.

Abelian Varieties.

sep. ft, int.

Def'n: An abelian variety over k. $X \in \mathrm{Var}_k$.

i) $X \in \mathrm{ComGrp}(\mathrm{Var}_k)$

ii) X is proper.

Let AVar_k be the category of abelian variety.

Cor: X is smooth.

Ex: $k = \mathbb{C}$. $X(\mathbb{C})^{an} \underset{\text{GAGA.}}{\simeq} \mathbb{C}^g / \Lambda \simeq (S^1)^g$

lattice, topologically.

$\therefore$ Abelian varieties $/\mathbb{C}$ are complex torus of dim g. [if X has dim $= g$].

Ex: $\dim X = 1$. $\mathrm{AVar}_k \simeq \mathrm{Ell}_k$ is the same as cat. of elliptic curves.

Chow Groups.

- algebraic versions of singular cohomology.
  $(Z \hookrightarrow X)$ $\qquad\qquad$ $(\Delta^j \to X)$
- let $X \in \mathrm{AVar}_k$ (defn work for any Var_k)

Defn:

$$Z_j(X) := \bigoplus_{\substack{Z \hookrightarrow X \\ \dim j}} \mathbb{Z}[Z] = \left\{ \sum_Z a_Z [Z] : a_Z \in \mathbb{Z} \right\}$$

"j-cycles"

$$Z^j(X) := \bigoplus_{\substack{Z \hookrightarrow X \\ \mathrm{codim} = j}} \mathbb{Z}[Z].$$

"j-cocycles"

Problem: these are quite big. There are var. equivalences. We consider rational equivalence.

It is the equivalence gen. by the following quadruples (V, p, q, f).

$$\begin{tikzcd}
f^{-1}(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \hookrightarrow X \times \mathbb{P}^1 \arrow[d, "f"] \arrow[dr, phantom, "\llcorner"] & f^{-1}(q) \arrow[l] \arrow[d] \\
\{p\} \arrow[r] & \mathbb{P}^1 & \{q\} \arrow[l]
\end{tikzcd}$$

- $V \in Z_{j+1}(X \times \mathbb{P}^1)$ $(j+1)$-dim subvariety in $X \times \mathbb{P}^1$.
- $p, q \in \mathbb{P}^1$
- $f: V \longrightarrow \mathbb{P}^1$ is a dominant map.

If we take $f^{-1}(p)$, $f^{-1}(q)$ these are two j-dim. sub var of X. (i.e. $f^{-1}(p), f^{-1}(q) \in Z_j(X)$)

We demand these to be equivalent

$$f^{-1}(p) \sim_{rat} f^{-1}(q)$$

Defn: $CH^{\bullet}(X) := \bigoplus_j Z^j(X)/\sim_{rat}$

$$=: \bigoplus_j CH^j(X).$$

Ex: $CH^1(X) = \{ \text{1-cocycles} \} / \sim_{rat}$

$\cong \operatorname{Div}(X) / \text{principal divisors}$

$\cong \operatorname{Pic}(X)$

(line bdles on X) $/ \sim$ iso.

Ex: if X is dim g.

$$CH^0(X) = \mathbb{Z}[X].$$

Chow groups have two ring str.

Note: if $f: X \to Y$ is a map

then $\exists$ a map $f^*: CH(Y) \to CH(X)$.

i) The intersection product $\cap$

$$CH(X) \times CH(X) \to CH(X \times X) \xrightarrow{\Delta^*} CH(X)$$

$$[Z], [W] \mapsto [Z \times W] \mapsto \Delta^*([Z \times W])$$

This is characterized by:

if Z, W intersect transversally, then

$[Z] \cap [W] := \Delta^*([Z \times W]) = [Z \cap W]$.

ii) The convolution structure, $*$

$$CH(X) \times CH(X) \to CH(X \times X) \xrightarrow{m} CH(X)$$

$$[Z], [W] \mapsto [Z \times W] \mapsto m_*([Z \times W])$$

where m: is the multiplication $X \times X \to X$

$[Z] * [W] := m_*([Z \times W])$.

Understanding via Chern character map.

Def'n: let $K(X)$ be the Grothendieck group of coherent sheaves on X.

- let $\mathrm{Pic}(X)$ be the group ($\otimes$) of line bdles on X upto isomorphism.

"The free group on vector bdles on X".

- Need to know: if $\mathcal{E}$ is v.b. on X, then regard it as locally free sheaf on X, giving an element $[\mathcal{E}] \in K(X)$.

There is a chern character map is a ring map.

$$(K(X)_{\mathbb{Q}}, \otimes) \xrightarrow{ch} (CH(X)_{\mathbb{Q}}, \cap)$$

Induced from

$$(\mathrm{Pic}(X)_{\mathbb{Q}}, \otimes) \xrightarrow[\sim]{ch_1} (CH^1(X)_{\mathbb{Q}}, \cap)$$

$$[\mathcal{O}(D) \simeq \mathcal{L}] \longmapsto [D]$$

We set

$$ch(\mathcal{L}) := e^{\mathcal{L}} = \sum_{n=0}^{\infty} \frac{ch_1(\mathcal{L})^n}{n!}$$

only has finitely many terms.

$$ch_1(\mathcal{L})^n := ch_1(\mathcal{L}) \cap ch_1(\mathcal{L}) \cap \dots \cap ch_1(\mathcal{L}).$$

The integral Fourier Transform.

Defn: let $X \in AVar_k^{\dim = g}$.

- There $\exists$ a canonical line bdle on $X \times X^\vee$, $\mathcal{P}$ where $X^\vee$ is the dual abelian variety. (Pic^0_X)
- we obtain $ch(\mathcal{P}) \in CH(X \times X^\vee)$. from chern character map.

Consider diagram:

$$\begin{tikzcd} & X \times X^\vee \arrow[dl, "p_X"'] \arrow[dr, "p_{X^\vee}"] & \\ X & & X^\vee \end{tikzcd}$$

i) let $\Lambda := \mathbb{Z}\left[\frac{1}{(2g+1)!}\right]$, $\gamma := 2g!\, ch(\mathcal{P})$

$\gamma^\vee := (2g)!\, ch(\mathcal{P}^\vee)$

$$\begin{aligned} &\quad\;\; CH(X) \otimes \Lambda \\ &\quad\;\; \| \\ \mathcal{F} :\; & CH(X)_\Lambda \longrightarrow CH(X^\vee)_\Lambda \\ & \alpha \longmapsto \frac{1}{(2g+1)!} p_{*,X^\vee}(p_X^*(\alpha) \cap \gamma) \end{aligned}$$

Fourier Mukai

$$\begin{aligned} \mathcal{F}^\vee :\; & CH(X^\vee)_\Lambda \longrightarrow CH(X)_\Lambda \\ & \alpha \longmapsto \frac{1}{(2g+1)!} p_{*,X}(p_{X^\vee}^*(\alpha) \cap \gamma^\vee). \end{aligned}$$

Then: we obtain

$$(CH(X)_\Lambda, *) \underset{F}{\overset{F^\vee}{\rightleftarrows}} (CH(X^\vee)_\Lambda, \cap)$$

and we proved that

$$F^\vee \circ F = (-1)^g \circ [-1]^*$$

where $[-1]^* : CH(X)_\Lambda \to CH(X)_\Lambda$

induced from $X \xrightarrow{-1} X$

($x \mapsto -x$ at level of points)

History: this was done by Beauville over $\mathbb{Q}$.

Cor: can study torsion aspects.